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ω-TOPOLOGY AND α-TOPOLOGY

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Abstract: The aim of this paper is to introduce and investigate the new notions called b-ω_α-open sets, α-ω_α-open sets and pre-ω_α-open sets which are weaker than ω-open sets. Moreover decompositions of continuity are obtained by using these new notions.

Keywords and Phrases: α-ω_α-open set, ω-closed set, semi-open set, ω-open set.

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1. Introduction

By a space (X, τ), it means a topological space (X, τ) with no separation properties assumed. If $H \subset X$, $cl(H)$ and $int(H)$ will, respectively, denote the closure and interior of H in (X, τ).

Definition 1.1. [11] *A subset H of a space (X, τ) is called*
1. α-closed if $cl(int(cl(H))) \subset H$,
2. α-open if $X \setminus H$ is α-closed, or equivalently, if $H \subset int(cl(int(H)))$.
For a subset H of (X, τ), the intersection of all α-closed subsets of X containing H is called the α-closure of H and is denoted by $cl_\alpha(H)$. It is known that $cl_\alpha(H) = H \cup cl(int(cl(H)))$ and $cl_\alpha(H) \subset cl(H)$. The union of all α-open subsets of X contained in H is called the α-interior of H and is denoted by $int_\alpha(H)$

In 1982, the notions of ω-closed sets and ω-open sets were introduced and studied by Hdeib [6]. In 2009, Noiri et al [12] introduced some generalizations of

ω-open sets and investigated some properties of the sets. Moreover, they used them to obtain decompositions of continuity.

The aim of this paper is to introduce and investigate the new notions called b-ω_α-open sets, α-ω_α-open sets and pre-ω_α-open sets which are weaker than ω-open sets. Moreover decompositions of continuity are obtained by using the new notions.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}^\star$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers). τ_u denotes the usual topology.

Definition 2.1. *A subset H of a space (X, τ) is said to be*
1. semi-open [9] if $H \subset cl(int(H))$
2. pre-open [10] if $H \subset int(cl(H))$,
3. β-open [1] if $H \subset cl(int(cl(H)))$,
4. b-open [4] if $H \subset int(cl(H)) \cup cl(int(H))$.

Properties. [8] Let H be a semi-open set of a space (X, τ). Then $cl_\alpha(H) = cl(H)$.

Definition 2.2. [13] *Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.3. [6] *A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open.*

It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.4. [6] *Let H be a subset of a space (X, τ). Then*
1. H is ω-closed in X if and only if $H = cl_\omega(H)$.
2. $cl_\omega(X \backslash H) = X \backslash int_\omega(H)$.
3. $cl_\omega(H)$ is ω-closed in X.
4. $x \in cl_\omega(H)$ if and only if $H \cap G \neq \emptyset$ for each ω-open set G containing x.
5. $cl_\omega(H) \subset cl(H)$.
6. $int(H) \subset int_\omega(H)$.

Remark [2, 6] In a space $(X, \tau,)$ every closed set is ω-closed but not conversely.

Definition 2.5. [12] *A subset H of a space (X, τ) is said to be*
1. α-ω-open if $H \subset int_\omega(cl(int_\omega(H)))$,
2. pre-ω-open if $H \subset int_\omega(cl(H))$,

3. β-ω-open if $H \subset cl(int_\omega(cl(H)))$,
4. b-ω-open if $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$.

Definition 2.6. [3] *A space* (X, τ) *is said to be anti-locally countable if every non-empty open set is uncountable.*

Lemma 2.7. [3] *If* (X, τ) *is an anti-locally countable space, then* $int_\omega(H) = int(H)$ *for every ω-closed set* H *of* X *and* $cl_\omega(H) = cl(H)$ *for every ω-open set* H *of* X.

3. Weak forms of ω-open sets

In this section the following notions are newly defined.

Definition 3.1. *A subset* H *of a space* (X, τ) *is said to be*
1. α-ω_α*-open if* $H \subset int_\omega(cl_\alpha(int_\omega(H)))$,
2. *semi-*ω_α*-open if* $H \subset cl_\alpha(int_\omega(H))$,
3. *pre-*ω_α*-open if* $H \subset int_\omega(cl_\alpha(H))$,
4. β-ω_α*-open if* $H \subset cl_\alpha(int_\omega(cl_\alpha(H)))$,
5. *b-*ω_α*-open if* $H \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H))$.

Properties. Let H be a subset of a space (X, τ). Then H is α-ω_α-open if and only if it is semi-ω_α-open and pre-ω_α-open.
Proof. (⇒) Let H be an α-ω_α-open set. Then $H \subset int_\omega(cl_\alpha(int_\omega(H))) \subset int_\omega(cl_\alpha(H))$ and $H \subset int_\omega(cl_\alpha(int_\omega(H))) \subset cl_\alpha(int_\omega(H))$. Thus H is pre-ω_α-open and semi-ω_α-open. (⇐) Let H be a pre-ω_α-open set. Then $H \subset int_\omega(cl_\alpha(H))$. Also, $H \subset cl_\alpha(int_\omega(H))$, since H is semi-ω_α-open. We have $H \subset int_\omega(cl_\alpha(int_\omega(H)))$. Thus H is α-ω_α-open.

Remark. The following Examples shows that the concepts of semi-ω_α-openness and pre-ω_α-openness are independent.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset H$. But H is not semi-$\omega_\alpha$-open, since $cl_\alpha(int_\omega(H)) = cl_\alpha(\phi) = \phi \not\supseteq H$.

Example. In $(\mathbb{R}, \tau_u)$, $H = (0, 1]$ is semi-ω_α-open, since $cl_\alpha(int_\omega(H)) = cl_\alpha((0, 1)) = [0, 1] \supset H$. But H is not pre-$\omega_\alpha$-open, since $int_\omega(cl_\alpha(H)) = int_\omega([0, 1]) = (0, 1) \not\supseteq H$.
Theorem 3.2. *For a subset of a space* (X, τ)*, the following properties hold:*
1. *Every ω-open set is* α-ω_α*-open.*
2. *Every* α-ω_α*-open set is pre-*ω_α*-open.*
3. *Every pre-*ω_α*-open set is b-*ω_α*-open.*
4. *Every b-*ω_α*-open set is* β-ω_α*-open.*

Proof.
1. If H is an ω-open set, then $H = int_\omega(H)$. Since $H \subset cl_\alpha(H)$, $H \subset cl_\alpha(int_\omega(H))$ and $int_\omega(H) \subset int_\omega(cl_\alpha(int_\omega(H)))$. Therefore $H \subset int_\omega(cl_\alpha(int_\omega(H)))$ and H is α-ω_α-open.
2. If H is an α-ω_α-open set, then $H \subset int_\omega(cl_\alpha(int_\omega(H))) \subset int_\omega(cl_\alpha(H))$. Therefore H is pre-ω_α-open.
3. If H is a pre-ω_α-open set, then $H \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H))$.

Therefore H is b-ω_α-open.
4. If H is a b-ω_α-open set, then $H \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) \subset cl_\alpha(int_\omega(cl_\alpha(H))) \cup cl_\alpha(int_\omega(H)) \subset cl_\alpha(int_\omega(cl_\alpha(H)))$. Therefore H is β-ω_α-open.

Example.
1. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is α-ω_α-open, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(\mathbb{Q})) = int_\omega(\mathbb{R}) = \mathbb{R} \supset H$. But $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is not ω-open, since $int_\omega(H) = \mathbb{Q} \neq H$.
2. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset H$. But H is not α-ω_α-open, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(\phi)) = \phi \not\supseteq H$.
3. In $(\mathbb{R}, \tau_u)$, $H = (0, 1]$ is b-ω_α-open, since $int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) = int_\omega(cl_\alpha(H)) \cup cl_\alpha((0, 1)) = int_\omega([0, 1]) \cup cl_\alpha((0, 1)) = (0, 1) \cup [0, 1] = [0, 1] \supset H$. But $H = (0, 1]$ is not pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega([0, 1]) = (0, 1) \not\supseteq (0, 1] = H$.
4. In $(\mathbb{R}, \tau_u)$, $H = [0, 1] \cap \mathbb{Q}$ is β-ω_α-open, since $cl_\alpha(int_\omega(cl_\alpha(H))) = cl_\alpha(int_\omega([0, 1])) = cl_\alpha((0, 1)) = [0, 1] \supset H$. But $H = [0, 1] \cap \mathbb{Q}$ is not b-ω_α-open, since $int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) = int_\omega([0, 1]) \cup \phi = (0, 1) \cup \phi = (0, 1) \not\supseteq H$.

Theorem 3.3. *For a subset of a space (X, τ), the following properties hold:*
1. Every α-ω_α-open set is α-ω-open.
2. Every pre-ω_α-open set is pre-ω-open.
3. Every b-ω_α-open set is b-ω-open.
4. Every β-ω_α-open set is β-ω-open.

Proof.
1. If H is an α-ω_α-open set, then $H \subset int_\omega(cl_\alpha(int_\omega(H))) \subset int_\omega(cl(int_\omega(H)))$. Hence H is α-ω-open.
2. If H is a pre-ω_α-open set, then $H \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl(H))$. Therefore H is pre-ω-open.
3. If H is a b-ω_α-open set, then $H \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$. Therefore H is b-ω-open.
4. If H is a β-ω_α-open set, then $H \subset cl_\alpha(int_\omega(cl_\alpha(H))) \subset cl(int_\omega(cl(H)))$. Therefore H is β-ω-open.

Definition 3.4. *A subset H of a space (X, τ) is called*
1. α-α-open if $H \subset int(cl_\alpha(int(H)))$.
2. semi-α-open if $H \subset cl_\alpha(int(H))$.
3.pre-α-open if $H \subset int(cl_\alpha(H))$.
4. b-α-open if $H \subset int(cl_\alpha(H)) \cup cl_\alpha(int(H))$.
5. β-α-open if $H \subset cl_\alpha(int(cl_\alpha(H)))$.

Example. Let H be a subset of a space (X, τ). Then H is α-α-open if and only if it is semi-α-open and pre-α-open.

Proof. ($\Rightarrow$) Let H be an α-α-open set. Then $H \subset int(cl_\alpha(int(H)))$. It implies $H \subset int(cl_\alpha(int(H))) \subset int(cl_\alpha(H))$ and $H \subset int(cl_\alpha(int(H))) \subset cl_\alpha(int(H))$. Thus H is pre-α-open and semi-α-open. ($\Leftarrow$) Let H be a semi-α-open set. Then $H \subset cl_\alpha(int(H))$ and $cl_\alpha(H) \subset cl_\alpha(int(H))$. Also $H \subset int(cl_\alpha(H))$, since H is pre-α-

open. Hence $H \subset int(cl_\alpha(H)) \subset int(cl_\alpha(int(H)))$ and H is α-α-open.

Remark. The following Examples show that the concepts of semi-α-openness and pre-α-openness are independent.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is pre-α-open, since $int(cl_\alpha(H)) = int(\mathbb{R}) = \mathbb{R} \supset H$. But H is not semi-$\alpha$-open, since $cl_\alpha(int(H)) = cl_\alpha(\phi) = \phi \not\supseteq H$.

Example. In $(\mathbb{R}, \tau_u)$, $H = (0, 1]$ is semi-α-open, since $cl_\alpha(int(H)) = cl_\alpha((0, 1)) = [0, 1] \supset H$. But H is not pre-α-open, since $int(cl_\alpha(H)) = int([0, 1]) = (0, 1) \not\supseteq H$.

Properties. For a subset of a space (X, τ), the following properties hold:
1. Every α-α-open set is α-ω_α-open.
2. Every pre-α-open set is pre-ω_α-open.
3. Every b-α-open set is b-ω_α-open.
4. Every β-α-open set is β-ω_α-open.

Proof.
1. Let H be an α-α-open set. Then $H \subset int(cl_\alpha(int(H))) \subset int_\omega(cl_\alpha(int_\omega(H)))$. This shows that H is α-ω_α-open.
2. Let H be a pre-α-open set. Then $H \subset int(cl_\alpha(H)) \subset int_\omega(cl_\alpha(H))$. This shows that H is pre-ω_α-open.
3. Let H be a b-α-open set. Then $H \subset int(cl_\alpha(H)) \cup cl_\alpha(int(H)) \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H))$. This shows that H is b-ω_α-open.
4. Let H be a β-α-open set. Then $H \subset cl_\alpha(int(cl_\alpha(H))) \subset cl_\alpha(int_\omega(cl_\alpha(H)))$. This shows that H is β-ω_α-open.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$,
1. $H = \mathbb{Q}^\star$ is α-ω_α-open, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(H)) = int_\omega(H) = H \supset H$. But $H = \mathbb{Q}^\star$ is not α-α-open, since $int(cl_\alpha(int(H))) = int(cl_\alpha(\phi)) = \phi \not\supseteq H$.
2. $H = \mathbb{Q}^\star$ is pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega(H) = H \supset H$. But $H = \mathbb{Q}^\star$ is not pre-α-open, since $int(cl_\alpha(H)) = int(H) = \phi \not\supseteq H$.
3. $H = \mathbb{Q}^\star$ is b-ω_α-open, since $int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) = int_\omega(H) \cup cl_\alpha(H) = H \cup H = H \supset H$. But $H = \mathbb{Q}^\star$ is not b-α-open, since $int(cl_\alpha(H)) \cup cl_\alpha(int(H)) = int(H) \cup cl_\alpha(\phi) = \phi \cup \phi = \phi \not\supseteq H$.
4. $H = \mathbb{Q}^\star$ is β-ω_α-open, since $cl_\alpha(int_\omega(cl_\alpha(H))) = cl_\alpha(int_\omega(H)) = cl_\alpha(H) = H \supset H$. But $H = \mathbb{Q}^\star$ is not β-α-open, since $cl_\alpha(int(cl_\alpha(H))) = cl_\alpha(int(H)) = cl_\alpha(\phi) = \phi \not\supseteq H$.

Example. For a subset of a space (X, τ), the following properties hold:
1. Every open set is α-α-open. 2. Every α-α-open set is pre-α-open.
3. Every pre-α-open set is b-α-open.
4. Every b-α-open is β-α-open.

Proof.
1. Let H be an open set. Then $H = int(H)$. Since $H \subset cl_\alpha(H)$, $H = int(H) \subset int(cl_\alpha(H)) = int(cl_\alpha(int(H)))$. This shows that H is α-α-open.

2. Let H be an α-α-open set. Then $H \subset int(cl_\alpha(int(H))) \subset int(cl_\alpha(H))$. This shows that H is pre-α-open.
3. Let H be a pre-α-open set. Then $H \subset int(cl_\alpha(H)) \subset int(cl_\alpha(H)) \cup cl_\alpha(int(H))$. This shows that H is b-α-open.
4. Let H be a b-α-open set. Then $H \subset int(cl_\alpha(H)) \cup cl_\alpha(int(H)) \subset cl_\alpha(int(cl_\alpha(H))) \cup cl_\alpha(int(H)) = cl_\alpha(int(cl_\alpha(H)))$. This shows that H is β-α-open.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is α-α-open, since $int(cl_\alpha(int(H))) = int(cl_\alpha(\mathbb{Q})) = int(\mathbb{R}) = \mathbb{R} \supset H$. But $H = \mathbb{Q} \cup \{\sqrt{2}\}$ is not open, since $int(H) = \mathbb{Q} \neq H$.

Example. In $(\mathbb{R}, \tau_u)$,
1. $H = \mathbb{Q}^\star$ is pre-α-open, since $int(cl_\alpha(H)) = int(\mathbb{R}) = \mathbb{R} \supset H$. But $H = \mathbb{Q}^\star$ is not α-α-open, since $int(cl_\alpha(int(H))) = int(cl_\alpha(\phi)) = int(\phi) = \phi \not\supseteq H$.
2. $H = (0, 1]$ is b-α-open, since $int(cl_\alpha(H)) \cup cl_\alpha(int(H)) = int([0, 1]) \cup cl_\alpha((0, 1)) = (0, 1) \cup [0, 1] = [0, 1] \supset H$. But $H = (0, 1]$ is not pre-α-open, since $int(cl_\alpha(H)) = int([0, 1]) = (0, 1) \not\supseteq H$.
3. $H = [0, 1] \cap \mathbb{Q}$ is β-α-open, since $cl_\alpha(int(cl_\alpha(H))) = cl_\alpha(int([0, 1])) = cl_\alpha((0, 1)) = [0, 1] \supset H$. But $H = [0, 1] \cap \mathbb{Q}$ is not b-α-open, since $int(cl_\alpha(H)) \cup cl_\alpha(int(H)) = int([0, 1]) \cup cl_\alpha(\phi) = (0, 1) \cup \phi = (0, 1) \not\supseteq H$.

Remark. Since every open set is ω-open, we have the following diagram for properties of subsets.

Diagram-2

$$\begin{array}{ccccccccc}
\textbf{open} & \longrightarrow & \alpha\text{-}\alpha\textbf{-open} & \longrightarrow & \textbf{pre-}\alpha\textbf{-open} & \longrightarrow & \textbf{b-}\alpha\textbf{-open} & \longrightarrow & \beta\text{-}\alpha\textbf{-open} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\omega\textbf{-open} & \longrightarrow & \alpha\text{-}\omega_\alpha\textbf{-open} & \longrightarrow & \textbf{pre-}\omega_\alpha\textbf{-open} & \longrightarrow & \textbf{b-}\omega_\alpha\textbf{-open} & \longrightarrow & \beta\text{-}\omega_\alpha\textbf{-open}
\end{array}$$

The converses of the above implications are not true in general as can be seen from the above Examples.

Theorem 3.5. *If H is a pre-ω_α-open subset of a space (X, τ) such that $U \subset H \subset cl_\alpha(U)$ for a subset U of X, then U is a pre-ω_α-open set.*
Proof. Since $H \subset int_\omega(cl_\alpha(H))$, $U \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl_\alpha(U))$ since $cl_\alpha(H) \subset cl_\alpha(U)$. Thus U is a pre-ω_α-open set.

Theorem 3.6. *A subset H of a space (X, τ) is semi-α-open if and only if H is β-ω_α-open and $int_\omega(cl_\alpha(H)) \subset cl_\alpha(int(H))$.*
Proof. Let H be semi-α-open. Then $H \subset cl_\alpha(int(H)) \subset cl_\alpha(int_\omega(cl_\alpha(H)))$ and hence H is β-ω_α-open. In addition $cl_\alpha(H) \subset cl_\alpha(int(H))$ and hence $int_\omega(cl_\alpha(H)) \subset int_\omega(cl_\alpha(int(H))) \subset cl_\alpha(int(H))$. Conversely let H be β-ω_α-open and $int_\omega(cl_\alpha(H)) \subset cl_\alpha(int(H))$. Then $H \subset cl_\alpha(int_\omega(cl_\alpha(H))) \subset cl_\alpha(cl_\alpha(int(H))) = cl_\alpha(int(H))$ and hence H is semi-α-open.

Lemma 3.7. *Let H be a subset of a space (X, τ). Then*
1. H is α-closed in X if and only if $H = cl_\alpha(H)$.
2. $cl_\alpha(X \backslash H) = X \backslash int_\alpha(H)$. 3. $cl_\alpha(H)$ is α-closed in X.
4. $x \in cl_\alpha(H)$ if and only if $H \cap G \neq \phi$ for each α-open set G containing x.
5. $int(H) \subset int_\alpha(H)$.
Proof. Since the family of all α-open subsets of a space (X, τ) forms a topology on X, we get the lemma.

Property. If U is an α-open set of a space (X, τ), then
1. $cl_\alpha(U \cap H) = cl_\alpha(U \cap cl_\alpha(H))$ and
2. $U \cap cl_\alpha(H) \subset cl_\alpha(U \cap H)$ for any subset H.
Proof. (i) ($\Rightarrow$) Since $H \subset cl_\alpha(H)$, $U \cap H \subset U \cap cl_\alpha(H)$ and $cl_\alpha(U \cap H) \subset cl_\alpha(U \cap cl_\alpha(H))$. ($\Leftarrow$) Let $x \notin cl_\alpha(U \cap H)$. Then there exists an α-open set U such that $x \in U$ and $U \cap (U \cap H) = \phi$. Now $U \cap H = \phi$ implies $U \subset X - H$ and $int_\alpha(U) \subset int_\alpha(X - H) = X - cl_\alpha(H) \Rightarrow int_\alpha(U) \cap cl_\alpha(H) = \phi \Rightarrow U \cap cl_\alpha(H) = \phi$ $\Rightarrow U \cap (U \cap cl_\alpha(H)) = \phi$. Then, $x \notin cl_\alpha(U \cap cl_\alpha(H))$. Thus we obtain $cl_\alpha(U \cap H)$ $= cl_\alpha(U \cap cl_\alpha(H))$. (ii) We have $U \cap cl_\alpha(H) \subset cl_\alpha(U \cap cl_\alpha(H)) = cl_\alpha(U \cap H)$ by (i). Thus we obtain (ii).

Properties. The intersection of a pre-ω_α-open set and an open set is pre-ω_α-open.
Proof. Let H be a pre-ω_α-open set and U an open set. Then $U = int_\omega(U)$ and $H \subset int_\omega(cl_\alpha(H))$. Since every open set is α-open, $U \cap H \subset int_\omega(U) \cap int_\omega(cl_\alpha(H))$ $= int_\omega(U \cap cl_\alpha(H)) \subset int_\omega(cl_\alpha(U \cap H))$ by Proposition **??**. This shows that $U \cap H$ is pre-ω_α-open.

Properties. The intersection of a β-ω_α-open set and an open set is β-ω_α-open.
Proof. Let H be a β-ω_α-open set and U an open set. Then $U = int_\omega(U)$ and $H \subset cl_\alpha\ (int_\omega(cl_\alpha(H)))$. Since every open set is α-open, $U \cap H \subset U$ $\cap\ cl_\alpha(int_\omega(cl_\alpha(H))) \subset cl_\alpha(U \cap int_\omega(cl_\alpha(H))) \subset cl_\alpha(int_\omega(U) \cap int_\omega(cl_\alpha(H))) =$ $cl_\alpha(int_\omega(U \cap cl_\alpha(H))) \subset cl_\alpha(int_\omega(cl_\alpha(U \cap H)))$ by Proposition **??**. This shows that $U \cap H$ is β-ω_α-open.

Properties. The intersection of a b-ω_α-open set and an open set is b-ω_α-open.
Proof. Let H be a b-ω_α-open and U an open set. Then $U = int_\omega(U)$ and H $\subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H))$. Since every open set is α-open, $U \cap H \subset U \cap$ $[int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H))] = [U \cap int_\omega(cl_\alpha(H))] \cup [U \cap cl_\alpha(int_\omega(H))] = [int_\omega(U)$ $\cap\ int_\omega(cl_\alpha(H))] \cup [U \cap cl_\alpha(int_\omega(H))] \subset [int_\omega(U \cap cl_\alpha(H))] \cup [cl_\alpha(U \cap int_\omega(H))]$ by Proposition **??**. Thus $U \cap H \subset [int_\omega(cl_\alpha(U \cap H))] \cup [cl_\alpha(int_\omega(U \cap H))]$. This shows that $U \cap H$ is b-ω_α-open.

Remark. The intersection of two pre-ω_α-open (resp. b-ω_α-open, β-ω_α-open) sets need not be pre-ω_α-open (resp. b-ω_α-open, β-ω_α-open) as can be seen from the following Example.

Example. In $(\mathbb{R}, \tau_u)$,
1. $A = \mathbb{Q}$ is pre-ω_α-open, since $int_\omega(cl_\alpha(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset A$. Also $B = \mathbb{Q}^\star \cup \{1\}$ is pre-ω_α-open, since $int_\omega(cl_\alpha(B)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset B$. But $A \cap B = \{1\}$ is not pre-ω_α-open, since $int_\omega(cl_\alpha(A \cap B)) = int_\omega(cl_\alpha(\{1\})) = int_\omega(\{1\}) = \phi \not\supseteq A \cap B$.
2. $A = \mathbb{Q}$ and $B = \mathbb{Q}^\star \cup \{1\}$ are b-ω_α-open, by (1) of Example 3 and Theorem 3.2 (3). But $A \cap B = \{1\}$ is not b-ω_α-open, since $int_\omega(cl_\alpha(\{1\})) \cup cl_\alpha(int_\omega(\{1\})) = \phi \cup cl_\alpha(\phi) = \phi \cup \phi = \phi \not\supseteq \{1\} = A \cap B$.
3. $A = \mathbb{Q}$ and $B = \mathbb{Q}^\star \cup \{1\}$ are β-ω_α-open by (2) of Example 3 and Theorem 3.2 (4). But $A \cap B = \{1\}$ is not β-ω_α-open, since $cl_\alpha(int_\omega(cl_\alpha(\{1\}))) = cl_\alpha(int_\omega(\{1\})) = cl_\alpha(int_\omega(\{1\})) = cl_\alpha(\phi) = \phi \not\supseteq \{1\} = A \cap B$.

Properties. The intersection of an α-ω_α-open set and an open set is α-ω_α-open.
Proof. Let H be α-ω_α-open and U be open. Then $U = int_\omega(U)$ and $H \subset int_\omega(cl_\alpha(int_\omega(H)))$. $U \cap H \subset int_\omega(U) \cap [int_\omega(cl_\alpha(int_\omega(H))] = int_\omega[U \cap cl_\alpha(int_\omega(H))] \subset int_\omega[cl_\alpha(U \cap int_\omega(H))]$ by Proposition **??**. Thus $U \cap H \subset int_\omega[cl_\alpha(int_\omega(U) \cap int_\omega(H))] = int_\omega[cl_\alpha(int_\omega(U \cap H))]$ which implies $U \cap H$ is α-ω_α-open.

Properties. If $\{H_\alpha : \alpha \in \triangle\}$ is a collection of pre-ω_α-open sets of a space (X, τ), then $\cup_{\alpha\in\triangle} H_\alpha$ is pre-ω_α-open.
Proof. Since $H_\alpha \subset int_\omega(cl_\alpha(H_\alpha))$ for every $\alpha \in \triangle$, $\cup_{\alpha\in\triangle} H_\alpha \subset \cup_{\alpha\in\triangle} int_\omega(cl_\alpha(H_\alpha)) \subset int_\omega(\cup_{\alpha\in\triangle} cl_\alpha(H_\alpha)) = int_\omega(cl_\alpha(\cup_{\alpha\in\triangle} H_\alpha))$. Therefore, $\cup_{\alpha\in\triangle} H_\alpha$ is pre-ω_α-open.

Theorem 3.8. *If $\{H_\alpha : \alpha \in \triangle\}$ is a collection of b-ω_α-open (resp. β-ω_α-open) sets of a space (X, τ), then $\cup_{\alpha\in\triangle} H_\alpha$ is b-ω_α-open (resp. β-ω_α-open).*
Proof. We prove only the first result since the other result follows similarly. Since H_α is b-ω_α-open for every $\alpha \in \triangle$, $H_\alpha \subset int_\omega(cl_\alpha(H_\alpha)) \cup cl_\alpha(int_\omega(H_\alpha))$ for every $\alpha \in \triangle$.

$$\begin{aligned} Then \cup_{\alpha\in\triangle} H_\alpha &\subset \cup_{\alpha\in\triangle}[int_\omega(cl_\alpha(H_\alpha)) \cup cl_\alpha(int_\omega(H_\alpha))] \\ &= [\cup_{\alpha\in\triangle} int_\omega(cl_\alpha(H_\alpha))] \cup [\cup_{\alpha\in\triangle} cl_\alpha(int_\omega(H_\alpha))] \\ &\subset [int_\omega(\cup_{\alpha\in\triangle} cl_\alpha(H_\alpha))] \cup [cl_\alpha(\cup_{\alpha\in\triangle} int_\omega(H_\alpha))] \\ &\subset [int_\omega(cl_\alpha(\cup_{\alpha\in\triangle} H_\alpha))] \cup [cl_\alpha(int_\omega(\cup_{\alpha\in\triangle} H_\alpha))]. \end{aligned}$$

Therefore, $\cup_{\alpha\in\triangle} H_\alpha$ is b-ω_α-open.

Properties. Let (X, τ) be a space and $H \subset X$. Let H be a b-ω_α-open set such that $int_\omega(H) = \phi$. Then H is pre-ω_α-open. Recall that a space (X, τ) is called a door space if every subset of X is open or closed.

Properties. If (X, τ) is a door space, then every pre-ω_α-open set in (X, τ) is ω-open.

Proof. Let H be a pre-ω_α-open set. If H is open, then H is ω-open. Otherwise, H is closed and hence $H \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl(H)) = int_\omega(H) \subset H$. Therefore, $H = int_\omega(H)$ and thus H is an ω-open set.

Theorem 3.9. *Let (X, τ) be an anti-locally countable space and H a subset of (X, τ). Then the following properties hold:*
1. if H is pre-ω_α-open, then it is pre-open.
2. if H is b-ω_α-open and ω-closed, then it is b-open.
3. if H is β-ω_α-open, then it is β-open.

Proof. 1. Let H be a pre-ω_α-open set. Then $H \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl(H)) = int(cl(H))$ by Lemma 2.7, since every closed set is ω-closed. This shows that H is pre-open.
2. Let H be a b-ω_α-open and ω-closed set. Since H and $cl(H)$ are ω-closed, $int_\omega(cl_\alpha(H)) \subset int_\omega(cl(H)) = int(cl(H))$ and $cl_\alpha(int_\omega(H)) \subset cl(int(H))$ by Lemma 2.7. Since H is b-ω_α-open, $H \subset int_\omega(cl_\alpha(H)) \cup cl_\alpha(int_\omega(H)) \subset int(cl(H)) \cup cl(int(H))$. This shows that H is b-open.
3. Let H be a β-ω_α-open set. Then $H \subset cl_\alpha(int_\omega(cl_\alpha(H))) \subset cl(int_\omega(cl(H))) = cl(int(cl(H)))$ by Lemma 2.7. This shows that H is β-open.

Definition 3.10. [5] *A space (X, τ) is called α-space if every α-closed set in X is closed in X.*

Remark. For an α-space (X, τ) and $H \subset X$, H is pre-ω_α-open (resp. b-ω_α-open) if and only if H is pre-ω-open (resp. b-ω-open).

Remark. If H is a semi-open subset of a space (X, τ), then H is pre-ω_α-open (resp. b-ω_α-open) if and only if H is pre-ω-open (resp. b-ω-open).

4. Decompositions of continuity

Definition 4.1. *A subset H of a space (X, τ) is called*
1. a t-ω_α-set if $int(H) = int_\omega(cl_\alpha(H))$;
2. a B-ω_α-set if $H = U \cap V$, where $U \in \tau$ and V is an t-ω_α-set.

Example. 1. In $(\mathbb{R}, \tau_u)$, $H = \mathbb{Q}$ is not a t-ω_α-set, since $int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \phi = int(H)$.
2. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is a t-ω_α-set, since $int_\omega(cl_\alpha(H)) = int_\omega(H) = \phi = int(H)$.

Remark. In a space (X, τ),
1. Every open set is a B-ω_α-set.

2. Every t-ω_α-set is a B-ω_α-set.

The converses of (1) and (2) in Remark 4 are not true in general as illustrated in the following Examples.

Example. In Example 4 (2), $H = \mathbb{Q}$ is a t-ω_α-set and hence by (2) of Remark 4, $H = \mathbb{Q}$ is a B-ω_α-set. But $H = \mathbb{Q}$ is not open, since $\mathbb{Q} \notin \tau$.

Example. In Example 4 (2), $H = \mathbb{Q}^\star$ is open in $\mathbb{R}$ and hence by (1) of Remark 4, H is a B-ω_α-set. But $int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \mathbb{Q}^\star = int(H)$. Thus $H = \mathbb{Q}^\star$ is not a t-ω_α-set.

Example. In $(\mathbb{R}, \tau_u)$, $H = \mathbb{Q}$ is not a B-ω_α-set. If $H = U \cap V$, where $U \in \tau$ and V is t-ω_α-set, then $H \subset U$. But $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$ which is a contradiction, since $H = V$ is not a t-ω_α-set by Example 4 (1). This proves that $H = \mathbb{Q}$ is not a B-ω_α-set.

Properties. Let A and B be subsets of a space (X, τ). If A and B are t-ω_α-sets, then $A \cap B$ is a t-ω_α-set.

Proof. Let A and B be t-ω_α-sets. Then we have $int(A \cap B) \subset int_\omega(cl_\alpha(A \cap B)) \subset int_\omega(cl_\alpha(A) \cap cl_\alpha(B)) = int_\omega(cl_\alpha(A)) \cap int_\omega(cl_\alpha(B)) = int(A) \cap int(B) = int(A \cap B)$. Then $int(A \cap B) = int_\omega(cl_\alpha(A \cap B))$ and hence $A \cap B$ is a t-ω_α-set.

Properties. For a subset H of a space (X, τ), the following properties are equivalent:

1. H is open;
2. H is pre-ω_α-open and a B-ω_α-set.

Proof. (1) $\Rightarrow$ (2): Let H be open. Then $H = int(H) \subset int_\omega(cl_\alpha(H))$ and H is pre-ω_α-open. Also by Remark 4 H is a B-ω_α-set. (2) $\Rightarrow$ (1): Given H is a B-ω_α-set. So $H = U \cap V$ where $U \in \tau$ and $int(V) = int_\omega(cl_\alpha(V))$. Then $H \subset U = int(U)$. Also, H is pre-ω_α-open implies $H \subset int_\omega(cl_\alpha(H)) \subset int_\omega(cl_\alpha(V)) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open.

Remark. The following Examples show that the concepts of pre-ω_α-openness and being a B-ω_α-set are independent.

Example. In $(\mathbb{R}, \tau_u)$, $H = \mathbb{Q}$ is pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q} = H$. But $H = \mathbb{Q}$ is not a B-ω_α-set by Example 4.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is a t-ω_α-set, by (2) of Example 4. Hence $H = \mathbb{Q}$ is a B-ω_α-set by (2) of Remark 4. But $H = \mathbb{Q}$ is not pre-ω_α-open, since $int_\omega(cl_\alpha(H)) = int_\omega(H) = \phi \not\supseteq \mathbb{Q} = H$.

Definition 4.2. *A subset H of a space (X, τ) is called*
1. a t_α-ω_α-set if $int(H) = int_\omega(cl_\alpha(int_\omega(H)))$;
2. a B_α-ω_α-set if $H = U \cap V$, where $U \in \tau$ and V is a t_α-ω_α-set.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q}^\star$ is not a t_α-ω_α-set, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(H)) = int_\omega(H) = H \neq \phi = int(H)$.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star$ is a t_α-ω_α-set, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(\phi)) = int_\omega(\phi) = \phi = int(H)$.

Remark. In a space (X, τ),
1. Every open set is a B_α-ω_α-set.
2. Every t_α-ω_α-set is a B_α-ω_α-set.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is a t_α-ω_α-set, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(\mathbb{N})) = int_\omega(H) = \mathbb{N} = int(H)$. Hence by (2) of Remark 4, $H = \mathbb{Q}$ is a B_α-ω_α-set. But $H = \mathbb{Q}$ is not open, since $\mathbb{Q} \notin \tau$.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star$ is open, since $H \in \tau$ and hence $H = \mathbb{Q}^\star$ is a B_α-ω_α-set by (1) of Remark 4. But $H = \mathbb{Q}^\star$ is not a t_α-ω_α-set, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \mathbb{Q}^\star = H = int(H)$.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q}^\star$ is not a B_α-ω_α-set. If $H = U \cap V$ where $U \in \tau$ and V is t_α-ω_α-set, then $H \subset U$. But $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$ which is a contradiction, since $H = V$ is not a t_α-ω_α-set by Example 4. This proves that $H = \mathbb{Q}^\star$ is not a B_α-ω_α-set.

Properties. *If* A and B are t_α-ω_α-sets of a space (X, τ), then $A \cap B$ is a t_α-ω_α-set.
Proof. Let A and B be t_α-ω_α-sets. Then we have $int(A \cap B) \subset int_\omega(cl_\alpha(int_\omega(A \cap B))) \subset int_\omega[cl_\alpha(int_\omega(A)) \cap cl_\alpha(int_\omega(B))] = int_\omega(cl_\alpha(int_\omega(A))) \cap int_\omega(cl_\alpha(int_\omega(B))) = int(A) \cap int(B) = int(A \cap B)$. Then $int(A \cap B) = int_\omega(cl_\alpha(int_\omega(A \cap B)))$ and hence $A \cap B$ is a t_α-ω_α-set.

Properties. For a subset H of a space (X, τ), the following properties are equivalent:
1. H is open;
2. H is α-ω_α-open and a B_α-ω_α-set.

Proof. (1) $\Rightarrow$ (2): Let H be open. Then $H = int_\omega(H) \subset cl_\alpha(int_\omega(H))$ and $H = int_\omega(H) \subset int_\omega(cl_\alpha(int_\omega(H)))$. Therefore H is α-ω_α-open. Also by (1) of Remark 4, H is a B_α-ω_α-set. (2) $\Rightarrow$ (1): Given H is a B_α-ω_α-set. So $H = U \cap V$ where $U \in \tau$ and $int(V) = int_\omega(cl_\alpha(int_\omega(V)))$. Then $H \subset U = int(U)$. Also H is α-ω_α-open implies $H \subset int_\omega(cl_\alpha(int_\omega(H))) \subset int_\omega(cl_\alpha(int_\omega(V))) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and H is open.

Remark. The following Examples show that the concepts of α-ω_α-openness and being a B_α-ω_α-set are independent.

Example. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, $H = \mathbb{Q}^\star$ is α-ω_α-open, since $int_\omega(cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha(H)) = int_\omega(H) = H \supset H$. But $H = \mathbb{Q}^\star$ is not a B_α-ω_α-set by Example 4.

Example. In $(\mathbb{R}, \tau_u)$, $H = (0, 1]$ is a t$_\alpha$-ω_α-set, since $int_\omega\ (cl_\alpha(int_\omega(H))) = int_\omega(cl_\alpha((0, 1))) = int_\omega([0, 1]) = (0, 1) = int(H)$. Hence $H = (0, 1]$ is a B_α-ω_α-set by (2) of Remark 4. But $H = (0, 1]$ is not α-ω_α-open, since $int_\omega(cl_\alpha(int_\omega(H))) = (0, 1) \not\supseteq (0, 1] = H$.

Definition 4.3. *A function $f : X \to Y$ is said to be ω-continuous [7] (resp. pre-ω_α-continuous, B-ω_α-continuous, α-ω_α-continuous, B_α-ω_α-continuous) if $f^{-1}(V)$ is ω-open (resp. pre-ω_α-open, a B-ω_α-set, an α-ω_α-open, a B_α-ω_α-set) for each open set V in Y.*

By Propositions 4 and 4 we have the immediate result.

Theorem 4.4. *For a function $f : X \to Y$, the following properties are equivalent:*
1. f is continuous.
2. f is pre-ω_α-continuous and B-ω_α-continuous.
3. f is α-ω_α-continuous and B_α-ω_α-continuous.

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