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Fractional Calculus: A New Look

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Abstract

In this article we give an overview of the recent developments in the area of fractional integrals and fractional derivatives. A new definition is given by this author in terms of Mellin convolutions of ratios and products in the case of real scalar variables and M-convolutions of ratios and products in the case of matrix variables, where one of the functions is a type-1 beta type so that all the definitions available in the literature for fractional integrals can be brought under one definition. Once the fractional integrals are defined, fractional derivatives can be defined as certain fractional integrals so that the results coming from fractional derivatives can describe global activities compared to integer order derivatives which can describe only local activities at a point. When fractional derivatives are defined as certain fractional integrals then these derivatives cover not only given points of interest but also their neighborhoods so that fractional derivatives become more useful in practical applications. An ideal situation may be a local activity but in reality the real-life situation may be in the neighborhood of the ideal case. The new definition is also extended to real matrix-variate case as well as to complex matrix-variate case. Thus, for the first time, fractional calculus of functions of complex variables is also given through the new definition.

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1. Introduction.

Even though fractional calculus is as old as integer order calculus itself, the area of fractional calculus was dormant all these years except for the last two decades.

Now it is a fast growing area with the investigations of new applications in various different fields from physical and engineering sciences, to biological and social sciences. The renewed interest came mostly because the solutions coming from fractional differential equations are found to describe practical situations much better compared to the solutions coming from integer order differential equations. The reason can be explained as follows: Integer order derivatives are local activities or they deal with a given point or instantaneous rate of change at a given point whereas an integral covers an interval in the real scalar variable case. Fractional derivatives are certain integrals and thus cover an interval covering the point which the integer order derivative is concerned with, plus its neighborhoods. Thus, naturally, the solutions coming from fractional order differential equations are found to be more suitable to describe practical situations. This is the main reason for the renewed interest in fractional calculus.

In the area of fractional calculus there are several definitions for fractional integrals, thereby for fractional derivatives, given by various authors from time to time. As a result, there are several types of notations to describe various fractional integrals and fractional derivatives. Thus, anyone looking at the area will find it a full jungle there and difficult to sort out things. Recently this author (Mathai, 2013, 2014) has given a geometrical interpretation for fractional integrals as fractions of certain total integrals. Earlier, the author had given the interpretation as a fraction of a total probability when fractional integrals were given interpretations in terms of statistical distribution theory. Consider a function $f(x_1)$ of one scalar variable x_1 and suppose that this is integrated out over a simplex which is a part of the n-dimensional cube $(b-a)\times(b-a)\times ...\times(b-a)$. Consider the plane $x_1=x_2=...=x_n$ and the simplex to the left or integrate over $a\le t\le x$. This will lead to left-sided or first kind fractional integrals. Consider the simplex to the right of the plane or integrate over $x\le t\le b$. Then we end up with the right-sided or second kind fractional integral, see also Mathai (2014). These left-sided and right-sided fractional integrals will contain a factor of the type $\frac{1}{\Gamma(n)}(1-\frac{x_1}{x})^{n-1}f(x_1)$ for the left-sided integral, where $f(x_1)$ is the arbitrary function of the one variable x_1, and a factor of the type $\frac{1}{\Gamma(n)}(1-\frac{x}{x_1})^{n-1}f(x_1), n=1,2,...$ for the right-sided integral. If the integer n is replaced by an arbitrary α with $\Re(\alpha)>0$ then we get the crucial factor in all the definitions of fractional integrals of order α. But, observe that the factor of the type $(1-\frac{1}{x}t)^{\alpha-1}, x>0$ or the form $(1-\frac{x}{t})^{\alpha-1}$ is a part of type-1 beta form. Also the structure $\int_a^x(1-\frac{t}{x})^{\alpha-1}f(t)\mathrm{d}t$ is the structure of a Mellin convolution of a ratio, which is of the form $\int_v \frac{v}{u^2}f_1(\frac{v}{u})f_2(v)\mathrm{d}v$. Similarly, the integral $\int_x^b(1-\frac{x}{t})^{\alpha-1}f(t)\mathrm{d}t$ has the structure $\int_v \frac{1}{v}f_1(\frac{u}{v})f_2(v)\mathrm{d}v$ which is in fact the structure of a Mellin convolution of a product. From these observations, this author has noted the following Mellin convolutions of a product and ratio.

2. Mellin Convolutions of Ratios and Products

These Mellin convolutions in the real scalar case are directly connected to statistical distributions and it is also easy to explain in terms of statistical densities of products and ratios. Hence consider two statistically independently distributed positive real scalar random variables $x_1 > 0, x_2 > 0$ with the densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Consider the product $u_2 = x_1x_2$. Let $v = x_2$. Then the Jacobian is v^{-1}. The joint density of x_1 and x_2 is the product $f_1(x_1)f_2(x_2)$ due to statistical independence. Let the joint density of u_2 and v be denoted by $g(u_2, v)$ and the marginal density of u_2 be denoted by $g_2(u_2)$. Then the density $g_2(u_2)$ is given by

$$g_2(u_2) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v. \tag{2.1}$$

Suppose that we take $f_1(x_1)$ as a type-1 beta density with the parameters $(\gamma+1, \alpha)$, that is,

$$f_1(x_1) = \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)\Gamma(\alpha)} x_1^{\gamma}(1-x_1)^{\alpha-1}$$

for $\Re(\alpha) > 0, \Re(\gamma) > -1, 0 \le x_1 \le 1$ and zero elsewhere. Let $f_2(x_2) = f(x_2)$ be an arbitrary density. Then (2.1) becomes the following:

$$\begin{aligned}
g_2(u_2) &= \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)\Gamma(\alpha)} u^{\gamma} \int_{v>u_2} v^{-\gamma-\alpha} \\
&\quad \times (v-u_2)^{\alpha-1} f(v)\mathrm{d}v \\
&= \frac{\Gamma(\gamma+1+\alpha)}{\Gamma(\gamma+1)} K_{2,u_2,\gamma}^{-\alpha} f \text{ where} \\
K_{2,u_2,\gamma}^{-\alpha} &= \frac{u_2^{\gamma}}{\Gamma(\alpha)} \int_{v>u_2} v^{-\gamma-\alpha}(v-u_2)^{\alpha-1} f(v)\mathrm{d}v.
\end{aligned} \tag{2.2}$$

This $K_{2,u_2,\gamma}^{-\alpha} f$ is known as Kober fractional integral of the second kind, or right-sided, of order α and with parameter γ. Note that $g_2(u_2)$ is a statistical density when f_1 and f_2 are densities. If they are not densities then $g_2(u_2)$ will be the Mellin convolution of a product of the functions $f_1(x_1)f_2(x_2)$ or Mellin convolution of the product x_1x_2.

Now let us look at the ratio $u_1 = \frac{x_2}{x_1}, v = x_2$. Then the Jacobian is $-\frac{v}{u_1^2}$. If we had taken $v = x_1$ then $x_2 = u_1 v$ and the Jacobian is v. We have taken it in the first way so that the first function f_1 will change into a convenient format. Then the density of u_1, going through the same steps as above, can be seen to be the following:

$$g_1(u_1) = \int_v \frac{v}{u_1^2} f(\frac{v}{u_1}) f_2(v)\mathrm{d}v. \tag{2.3}$$

Let f_1 be a type-1 beta density with the parameters (γ, α) or

$$f_1(x_1) = \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)\Gamma(\alpha)} x_1^{\gamma-1}(1-x_1)^{\alpha-1}$$

for $0 \le x_1 \le 1, \Re(\alpha) > 0, \Re(\gamma) > 0$ and zero elsewhere. Let $f_2 = f(x_2)$ be an arbitrary density. Then (2.3) becomes the following:

$$\begin{aligned} g_1(u_1) &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)} \frac{u_1^{-\gamma-\alpha}}{\Gamma(\alpha)} \int_{v<u_1} v^{\gamma} \\ &\times (u_1 - v)^{\alpha-1} f(v)\mathrm{d}v \\ &= \frac{\Gamma(\gamma+\alpha)}{\Gamma(\gamma)} K_{1,u_1,\gamma}^{-\alpha} f \text{ where} \\ K_{1,u_1,\gamma}^{-\alpha} f &= \frac{u_1^{-\gamma-\alpha}}{\Gamma(\alpha)} \int_{v<u_1} v^{\gamma}(u_1-v)^{\alpha-1} f(v)\mathrm{d}v. \end{aligned} \tag{2.4}$$

Here $K_{1,u_1,\gamma}^{-\alpha} f$ is called Kober fractional integral of the first kind, or left-sided, of order α and with parameter γ.

Thus from (2.2) and (2.4) it may be observed that Kober fractional integral operators of the first and second kind can be given direct interpretations in terms of statistical densities of ratio and product respectively. If f_1 and f_2 are not statistical densities then (2.2) and (2.4) become Mellin convolutions of product and ratio respectively. Motivated by (2.2) and (2.4), this author has given a general definition for fractional integrals of the second and first kinds as Mellin convolutions of product and ratio in the real scalar variable case.

3. A General Definition for Fractional Integrals in the Real Scalar Case

Let us consider fractional integral of the second kind first. Let

$$f_1(x_1) = \frac{1}{\Gamma(\alpha)} \phi_1(x_1)(1-x_1)^{\alpha-1} \tag{3.1}$$

for $0 \leq x_1 \leq 1, \Re(\alpha) > 0$ and zero elsewhere, where $\phi_1(x_1)$ is some specified function, and let

$$f_2(x_2) = \phi_2(x_2) f(x_2) \tag{3.2}$$

where $f(x_2)$ is an arbitrary function and $\phi_2(x_2)$ is a specified function. Then take $u_2 = x_1 x_2$ and compute $g_2(u_2)$ as in (2.2). Then $g_2(u_2)$ will be of the following form, again denoted by g_2:

$$g_2(u_2) = \int_v \frac{1}{v} \phi_1(\frac{u}{v})(1 - \frac{u}{v})^{\alpha-1} \phi_2(v) f(v) \mathrm{d}v. \tag{3.3}$$

Suppose that

$$\phi_1(x_1) = \frac{\Gamma(\gamma + 1 + \alpha)}{\Gamma(\gamma + 1)\Gamma(\alpha)} x_1^{\gamma}$$

for $\Re(\gamma) > -1, \Re(\alpha) > 0$, and $\phi_2 = 1$. Then (3.3) reduces to a constant multiple of the Kober fractional integral operator of the second kind of order α, denoted by this author as $K_{2,u_2,\gamma}^{-\alpha} f$, as given in (2.2). Suppose that $\phi_1 = 1, \phi(x_2) = x_2^{-\alpha}$ then (3.3) becomes Weyl fractional integral operator of the second kind of order α, denoted by this author as $W_{2,u_1}^{-\alpha} f$, and given by

$$W_{2,u_2}^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_{v > u_2} (v - u_2)^{\alpha-1} f(v) \mathrm{d}v, \tag{3.4}$$

for $\Re(\alpha) > 0$. If v is bounded above by a constant b then (3.4) becomes Riemann-Liouville fractional integral operator of the second kind of order α, denoted by this author as $D_{2,(u_2,b)}^{-\alpha} f$. If we take $\phi_1(x_1)$ as a Gauss hypergeometric function for $0 \leq x_1 \leq 1$ and zero elsewhere then we can obtain Saigo fractional integral operator of the second kind, denoted by this author as $S_{2,u_2}^{-\alpha} f$ and its generalizations by taking the argument in the hypergeometric functions as $a x_1^{\delta_1}$ or $a(1 - x_1)^{\delta_2}$ or $a x_1^{\delta_3}(1 - x_1)^{\delta_4}$ where $a > 0, \delta_j > 0, j = 1, 2, 3, 4$. Some of these generalizations are given in Mathai and Haubold (2008), Mathai,Saxena and Haubold (2010). One can also obtain a pathway generalized form of fractional integrals of the second kind by replacing in (3.1) the factor $(1 - x_1)^{\alpha-1}$ by $(1 - bx^{\delta})^{\gamma}$ for $\delta > o, \gamma > 0$, see the details from Mathai (2013,2014).

Now, we will look into a general definition for fractional integrals of the first kind or left-sided fractional integrals. Let f_1 and f_2 be as defined in (3.1) and (3.2) respectively. Let $u_1 = \frac{x_2}{x_1}, v = x_2$ so that the Jacobian is $-\frac{v}{u_1^2}$, $x_1 = \frac{v}{u_1}$. Then proceed as in the derivation of (2.4). We end up in a $g_1(u_1)$ in the following form, again denoted by $g_1(u_1)$:

$$g_1(u_1) = \frac{1}{\Gamma(\alpha)} \int_v \frac{v}{u^2} \phi_1(\frac{v}{u_1})(1 - \frac{v}{u_1})^{\alpha-1} \phi_2(v) f(v) \mathrm{d}v \tag{3.5}$$

for $0 \le x_1 \le 1, \Re(\alpha) > 0$ and zero elsewhere. Let us look into some special cases. Let

$$\phi_1(x_1) = \frac{\Gamma(\alpha+\gamma)}{\Gamma(\gamma)} x_1^{\gamma-1}, \Re(\gamma) > -1, \Re(\alpha) > \frac{p-1}{2}$$

and $\phi_2 = 1$. Then $g_1(u_1)$ in (3.5) reduces to the Kober fractional integral of the first kind of order α, denoted by this author as $K_{1,u_1,\gamma}^{-\alpha} f$, as in (2.4). Let us look into another special case. Let $\phi_1(x_1) = x_1^{\alpha-1}$ and $\phi_2(x_2) = x_2^{-\alpha}$ then (3.5) reduces to the Weyl fractional integral of the first kind of order α, denoted by this author as $W_{1,u_1}^{-\alpha} f$, and given by

$$W_{1,u_1}^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_{v<u_1} (u_1 - v)^{\alpha-1} f(v) \mathrm{d}v \tag{3.6}$$

for $\Re(\alpha) > 0$. If v is bounded below at a, where a is a constant, then (3.6) reduces to Riemann-Liouville fractional integral of the first kind of order α, denoted by this author as $D_{1,(a,u_1)}^{-\alpha} f$ or $D_{1,u_1}^{-\alpha} f$ for $a = 0$. If we write $\phi_1(x_1)$ in terms of a Gauss hypergeometric function then (3.5) reduces to Saigo fractional integral operator of the first kind of order α in the real scalar variable case. A pathway generalized form for the first kind integral is also available by replacing the factor $(1 - x_1)^{\alpha-1}$ in (3.1) by $(1 - cx_1^{\rho})^{\eta}, \rho > 0, \eta > 0, c > 0$ and then specializing c, η.

4. Fractional Integrals for Real Matrix-variate Case

Fractional integrals of the first and second kind in the real scalar variable case are extended to real matrix-variate case, to complex matrix-variate case, to several real scalar variables case, to several real matrix-variates case and to several complex matrix-variates case. Out of these, we will consider here the situation of one real matrix variable case. Again, as in the real scalar variable case, it is easy to explain products and ratios of matrices and M-convolutions of products and ratios in terms of statistical densities.

In this section, all matrices appearing are $p \times p$ real positive definite unless stated otherwise. We will use the following standard notations. $|X|$ and $\mathrm{tr}(X)$ denote the determinant and trace of $X = (x_{ij})$ respectively. $X > O, X \ge O, X < O, X \le O$ denote positive definiteness, positive semidefiniteness, negative definiteness and negative semidefiniteness respectively. $\int_{A<X<B} f(X) \mathrm{d}X$ denotes the real-valued function $f(X)$ of the matrix argument X is integrated out over all X such that

$X > O, X - A > O, B - X > O, A > o, B > O$ where A and B are constant matrices. Here $\mathrm{d}X$ denotes the wedge product of all distinct differentials in X. Here X is symmetric $p\times p$ and hence there are only $p(p+1)/2$ distinct real variables, and then $\mathrm{d}X = \wedge_{i\geq j}\mathrm{d}x_{ij} = \wedge_{i\leq j}\mathrm{d}x_{ij}$. When $Y = (y_{ij})$ is a general $m \times n$ matrix then $\mathrm{d}Y = \wedge_{i=1}^{m} \wedge_{j=1}^{n} \mathrm{d}y_{ij}$. Note that $O < X < I$ means that X is positive definite and that all eigenvalues of X are in the open interval $(0, 1)$. We need a few Jacobians of matrix transformations in this section and these will be given as lemmas without proofs. For proofs and for other results on Jacobians, see Mathai (1997).

Lemma 4.1. *Let A be $m\times m$ nonsingular constant matrix, B be $n\times n$ nonsingular constant matrix and let X and Y be $m\times n$ matrices of distinct real scalar variables. Then*

$$\begin{aligned}
&Y = AX, |A| \neq 0 \Rightarrow \mathrm{d}Y = |A|^{n}\mathrm{d}X\\
&Y = XB, |B| \neq 0 \Rightarrow \mathrm{d}Y = |B|^{m}\mathrm{d}X\\
&Y = AXB, |A| \neq 0, |B| \neq 0 \Rightarrow \mathrm{d}Y = |A|^{n}|B|^{m}\mathrm{d}X.
\end{aligned}$$

Lemma 4.2. *Let $X = X'$, a $p \times p$ symmetric matrix of distinct real variables, except for symmetry. Let A be a $p \times p$ nonsingular constant matrix. Then*

$$Y = AXA', |A| \neq 0 \Rightarrow \mathrm{d}Y = \begin{cases} |A|^{p+1}\mathrm{d}X \text{ for } X = X' \\ |A|^{p-1}\mathrm{d}X \text{ for } X' = -X. \end{cases}$$

Lemma 4.3. *Let X be a nonsingular $p \times p$ matrix and let X^{-1} be its regular inverse. Then*

$$Y = X^{-1} \Rightarrow \mathrm{d}Y = \begin{cases} |X|^{-2p}\mathrm{d}X \text{ for a general } X \\ |X|^{-(p+1)}\mathrm{d}X \text{ for } X = X'. \end{cases}$$

Lemma 4.4. *Let X be a $p \times p$ real positive definite matrix with distinct real variables, except for symmetry, and let T be a lower triangular matrix of distinct elements and with positive diagonal elements. Then the transformation $X = TT'$ is one to one and*

$$\mathrm{d}X = 2^{p}\{\prod_{j=1}^{p} t_{jj}^{p+1-j}\}\mathrm{d}T.$$

Now, let us consider products and ratios of $p \times p$ real positive definite matrices X_1 and X_2. Let $X_2^{\frac{1}{2}}$ denote the positive definite square root of X_2. Consider the product $U_2 = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}, V = X_2$ or $X_1 = V^{-\frac{1}{2}} U_2 V^{-\frac{1}{2}}$, and the ratio $U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}, V = X_2$ or $X_1 = V^{\frac{1}{2}} U_1^{-1} V^{\frac{1}{2}}$. These U_2 and U_1 are called symmetric product and ratio respectively. From the above lemmas the Jacobians can be seen to be the following:

$$\mathrm{d}X_1 \wedge \mathrm{d}X_2 = |V|^{-\frac{p+1}{2}} \mathrm{d}U_2 \wedge \mathrm{d}V$$
$$\mathrm{d}X_1 \wedge \mathrm{d}X_2 = |V|^{\frac{p+1}{2}} |U_1|^{-(p+1)} \mathrm{d}U_1 \wedge \mathrm{d}V,$$

ignoring the sign. Let X_1 and X_2 be statistically independently distributed real $p \times p$ matrix-variate random variables with the real-valued scalar functions $f_1(X_1)$ and $f_2(X_2)$ as densities respectively. Then due to independence the joint density of X_1 and X_2 is the product $f_1(X_1) f_2(X_2)$. Let the joint density of U_2 and V be denoted by $g(U_2, V)$ and the marginal density of U_2 as $g_2(U_2)$. Then from standard procedures we can see that the density of U_2 has the following format:

$$g_2(U_2) = \int_V |V|^{-\frac{p+1}{2}} f_1(V^{-\frac{1}{2}} U V^{-\frac{1}{2}}) f_2(V) \mathrm{d}V. \tag{4.1}$$

If f_1 and f_2 are statistical densities then (4.1) gives a statistical density of the product $U_2 = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}$. In general, for arbitrary functions f_1 and f_2, including densities, (4.1) is called M-convolution of a product. Thus, the concept of M-convolutions defined in Mathai (1997) is given a proper interpretation in terms of statistical densities here. Suppose that we consider f_1 a real matrix-variate type-1 beta density of the following form:

$$f_1(X_1) = \frac{\Gamma_p(\gamma + \frac{p+1}{2} + \alpha)}{\Gamma_p(\gamma + \frac{p+1}{2}) \Gamma_p(\alpha)} |X_1|^{\gamma}$$
$$\times |I - X_1|^{\alpha - \frac{p+1}{2}}, O < X_1 < I \tag{4.2}$$

for $\Re(\alpha) > \frac{p-1}{2}, \Re(\gamma) > -1$ and zero elsewhere, where $\Gamma_p(\alpha)$ is the real matrix-variate gamma given by the following expression which has the following integral representation also:

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}$$
$$= \int_{X>O} |X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X. \tag{4.3}$$

Now, if (4.2) is substituted in (4.1) and if $f_2(X_2) = f(X_2)$ where f is an arbitrary density then $g_2(U_2)$ of (4.1) becomes the following, again denoted by $g_2(U_2)$:

$$g_2(U_2) = \frac{\Gamma_p(\gamma + \frac{p+1}{2} + \alpha)}{\Gamma_p(\gamma)} \frac{|U_2|^\gamma}{\Gamma_p(\alpha)} \int_{V>U_2} |V|^{-\gamma-\alpha}$$
$$\times |V - U_2|^{\alpha-\frac{p+1}{2}} f(V)\mathrm{d}V, \Re(\alpha) > \frac{p-1}{2}. \tag{4.4}$$

Note that for $p = 1$, (4.4) corresponds to Kober fractional integral of the second kind and of order α and parameter γ. Hence this author has called (4.4) as Kober fractional integral of order α and of the second kind with parameter γ in the real matrix-variate case. A corresponding result is also established in Mathai (2013) for the complex matrix-variate case, which will not be discussed here.

Now, let us consider the ratio of two matrix random variables in the real case. In this case we take for convenience as $U_1 = X_2^{\frac{1}{2}}X_1^{-1}X_2^{\frac{1}{2}}$ or $X_1 = V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}}$. Then the density of U_1, again denoted by $g_1(U_1)$ will be the following:

$$g_1(U_1) = \int_V |V|^{\frac{p+1}{2}}|U_1|^{-(p+1)} f_1(V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}})f(V)\mathrm{d}V. \tag{4.5}$$

Let us consider some special cases. Let us assume that f_1 is of the following form:

$$f_1(X_1) = \frac{\Gamma_p(\gamma + \alpha)}{\Gamma_p(\gamma)\Gamma_p(\alpha)}|X_1|^{\gamma-\frac{p+1}{2}}$$
$$\times |I - X_1|^{\alpha-\frac{p+1}{2}}, O < X_1 < I, \Re(\alpha) > \frac{p-1}{2}, \Re(\gamma) > \frac{p-1}{2}. \tag{4.6}$$

If we substitute (4.6) in (4.5) then we have the following form for the density $g_1(U_1)$:

$$\begin{aligned} g_1(U_1) &= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} \frac{|U_2|^{-\gamma-\alpha}}{\Gamma_p(\alpha)} \int_{V<U_1} |V|^{\gamma} \\ &\times |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V)\mathrm{d}V \\ &= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} K_{1,U_1,\gamma}^{-\alpha} f \text{ where} \end{aligned} \tag{4.7}$$

$$K_{1,U_1,\gamma}^{-\alpha} f = \frac{|U_1|^{-\gamma-\alpha}}{\Gamma_p(\alpha)} \int_{V<U_1} |V|^{\gamma} |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V)\mathrm{d}V. \tag{4.8}$$

Here, (4.8) for $p = 1$ corresponds to Kober fractional integral of order α and of the first kind and hence this author has called (4.8) as Kober fractional integral of order α of the first kind with parameter γ in the real matrix-variate case.

5. General Definitions in the Real Matrix-variate Case

A general definition is given by this author (Mathai 2013,2014) for fractional integrals of the first kind and second kind of order α in the light of the results in Section 4 above. Let f_1 and f_2 be of the following forms:

$$f_1(X_1) = \frac{1}{\Gamma_p(\alpha)} \phi_1(X_1) |I - X_1|^{\alpha - \frac{p+1}{2}} \tag{5.1}$$

$$O < X_1 < I, \Re(\alpha) > \frac{p-1}{2}, \text{ and zero elsewhere, and}$$

$$f_2(X_2) = \phi_2(X_2) f(X_2) \tag{5.2}$$

where ϕ_1 and ϕ_2 are specified functions and $f(X_2)$ is an arbitrary function. Note that X_1 and X_2 are $p \times p$ real positive definite matrices. If X_1 and X_2 are matrix random variables then they are assumed to be independently distributed and in that case f_1 and f_2 are the corresponding densities, otherwise they are not assumed to be densities. Again we look at M-convolutions of products and ratios. Let $U_2 = X_2^{\frac{1}{2}} X_1 X_2^{\frac{1}{2}}, V = X_2$ and $U_1 = X_2^{\frac{1}{2}} X_1^{-1} X_2^{\frac{1}{2}}, V = X_2$ be the symmetric product and symmetric ratio of the matrices X_1 and X_2. The Jacobians are already evaluated in Section 4. If we denote the M-convolution of product as $g_2(U_2)$ and that of ratio as $g_1(U_1)$, then they are the following for f_1 and f_2 as defined in (5.1) and (5.2).

$$\begin{aligned}
g_2(U_2) &= \frac{1}{\Gamma_p(\alpha)}\int_{V>U_2}|V|^{-\frac{p+1}{2}}\phi_1(V^{-\frac{1}{2}}U_2V^{-\frac{1}{2}}) \\
&\quad\times|I-V^{-\frac{1}{2}}U_2V^{\frac{1}{2}}|^{\alpha-\frac{p+1}{2}}\phi_2(V)f(V)\mathrm{d}V \qquad (5.3)\\
g_1(U_1) &= \frac{1}{\Gamma_p(\alpha)}\int_{V<U_1}|V|^{\frac{p+1}{2}}U_1^{-(p+1)}\phi_1(V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}}) \\
&\quad\times|I-V^{\frac{1}{2}}U_1^{-1}V^{\frac{1}{2}}|^{\alpha-\frac{p+1}{2}}\phi_2(V)f(V)\mathrm{d}V \qquad (5.4)
\end{aligned}$$

for $\Re(\alpha)>\frac{p-1}{2}$.

Now, let us look at some special cases. First, we will consider Kober type fractional integrals, which are directly connected to statistical distributions of product and ratio of independently distributed real matrix random variables. Let

$$\phi_1(X_1)=\frac{\Gamma_p(\gamma+\frac{p+1}{2}+\alpha)}{\Gamma_p(\gamma+\frac{p+1}{2})}|X_1|^{\gamma}$$

and $\phi_2=1$. Then substituting these in (5.3) for the M-convolution of a product $U_2=X_2^{\frac{1}{2}}X_1X_2^{\frac{1}{2}}, V=X_2$ it is easily seen that (5.3) reduces to the following form:

$$\begin{aligned}
g_2(U_2) &= \frac{\Gamma_p(\gamma+\frac{p+1}{2}+\alpha)}{\Gamma_p(\gamma+\frac{p+1}{2})}\frac{U_2^{\gamma}}{\Gamma_p(\alpha)}\int_{V>U_2}|V|^{-\gamma-\alpha} \\
&\quad\times|V-U_2|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \\
&= \frac{\Gamma_p(\gamma+\frac{p+1}{2}+\alpha)}{\Gamma_p(\gamma+\frac{p+1}{2})}K_{2,U_2,\gamma}^{-\alpha}f \text{ where} \\
K_{2,U_2,\gamma}^{-\alpha}f &= \frac{|U_2|^{\gamma}}{\Gamma_p(\alpha)}\int_{V>U_2}|V|^{-\gamma-\alpha}|V-U_2|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \qquad (5.5)
\end{aligned}$$

for $\Re(\gamma)>-1,\Re(\alpha)>\frac{p-1}{2}$, where $K_{2,U_2,\gamma}^{-\alpha}f$ for $p=1$ corresponds to Kober fractional integral of order α of the second kind with parameter γ and hence this author called (5.5) as Kober fractional integral of order α of the second kind with parameter γ in the real matrix-variate case. A corresponding definition is given in the complex matrix-variate case also, see Mathai (2013). Let us consider another special case with $\phi_1=1$ and $\phi_2(X_2)=|X_2|^{\alpha}$. Then (5.3) reduces to the following form:

$$g_2(U_2)=\frac{1}{\Gamma_p(\alpha)}\int_{V>U_2}|V-U_2|^{\alpha-\frac{p+1}{2}}f(V)\mathrm{d}V \qquad (5.6)$$

for $\Re(\alpha) > \frac{p-1}{2}$, which for $p = 1$ is the Weyl fractional integral of order α of the second kind. Hence this author has called (5.6) as the Weyl fractional integral of the second kind of order α in the real matrix-variate case and it is denoted by him as $W_{2,U_2}^{-\alpha} f$. If V is bounded above by a constant positive definite $p \times p$ matrix B then this author has called (5.6) as the Riemann-Liouville fractional integral of the second kind of order α and upper matrix B in the real matrix-variate case. This author has also defined Saigo fractional integral of the second kind and its generalizations in the matrix-variate case by replacing $\phi_1(X_1)$ with a general hypergeometric series in the matrix-variate case (expansion in terms of zonal polynomials) with arguments AX_1 and $A(I - X_1)$, where A is a positive definite constant matrix, then specializing it as a ${}_2F_1$, Gauss hypergeometric series form. Expansions in terms of zonal polynomials and discussion of zonal polynomials may be seen from Mathai, Provost and Hayakawa (1995).

Now, let us look at some special cases for fractional integrals of the first kind in the real matrix-variate case. Let

$$\phi_1(X_1) = \frac{\Gamma_p(\gamma + \alpha)}{\Gamma_p(\gamma)} |X_1|^{\gamma - \frac{p+1}{2}}. \tag{5.7}$$

Then substituting (5.7) in (5.4) we have the following result, again denoted by $g_1(U_1)$:

$$\begin{aligned} g_1(U_1) &= \frac{\Gamma_p(\gamma + \alpha)}{\Gamma_p(\gamma)} \frac{|U_1|^{-\gamma-\alpha}}{\Gamma_p(\alpha)} \int_{V<U_1} |V|^{\gamma} \\ &\quad \times |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V) \mathrm{d}V \\ &= \frac{\Gamma_p(\gamma + \alpha)}{\Gamma_p(\gamma)} K_{1,U_1,\gamma}^{-\alpha} f \text{ where} \\ K_{1,U_1,\gamma}^{-\alpha} f &= \frac{|U_1|^{-\gamma-\alpha}}{\Gamma_p(\alpha)} \int_{V<U_1} |V|^{\gamma} |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V) \mathrm{d}V \end{aligned} \tag{5.8}$$

for $\Re(\alpha) > \frac{p-1}{2}, \Re(\gamma) > \frac{p-1}{2}$, where $K_{1,U_1,\gamma}^{-\alpha} f$ for $p = 1$ is Kober fractional integral of the first kind of order α and parameter γ and hence this author has called (5.8) as Kober fractional integral of the first kind of order α and parameter γ in the real matrix-variate case. A corresponding quantity for the complex matrix-variate case is also defined by this author, see Mathai (2013).

Let us consider another special case. Let

$$\phi_1(X_1) = |X_1|^{-\alpha - \frac{p+1}{2}} \text{ and } \phi_2(X_2) = |X_2|^{\alpha}$$

then (5.4) reduces to the following form, again denoted by $g_1(U_1)$:

$$g_1(U_1) = \frac{1}{\Gamma_p(\alpha)} \int_{V<U_1} |U_1 - V|^{\alpha - \frac{p+1}{2}} f(V) \mathrm{d}V. \tag{5.9}$$

This (5.9) for $p = 1$ is Weyl fractional integral of the first kind of order α and hence this author has called (5.9) as the Weyl fractional integral of the first kind of order α in the real matrix-variate case. If V is bounded below by a constant $p \times p$ positive definite matrix A then (5.9) is called the Riemann-Liouville fractional integral of the first kind of order α in the real matrix-variate case with lower parameter matrix A.

The above ideas are extended to fractional integrals of the first and second kinds for many scalar variables and many matrix variables, both in the real and complex cases. The real cases may be seen from Mathai and Haubold (2012, I-IV) and the complex cases in Mathai (2013,2014).

6. Fractional Differential Operators

Different authors have used different definitions for fractional derivatives in the real scalar variables case. In the Riemann-Liouville sense fractional derivatives are defined as certain fractional integrals. This is found to be very useful in practical applications. We will use the following notations. Derivatives of order α will be denoted by the exponent $+\alpha$ and the corresponding integrals with $-\alpha$ indicating integrals as antiderivatives. Let n be a positive integer, $n = 1, 2, ...$ such that $n - \Re(\alpha) > 0$. The smallest such n is given by $n = [\Re(\alpha)] + 1$ where $[(\cdot)]$ indicates the integer part of the real number $(\cdot)$. For example, if $\alpha = 2.7$ then $[\Re(\alpha)] = 2$ so that $n = 2 + 1 = 3$. If $\alpha = 1.5 + 3i, i = \sqrt{-1}$ then $[\Re(\alpha)] = 1$ then $n = 1 + 1 = 2$. Then the fractional derivative of order α in the Riemann-Liouville sense will be denoted symbolically as $D^\alpha = D^n D^{-(n-\alpha)}$, that is, the $(n - \alpha)$th order fractional integral is taken first and then integer order derivative is applied n times. For example, if Riemann-Liouville fractional integral of order $n - \alpha$ of the first kind $D_{1,(a,x)}^{-(n-\alpha)} f$ is taken and then n-th order derivative with respect to the parameter x is taken then we have Riemann-Liouville sense fractional derivative of the first kind of order α, denoted by $D_{1,x}^\alpha f$, and it is given by the following:

$$D_{1,x}^\alpha f = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \{ \frac{1}{\Gamma(n-\alpha)} \int_{v<x} [x - v]^{(n-\alpha)-1} f(v) \mathrm{d}v \}. \tag{6.1}$$

This indicates that the $(n - \alpha)$th order fractional integral is taken first and then it is differentiated n times with respect to the parameter x of the fractional integral.

Note that if it is the second kind Rieman-Liouville fractional integral then the factor corresponding to $[x - v]$ in (6.1) is $[v - x]$ and hence when differentiated n times a $(-1)^n$ will come out. Hence when defining the fractional derivative of the second kind of order α, it is usually defined as

$$D^{\alpha}_{2,x} f = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} D^{-(n-\alpha)}_{2,x} f.$$

Note that, symbolically, we have

$$D^{\alpha} = D^n D^{-(n-\alpha)} \text{ and } D^{\alpha} = D^{-(n-\alpha)} D^n.$$

In the second case above we have differentiated the arbitrary function f, n times first and then the fractional integral of order $n - \alpha$ is taken. This will be called fractional derivatives in the Caputo sense and the former as the fractional derivatives in Riemann-Liouville sense. Hence for the real scalar variable case we have used the following general notations. First kind derivative or integral is denoted by 1 and the second kind by 2, fractional integral of order α with exponent $-\alpha$ and the derivative with $+\alpha$. Weyl integral or derivative is denoted by W, Caputo by C, Saigo by S, Kober by K etc and since Riemann-Liouville is the most popular one, D is used for that. For example

$$W^{\alpha}_{1,x} f = \begin{cases} D^n W^{-(n-\alpha)}_{1,x} f \text{ in Riemann-Liouville sense} \\ W^{-(n-\alpha)}_{1,x} D^n f \text{ in Caputo sense.} \end{cases}$$

For the matrix-variate case also the same notations are used but here D^n has to be defined. Recently, this author has given a definition of D^n which can operate on real-valued scalar functions of real and complex matrix argument of the following types: $|X|^{-\gamma}, |I + X|^{-\gamma}, |A + X|^{-\gamma}$ for A, X positive definite and $p \times p$ where A is a constant matrix, $\Re(\gamma) > \frac{p-1}{2}$; in the exponential types $\mathrm{e}^{\pm \mathrm{tr}(X)}$; power function type $\frac{|I-X|^{\alpha - \frac{p+1}{2}}}{\Gamma_p(\alpha)}, \Re(\alpha) > \frac{p-1}{2}$. But a differential operator which operates universally on all functions is not yet obtained. For $X = (x_{ij}) > O$, that is, $p \times p$ and real positive definite consider the differential operator $\frac{\partial}{\partial X} = (\frac{\partial}{\partial x_{ij}})$ that is, the partial differential operator of the corresponding elements. Consider the determinant of this operator, that is $|\frac{\partial}{\partial X}|$. Consider the function $f(Y) = \mathrm{e}^{\mathrm{tr}(XY)}$ where Y and X are symmetric and Y is such that its non-diagonal elements are multiplied by $\frac{1}{2}$ and diagonal elements by 1. Then $\mathrm{tr}(YX)$ will be the sum of products all corresponding elements coming once. If the non-diagonal elements of Y are not multiplied by $\frac{1}{2}$ then the $x_{ij} y_{ij}, i \neq j$ will be coming twice and the diagonal elements only once.

Note that the determinant operator operating on this exponential function will give the following result:

$$|\frac{\partial}{\partial X}|\mathrm{e}^{\mathrm{tr}(XY)} = |Y|\mathrm{e}^{\mathrm{tr}(XY)}$$

and then this operator operating repeatedly n times brings $|Y|^n$ outside. Consider the following identity:

$$|X|^{-\alpha} \equiv \frac{1}{\Gamma_p(\alpha)} \int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(YX)} \mathrm{d}Y, \Re(\alpha) > \frac{p-1}{2}. \qquad (a)$$

Operate on both sides with $\tilde{D}_X^n = (-1)^n |\frac{\partial}{\partial X}|^n$. Then we have

$$\begin{aligned}
\tilde{D}_X^n |X|^{-\alpha} &= \tilde{D}_X^n \frac{1}{\Gamma_p(\alpha)} \int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(YX)} \mathrm{d}Y \\
&= \frac{1}{\Gamma_p(\alpha)} \int_{Y>O} |Y|^{\alpha-\frac{p+1}{2}} |Y|^n \mathrm{e}^{-\mathrm{tr}(YX)} \mathrm{d}Y \\
&= |X|^{-(\alpha+n)}
\end{aligned}$$

interpreting the right side integral by using the identity in (a). The above is one such result. Similar results are obtained by this author recently. Since the area of fractional derivatives in the matrix-variate case is not fully developed, further discussion is omitted.

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