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DISTANCE BASED INDICES FOR COMPLEMENT OF MYCIELSKIAN GRAPHS

S. Parameswari and K. Balasangu*

Department of Mathematics,
Arignar Anna Govt Arts college,
Villupuram - 605752, Tamil Nadu, INDIA

E-mail : eswarip651@gmail.com

*Department of Mathematics,
Thiru kolanjiappar Govt Arts college,
Viruthachalam - 606001, Tamil Nadu, INDIA

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Abstract: In this paper, we establish an explicit formula to calculate the several graph indices based on distance for the complement of generalized Mycielskian of any graph G.

Keywords and Phrases: Graph index, distance, Mycielskian graph.

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1. Introduction and Preliminaries

For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *diameter* of the graph G is $max\{d_G(u, v)|u, v \in V(G)\}$. A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

Dobrynin and Kochetova [2] and Gutman [3] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for a connected graph G as $DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$. The *additively weighted Harary index*(H_A) or *reciprocal degree distance*(RDD) is defined in [1] as $RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$.

The *generalized degree distance*, denoted by $H_\lambda(G)$, is defined as $H_\lambda(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v)$, where λ is a any real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD(G)$. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et. al [4], [5]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. The first Zagreb index is defined for a connected graph G as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2 = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have applications in QSPR and QSAR studies as well. In this paper, we establish an explicit formula to calculate the several graph indices based on distance for the complement of generalized Mycielskian of a given graph G.

2. Main Results

Let G be (s,e)-graph (that is, G has s vertices and e edges). The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices of G in which two vertices u and v are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. Obviously, $E(G) \cup E(\overline{G}) = E(K_s)$ and $\overline{e} = \left|E(\overline{G})\right| = \frac{s(s-1)}{2} - e$. The degree of a vertex v in G is denoted by $d_G(v)$; the degree of the same vertex in $\overline{G}$ is given by $d_{\overline{G}}(v) = s - 1 - d_G(v)$.

Mycielski [6] developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, which is called the Mycielskian of G, in a search for triangle for triangle-free graphs with arbitrarily large chromatic number. The generalized Mycielskian is natural generalization of Mycielskian graph, which is also called by Tardif cones over graphs. Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. Given an integer $r \geq 1$, the r-Mycielskian of G, denoted by $\mu_r(G)$, is the graph with vertex set $V^0 \cup V^1 \cup \ldots \cup V^r \cup \{r\}$, where $V^i = \{v_j^i : V^0 \in V^0\}$ is the i^{th} distinct copy of V^0 for $i = 1, 2, \ldots, r$ and edge set $E^0 \cup \Big(\cup_{i=0}^{r-1} v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0 \Big) \cup \{v_j^r u : v_j^r \in V^r\}$.

The proof of the following lemmas are easily follows from the structure of the graph $\overline{\mu}_r(G)$.

Lemma 2.1. *Let G be a (n,e)-graph. If $\overline{\mu}_r(G)$ is the generalized Mycielkian graph* *of G, then for any two vertices $a, b \in V(\overline{\mu}_r(G))$,*

(i) $d_{\overline{\mu}_r(G)}(a,b) = 1$, *if* $a = a_i^t, \quad b = a_j^t, \quad t = 1,2,...,r; \quad i,j = 1,2,...,s.$

(ii) $d_{\overline{\mu}_r(G)}(a,b) = 1$, *if* $a = a_i^t, \quad b = w, \quad t = 0,1,2,...,r-1; \quad i = 1,2,...,s.$

(iii) $d_{\overline{\mu}_r(G)}(a,b) = 1$, *if* $a = a_i^t, \quad b = a_i^z, \quad t,z = 1,2,...,r; \quad i = 1,2,...,s, \quad t \neq z.$

(iv) $d_{\overline{\mu}_r(G)}(a,b) = \begin{cases} 1 \quad \text{if} \quad a = a_i^0, \quad b = a_j^0, \quad d_{\mu_r(G)}(a_i^0, a_j^0) > 1; \quad i,j = 1,2,...,s, \\ 2, \quad \text{if} \quad a = a_i^0, \quad b = a_j^0, \quad d_{\mu_r(G)}(u_i^0, u_j^0) = 1; \quad i,j = 1,2,...,s. \end{cases}$

(v) $d_{\overline{\mu}_r(G)}(a,b) = \begin{cases} 1 \quad \text{if} \quad a = a_i^t, \quad b = a_j^z, \quad d_{\mu_r(G)}(a_i^t, a_j^z) > 1; \\ \quad t \neq z; \quad i \neq j; \quad t,z = 0,1,2,...,r; \quad i,r = 1,2,...,s. \\ 2, \quad \text{if} \quad a = a_i^t, \quad b = a_j^z, \quad d_{\mu_r(G)}(a_i^t, a_j^z) = 1; \\ \quad t \neq z; \quad i \neq j; \quad t,z = 0,1,2,...,r; \quad i,r = 1,2,...,s. \end{cases}$

(vi) $d_{\overline{\mu}_r(G)}(a,b) = 2$, *if* $a = a_i^r, \quad b = w, \quad i = 1,2,...,s.$

Lemma 2.2. *Let G ba a graph with s vertices. Then for any vertex $a \in V(\overline{\mu}_r(G))$,*

$$d_{\overline{\mu}_r(G)}(a) = \begin{cases} rs \quad \text{if } a = w; \\ s(r+1) - 2d_G(a_i^0), \quad \text{if} \quad a = a_i^t; \ t = 0,1,2,...,r-1; \ i = 1,2,...,s. \\ s(r+1) - d_G(a_i^0) - 1, \quad \text{if} \quad a = a_i^r; \quad i = 1,2,...,s. \end{cases}$$

Theorem 2.3. *Let G be a (n,e)-graph. If diameter of G is 2, then* $H_\lambda(\overline{\mu}_r(G)) = [r(r-1)+2]M_1(G) + r(2r+1)(s^2-2e) + s(r+1)(rs(r+1)-e) - 4er(s-1) - 4er + \frac{s(s-1)}{2}\Big[3s(r+1) + 2s(r^2-1) - 2\Big] - 2^\lambda\Big[(r(r-1)+4)M_1(G) - es(r+1)(4r+1) - s(2rs+s-1) + 4e\Big]$.

Proof. To find $H_\lambda(\overline{\mu}_r(G))$, we consider the following cases of adjacent and nonadjacent pairs of vertices in $\overline{\mu_r}(G)$.

Case 1. If $\{a,b\} \subseteq V(\overline{\mu_r}(G)), \quad a = a_i^0, \quad b = a_j^0 \in V^0$, then by Lemmas 2.1 and 2.2,

$$\begin{aligned} H_\lambda(\overline{\mu_r}(G)) &= \frac{1}{2} \sum_{a=a_i^0, \ b=a_j^0 \in V^0} \Big(d_{\overline{\mu_r}(G)}(a_i^0) + d_{\overline{\mu_r}(G)}(a_j^0)\Big) d^\lambda_{\overline{\mu_r}(G)}(a_i^0, a_j^0) \\ = & \ \frac{1}{2} \sum_{a=a_i^0, \ b=a_j^0 \in V^0; \ a_i^0, a_j^0 \notin E(\mu_r(G))} \Big(s(r+1) - 2d_G(a_i^0) + s(r+1) - 2d_G(a_j^0)\Big) \\ + & \ \frac{1}{2} \sum_{a=a_i^0, \ b=a_j^0 \in V^0; \ a_i^0, a_j^0 \in E(\mu_r(G))} \Big(s(r+1) - 2d_G(a_i^0) + s(r+1) - 2d_G(a_j^0)\Big) 2^\lambda \end{aligned}$$

$$= \frac{1}{2} \sum_{a=a_i^0,\ b=a_j^0 \in V^0;\ a_i^0, a_j^0 \notin E(\mu_r(G))} 2\Big(s(r+1) - (d_G(a_i^0) + d_G(a_j^0))\Big)$$
$$+ \frac{1}{2} \sum_{a=a_i^0,\ b=a_j^0 \in V^0;\ a_i^0, a_j^0 \in E(\mu_r(G))} 2(2^\lambda)\Big(s(r+1) - (d_G(a_i^0) + d_G(a_j^0))\Big)$$
$$= \Big[\Big(\frac{s(s-1)}{2} - e\Big)(r+1)s - 2e(s-1) - M_1(G)\Big] + 2^\lambda\Big[(r+1)se - M_1(G)\Big].$$

Case 2. If $\{a, b\} \subseteq V(\overline{\mu_r}(G))$, $a = a_i^t$, $b = a_j^t \in V^t$; $t = 1, 2, ..., r$, then by Lemmas 2.1 and 2.2,

$$H_\lambda(\overline{\mu}_r(G)) = \frac{1}{2} \sum_{a=a_i^t,\ b=a_j^t \in V^t,\ t=1,2,\ldots,r} \Big(d_{\overline{\mu_r}(G)}(a_i^t) + d_{\overline{\mu_r}(G)}(a_j^t)\Big) d^\lambda_{\overline{\mu_r}(G)}(a_i^t, a_j^t)$$
$$= \frac{1}{2} \sum_{a=a_i^t;\ b=a_j^t \in V^t;\ t=1,2,\ldots,r-1} \Big[s(r+1) - 2d_G(a_i^0) + s(r+1) - 2d_G(a_j^0)\Big]$$
$$+ \frac{1}{2} \sum_{a=a_i^r,\ b=a_j^r \in V^r} \Big[s(r+1) - d_G(a_i^0) - 1 + s(r+1) - d_G(a_j^0) - 1\Big]$$
$$= \frac{1}{2} \sum_{a=a_i^t;\ b=a_j^t \in V^t;\ t=1,2,\ldots,r-1} 2(s(r+1) - (d_G(a_i^0) + d_G(a_j^0)))$$
$$+ \frac{1}{2} \sum_{a=a_i^r,\ b=a_j^r \in V^r} 2(s(r+1) - (d_G(a_i^0) + d_G(a_j^0)))$$
$$= 2\Big[\frac{s(s-1)}{2}(r-1)s(r+1) - 2e(s-1)(r-1)\Big] + 2\Big[\frac{s(s-1)}{2}(s(r+1)-1) - e(s-1)\Big]$$
$$= s(s-1)\Big(s(r-1)(r+1) + (s(r+1)-1)\Big) - 2e(s-1)(2(r-1)+1).$$

Case 3. If $\{a, b\} \subseteq V(\overline{\mu_r}(G))$, $a = a_i^r \in V^r$, $b = w$ and $i = 1, 2, ..., s$, then by Lemmas 2.1 and 2.2,

$$H_\lambda(\overline{\mu_r}(G)) = \frac{1}{2} \sum_{a=a_i^r \in V^r,\ b=w,\ i=1,2,\ldots,s} \Big(d_{\overline{\mu_r}(G)}(a_i^r) + d_{\overline{\mu_r}(G)}(w)\Big) d^\lambda_{\overline{\mu_r}(G)}(a_i^r, w)$$
$$= \frac{1}{2} \sum_{a=a_i^r \in V^r,\ b=w,\ i=1,2,\ldots,s} \Big(s(r+1) - 1 - d_G(a_i^0) + rs\Big)(2^\lambda)$$
$$= 2^{\lambda-1}\Big(\sum_{a=a_i^r \in V^r,\ b=w,\ i=1,2,\ldots,s} (rs + s(r+1) - 1) - \sum_{a=a_i^r \in V^r,\ b=w,\ i=1,2,\ldots,s} d_G(a_i^0)\Big)$$
$$= 2^{\lambda-1}(2)\Big(s(2rs + s - 1) - 2e\Big) = 2^\lambda\Big(s(2rs + s - 1) - 2e\Big).$$

Case 4. If $\{a, b\} \subseteq V(\overline{\mu_r}(G))$, $a = a_i^t \in V^t$, $b = w$, $t = 0, 1, ..., r-1$,

$i=1,2,...,s$, then by Lemmas 2.1 and 2.2,

$$H_\lambda(\overline{\mu_r}(G)) = \frac{1}{2}\sum_{a=a_i^t\in V^t,\ b=w,\ t=0,1,\ldots,r-1,\ i=1,2,\ldots,s}\Big(d_{\overline{\mu_r}(G)}(a_i^t)+d_{\overline{\mu_r}(G)}(w)\Big)d^{\lambda}_{\overline{\mu_r}(G)}(a_i^t,w)$$

$$= \frac{1}{2}\sum_{a=a_i^t\in V^t,\ b=w,\ t=0,1,\ldots,r-1,\ i=1,2,\ldots,s}\Big(s(r+1)-2(d_G(a_i^0)+rs)\Big)$$

$$= \frac{1}{2}\Big(2rs+s-2d_G(a_i^0)\Big) = \frac{1}{2}\Big(2rs(2rs+s)-4(2er)\Big) = rs(2rs+s)-4er.$$

Case 5. If $\{a,b\}\subseteq V(\overline{\mu_r}(G))$, $a=a_i^t$, $b=a_i^z$, $t\neq z$, $t,z\in\{0,1,...,r-1\}$, $i=1,2,...,s$, then by Lemmas 2.1 and 2.2,

$$H_\lambda(\overline{\mu_r}(G)) = \frac{1}{2}\sum_{a=a_i^t,\ b=a_i^z,\ t\neq z\in\{0,1,\ldots,r-1\},\ i\in\{1,2,\ldots,s\}}\Big(d_{\overline{\mu_r}(G)}(a_i^t)+d_{\overline{\mu_r}(G)}(a_i^z)\Big)d^{\lambda}_{\overline{\mu_r}(G)}(a_i^t,a_i^z)$$

$$= \frac{1}{2}\sum_{a=a_i^t,\ b=a_i^z,\ t\neq z,\ t,z\in\{0,1,\ldots,r-1\}}\Big(s(r+1)-2d_G(a_i^0)+s(r+1)-2d_G(a_i^0)\Big)$$

$$+\frac{1}{2}\sum_{a=a_i^t,\ b=a_i^r,\ t=0,1,\ldots,r-1,\ i=1,2,\ldots,s}\Big(s(r+1)-2d_G(a_i^0)+s(r+1)-2d_G(a_i^0)-1\Big)$$

$$= \frac{1}{2}\sum_{a=a_i^t,\ b=a_i^z,\ t\neq z,\ t,z\in\{0,1,\ldots,r-1\}}\Big(2s(r+1)-4d_G(a_i^0)\Big)$$

$$+\frac{1}{2}\sum_{a=a_i^t,\ b=a_i^r,\ t=0,1,\ldots,r-1,\ i=1,2,\ldots,s}\Big(2s(r+1)-3d_G(a_i^0)-1\Big)$$

$$= \Big(r(r-1)s^2(r+1)-4r(r-1)e\Big)+\Big(rs(2s(r+1)-1)-6re\Big)$$

$$= rs^2(r+1)^2-2re(2r+1).$$

Case 6. If $\{a,b\}\subseteq V(\overline{\mu_r}(G))$, $a=a_i^t$, $b=a_j^z$, $t\neq z$, $t,z\in\{0,1,...,r-1\}$, $i\neq j, i\in\{1,2,...,s\}$, then by Lemmas 2.1 and 2.2,

$$H_\lambda(\overline{\mu_r}(G)) = \frac{1}{2}\sum_{a=a_i^t,\ b=a_j^z,\ t\neq z\in\{0,1,\ldots,r-1\}}\Big(s(r+1)-2d_G(a_i^0)+s(r+1)-2d_G(a_j^0)\Big)d^{\lambda}_{\overline{\mu_r}(G)}(a_i^0,a_j^0)$$

$$+\ \frac{1}{2}\sum_{a=a_i^t,\ b=a_j^r,\ t=0,1,\ldots,r-1,\ i,j=1,2,\ldots,s}\Big(s(r+1)-2d_G(a_i^0)+s(r+1)-2d_G(a_j^0)-1\Big)d^{\lambda}_{\overline{\mu_r}(G)}(a_i^0,a_j^0)$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{a=a_i^t,\ b=a_j^z,\ t\neq z,\ i\neq j,\ i,j\in\{0,1,\ldots,s\},\ t,z\in\{0,1,\ldots,r-1\},\ a_i^t a_j^z \notin E(\mu_r(G))} (2s(r+1) - 2(d_G(a_i^0) + d_G(a_j^0))) \\
&+ \frac{1}{2} \sum_{a=a_i^t,\ b=a_j^z,\ t\neq z,\ i\neq j,\ i,j\in 0,1,\ldots,s,\ t,z\in\{0,1,\ldots,r-1\},\ a_i^t a_j^z \in E(\mu_r(G))} (2s(r+1) - 2(d_G(a_i^0) + d_G(a_j^0)))(2^\lambda) \\
&+ \frac{1}{2} \sum_{a=a_i^t,\ b=a_j^r,\ t\neq r,\ i\neq j,\ i,j\in 0,1,\ldots,s,\ t,r\in\{0,1,\ldots,r-1\},\ a_i^t a_j^r \notin E(\mu_r(G))} (2s(r+1) - 1 - 2(d_G(a_i^0) - d_G(a_j^0))) \\
&+ \frac{1}{2} \sum_{a=a_i^t,\ b=a_j^r,\ t\neq r,\ i\neq j,\ i,j\in 0,1,\ldots,s,\ t,r\in\{0,1,\ldots,r-1\},\ a_i^t a_j^r \in E(\mu_r(G))} (2s(r+1) - 1 - 2d_G(a_i^0) - d_G(a_j^0))(2^\lambda) \\
&= \Big(\Big(s(s-1)\frac{r(r-1)}{2} - (r-1)2e\Big)2s(r+1) - 2\Big(\frac{r(r-1)}{2}(2e)(s-1) \\
&- M_1(G)\Big)\Big) + 2^\lambda\Big((r-1)2e(2s(r+1)) - 2\frac{r(r-1)}{2}M_1(G)\Big) \\
&+ \Big((s(s-1) - 2e + (r-1)s(s-1))(2s(r+1)-1) - 2(r(s-1)2e - M_1(G)) \\
&\quad -(r(s-1)2e - M_1(G))\Big) + 2^\lambda\Big(2e(2s(r+1)-1) - 2M_1(G) - M_1(G)\Big) \\
&= \Big(r(r-1)+3\Big)M_1(G) + s(r+1)\Big(s(s-1)(r-1)(r+2) - 4er + 2s(s-1)\Big) \\
&\quad -2er(s-1)(r-4) + 4e - rs(s-1) + 2^\lambda\Big(4esr(r+1) - 2e - (r(r-1)+3) \\
&\quad M_1(G)\Big) + 2^\lambda\Big(4esr(r+1) - 2e - (r(r-1)+3)M_1(G)\Big).
\end{aligned}
$$

Summarizing the total contributions of above cases of adjacent and non-adjacent pairs of vertices in $\overline{\mu_r}(G)$, we can obtain the desired result.

Using $\lambda = 1$ in Theorem 2.3, we obtain the degree distance of complement of generalized Mycielskian of a graph G.

Corollary 2.4. *Let G be a (n,e)-graph with diameter 2. Then* $DD(\overline{\mu_r}(G)) = 2es(r+1)(4r+1) + 2s(2rs+s-1) + r(2r+1)(s^2-2e) + s(r+1)(rs(r+1)-e) - 4e(r+2) - 4er(s-1) + \frac{s(s-1)}{2}\Big(3s(r+1) + 2s(r^2-1) - 2\Big) - (r(r-1)+6)M_1(G)$.

Using $\lambda = -1$ in Theorem 2.3, we obtain the following.

Corollary 2.5. *Let G be a (n, e)-graph with diameter 2. Then* $RDD(\overline{\mu_r}(G)) = \frac{es(r+1)(4r+1)}{2} + \frac{s(2rs+s-1)}{2} + r(2r+1)(s^2-2e) + s(r+1)(rs(r+1)-e) - 2e(2r+1) - 4er(s-1) + \frac{s(s-1)}{2}\Big(3s(r+1) + 2s(r^2-1) - 2\Big) + \frac{r(r-1)}{2}M_1(G).$

3. Conclusion:

In this paper, we have computed the general results related to degree and distance for generalized Mycielskian graph.

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