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On Certain Special Series and Continued Fractions

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Abstract: In this paper, making use of a known summation formulae for bilateral series, an attempt has been made to establish certain interesting results involving continued fractions.
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1. Introduction, Notations and Definitions

Here and in the sequel, we employ the customary notations for $|q| < 1$,

$$[a;q]_0 = 1, \qquad and \quad n \geq 1, \quad let$$

$$[a;q]_n = (1-a)(1-aq)...(1-aq^{n-1}),$$

$$[a;q]_\infty = \prod_{r=0}^{\infty}(1-aq^r)$$

and

$$[a_1, a_2, ..., a_r; q]_n = [a_1;q]_n[a_2;q]_n...[a_r;q]_n.$$

The 'Lost' notebook of Ramanujan contains several results involving Lambert series and continued fraction, we find the following summation formula,

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right] = a\frac{[ab, q/ab, b/a, aq/b; q]_\infty [q;q]_\infty^4}{[a, b, q/a, q/b; q]_\infty^2} \tag{1.1}$$

[Agarwal 1; (4.5) p.197]

It can be utilized to establish several interesting results involving continued fractions. Following results are also needed in our analysis.

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{[q;q]_n} = \frac{1}{[q, q^4; q^5]_\infty}. \tag{1.2}$$

[Andrews and Berndt 2; (4.3.3) p. 114]

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[q;q]_n} = \frac{1}{[q^2, q^3; q^5]_\infty}. \tag{1.3}$$

[Andrews and Berndt 2; (4.3.4) p. 114]

$$C(q) = \frac{H(q)}{G(q)} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+...}. \tag{1.4}$$

[Andrews and Berndt 2; (4.1.1),(4.1.2) p. 107]

$$\frac{[q, q^5; q^6]_\infty}{[q^3; q^6]^2_\infty} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+...}. \tag{1.5}$$

[Andrews and Berndt 2; (6.2.37) p. 154]

$$\frac{[q, q^7; q^8]_\infty}{[q^3, q^5; q^8]_\infty} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+...}. \tag{1.6}$$

[Andrews and Berndt 2; (6.2.38) p. 154]

$$\frac{[q; q^2]_\infty}{[q^2; q^4]^2_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\frac{q^2+q^4}{1+}\frac{q^5}{1+...}. \tag{1.7}$$

[Andrews and Berndt 2; (6.2.22) p. 150]

2. Main Results

In this section we shall establish our main results. Replacing q by q^k and $a = q^i, b = q^j$ in (1.1) we get,

$$\sum_{n=-\infty}^{\infty} \left[\frac{q^{kn+i}}{(1-q^{kn+i})^2} - \frac{q^{kn+j}}{(1-q^{kn+j})^2} \right]$$

$$= q^i \frac{[q^{i+j}, q^{k-i-j}, q^{j-i}, q^{k+i-j}; q^k]_\infty [q^k; q^k]^4_\infty}{[q^i, q^j, q^{k-i}, q^{k-j}; q^k]^2_\infty}, \tag{2.1}$$

provided $i, j \neq 0$ (mod k).

(i) Taking $i = 1, j = 2$ and $k = 5$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty} \left[\frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} \right] = q[q^5; q^5]^4_\infty H(q)G(q). \tag{2.2}$$

(ii) Taking $i = 2, j = 3$ and $k = 6$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+2}}{(1-q^{6n+2})^2}-\frac{q^{6n+3}}{(1-q^{6n+3})^2}\right]=\frac{q^2[q^6;q^6]_\infty^4[q,q^5;q^6]_\infty^2}{[q^2,q^4;q^6]_\infty^2[q^3;q^6]_\infty^4}. \tag{2.3}$$

$$=\frac{q^2[q^6;q^6]_\infty^4}{[q^2,q^4;q^6]_\infty^2}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}^2. \tag{2.4}$$

(iii) Taking $i = 1, j = 4$ and $k = 6$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{q^{6n+4}}{(1-q^{6n+4})^2}\right]=\frac{q[q^6;q^6]_\infty^4[q^3;q^6]_\infty^2}{[q^2,q^4;q^6]_\infty^2[q,q^5;q^6]_\infty}. \tag{2.5}$$

Dividing (2.3) by (2.5) and then using (1.5) we get

$$\frac{\displaystyle\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+2}}{(1-q^{6n+2})^2}-\frac{q^{6n+3}}{(1-q^{6n+3})^2}\right]}{\displaystyle\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{q^{6n+4}}{(1-q^{6n+4})^2}\right]}=q\frac{[q,q^5;q^6]_\infty^3}{[q^3;q^6]_\infty^6}$$

$$=q\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}^3. \tag{2.6}$$

Again comparing (2.4) and (2.6) we have

$$\left\{\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+2}}{(1-q^{6n+2})^2}-\frac{q^{6n+3}}{(1-q^{6n+3})^2}\right]\right\}\left\{\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{q^{6n+4}}{(1-q^{6n+4})^2}\right]\right\}^2$$

$$=\frac{q^4[q^6;q^6]_\infty^{12}}{[q^2,q^4;q^6]_\infty^6}. \tag{2.7}$$

(iv) Taking $i = 3, j = 4$ and $k = 8$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+3}}{(1-q^{8n+3})^2}-\frac{q^{8n+4}}{(1-q^{8n+4})^2}\right]=\frac{q^3[q^8;q^8]_\infty^4[q,q^7;q^8]_\infty^2}{[q^4;q^8]_\infty^4[q^3,q^5;q^8]_\infty^2}. \tag{2.8}$$

Making use of (1.6) in (2.8) we get

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+3}}{(1-q^{8n+3})^2}-\frac{q^{8n+4}}{(1-q^{8n+4})^2}\right]$$

$$= \frac{q^3[q^8;q^8]^4_\infty}{[q^4;q^8]^4_\infty}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+...}\right\}^2. \tag{2.9}$$

(v) Taking $i = 2, j = 5$ and $k = 8$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+2}}{(1-q^{8n+2})^2}-\frac{q^{8n+5}}{(1-q^{8n+5})^2}\right] = q^2\frac{[q^8;q^8]^4_\infty}{[q^2;q^4]^2_\infty}\frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty}. \tag{2.10}$$

Applying (1.6) in (2.10) we obtain

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+2}}{(1-q^{8n+2})^2}-\frac{q^{8n+5}}{(1-q^{8n+5})^2}\right]$$

$$= q^2\frac{[q^8;q^8]^4_\infty}{[q^2;q^4]^2_\infty}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+...}\right\}. \tag{2.11}$$

(vi) Dividing (2.9) by (2.11) we get

$$\frac{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+3}}{(1-q^{8n+3})^2}-\frac{q^{8n+4}}{(1-q^{8n+4})^2}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+2}}{(1-q^{8n+2})^2}-\frac{q^{8n+5}}{(1-q^{8n+5})^2}\right]}$$

$$= q\frac{[q^2;q^4]^2_\infty}{[q^4;q^8]^4_\infty}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+...}\right\}. \tag{2.12}$$

Applying (1.7) in (2.12) after replacing q by q^2 we get

$$\frac{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+3}}{(1-q^{8n+3})^2}-\frac{q^{8n+4}}{(1-q^{8n+4})^2}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+2}}{(1-q^{8n+2})^2}-\frac{q^{8n+5}}{(1-q^{8n+5})^2}\right]}$$

$$= q\left\{\frac{1}{1+}\frac{q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^6}{1+}\frac{q^4+q^8}{1+}\frac{q^{10}}{1+...}\right\}^2\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+...}\right\}. \tag{2.13}$$

(vii) Taking $i = 1, j = 6$ and $k = 8$ in (2.1) we find,

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+1}}{(1-q^{8n+1})^2}-\frac{q^{8n+6}}{(1-q^{8n+6})^2}\right] = q\frac{[q^8;q^8]^4_\infty}{[q^2;q^4]^2_\infty}\frac{[q^3,q^5;q^8]_\infty}{[q,q^7;q^8]_\infty}. \tag{2.14}$$

Dividing (2.10) by (2.14) and using (1.6) we get

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+2}}{(1-q^{8n+2})^2}-\frac{q^{8n+5}}{(1-q^{8n+5})^2}\right]=q\frac{[q,q^7;q^8]_\infty^2}{[q^3,q^5;q^8]_\infty^2}$$

$$=q\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+\ldots}\right\}^2. \tag{2.15}$$

Again, dividing (2.8) by (2.14) and using (1.6) and (1.7) we get

$$\frac{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+3}}{(1-q^{8n+3})^2}-\frac{q^{8n+4}}{(1-q^{8n+4})^2}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{q^{8n+1}}{(1-q^{8n+1})^2}-\frac{q^{8n+6}}{(1-q^{8n+6})^2}\right]}$$

$$=q^2\frac{[q^2;q^4]_\infty^2}{[q^4;q^8]_\infty^4}\frac{[q,q^7;q^8]_\infty^3}{[q^3,q^5;q^8]_\infty^3}$$

$$=q^2\left\{\frac{1}{1+}\frac{q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^6}{1+\ldots}\right\}^2\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+\ldots}\right\}^3. \tag{2.16}$$

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