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On Modular Identities and Evaluation of Theta-Functions

Rajnish Kant Upadhayay & *Jay Parkash Singh,
Department of Mathematics,
S.N.P.G. College, Azamgarh, (U.P.) India
*Department of Mathematics,
Deen Dayal Upadhyaya Govt. P.G. College,
Saidabad, Allahabad (U.P.) India.

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Abstract: . In this paper, making use of modular equations due to Ramanujan relations between α, β and the multiplier m have degree 3,5,7,9,13 and 25 we have established interesting P, Q identities.

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1. Introduction, Notations and Definitions

For real and complex q, $|q| < 1$, then

$$[\alpha; q]_\infty = \prod_{k=0}^{\infty}(1 - \alpha q^k),$$

where α is any complex number.
Also,

$$[a_1, a_2, a_3, ..., a_r; q]_\infty = [a_1; q]_\infty [a_2; q]_\infty ... [a_r; q]_\infty.$$

Ramanujan's defined the general theta function as,

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.1}$$

which by an appeal of Jacobi's triple product identity [Gasper and Rahman 2; App.11 (11.28)] yields,

$$f(a, b) = [ab, -a, -b; q]_\infty \tag{1.2}$$

The most important special cases of (1.1) are,

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q; -q]_\infty}{[q; -q]_\infty} = [q^2; q^2]_\infty [-q; q^2]_\infty. \tag{1.3}$$

$$\Psi(q)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{[q^2;q^2]_\infty}{[q;q^2]_\infty}, \tag{1.4}$$

$$f(-q)=\sum_{n=-\infty}^{\infty}(-)^n q^{n(3n-1)/2}=[q;q]_\infty, \tag{1.5}$$

and

$$\chi(-q)=[q;q^2]_\infty, \tag{1.6}$$

Let

$$z_r=z(r;x)={}_2F_1[1/r,(r-1)/r;1;x]$$

and

$$q_r=q_r(x)=exp\left[-rcosec\ \pi/r\frac{{}_2F_1[1/r,(r-1)/r;1;1-x]}{{}_2F_1[1/r,(r-1)/r;1;x]}\right]. \tag{1.7}$$

where r=2,3,4,5,6 and $|x|<1$.

$${}_2F_1[a,b;c;x]=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k,$$

with $(a)_k=a(a+1)(a+2)...(a+k-1);\quad a_0=1$.
Let n denote a fixed natural number and assume that

$$n\frac{{}_2F_1[1/r,(r-1)/r;1;1-\alpha]}{{}_2F_1[1/r,(r-1)/r;1;\alpha]}=\frac{{}_2F_1[1/r,(r-1)/r;1;1-\beta]}{{}_2F_1[1/r,(r-1)/r;1;\beta]} \tag{1.8}$$

where r=2,3,4 and 6. Then a modular equation of degree 'n' in the theory of elliptic function of signature 'r' is a relation between α and β induced by (1.8). We often say that β has degree n order α and $m(r)=z(r,\alpha)/z(r,\beta)$ is called multiplier.

We shall use the following modular equations due to Ramanujan in our analysis.
(i) If β and the multiplier m have degree 3, then

$$m^2=\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}. \tag{1.9}$$

$$\frac{9}{m^2}=\left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}. \tag{1.10}$$

[Andrews and Berndt 1; Entry (17.3.21) p. 391]

(ii) If β and the multiplier m have degree 5, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{1.11}$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{1.12}$$

[Andrews and Berndt 1; Entry (17.3.22) p. 391]

(iii) If β and the multiplier m have degree 7, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}. \tag{1.13}$$

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}. \tag{1.14}$$

[Andrews and Berndt 1; Entry (17.3.23) p. 391]

(iv) If β and the multiplier m have degree 9, then

$$m^{1/2} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}. \tag{1.15}$$

$$\frac{3}{m^{1/2}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}. \tag{1.16}$$

[Andrews and Berndt 1; Entry (17.3.24) p. 391]

(v) If β and the multiplier m have degree 13, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}. \tag{1.17}$$

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}. \tag{1.18}$$

[Andrews and Berndt 1; Entry (17.3.25) p. 391]

(vi) If β and the multiplier m have degree 25, then

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}. \quad (1.19)$$

$$\frac{5}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}. \quad (1.20)$$

[Andrews and Berndt 1; Entry (17.3.27) p. 391]

We shall make use of the following results due to Ramanujan,

$$f(q) = \frac{\sqrt{z}}{\sqrt[6]{z}}\left\{\frac{x(1-x)}{q}\right\}^{1/24}. \quad (1.21)$$

[Ramanujan 3; Chapter 17 entry 12 (i)]

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{x(1-x)}{q}}}. \quad (1.22)$$

[Ramanujan 3; Chapter 17 entry 12 (v)]

2. Modular Identities

In this section we establish certain modular identities,
(i) In modular equations (1.9) and (1.10), β is degree 3 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}}\left\{\frac{\alpha(1-\alpha)}{q}\right\}^{1/24}, \quad f(q^3) = \frac{\sqrt{z_3}}{\sqrt[6]{2}}\left\{\frac{\beta(1-\beta)}{q^3}\right\}^{1/24} \quad (2.1)$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^3) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\beta(1-\beta)}{q^3}}}. \quad (2.2)$$

Let us assume that

$$P = \frac{f(q)}{q^{1/12}f(q^3)} = \sqrt{m}\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \quad (2.3)$$

and

$$Q = \frac{\chi(q^3)}{q^{1/12}\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \quad (2.4)$$

Thus we have

$$\frac{P}{Q} = \sqrt{m}$$

Now, eliminating α, β and m from (1.9) and (1.10) using (2.3) and (2.4) we get the modular identity,

$$P^4 + P^8 Q^8 = P^4 Q^{12} + 9Q^4. \tag{2.5}$$

(ii) In the modular equations (1.11) and (1.12), β is degree 5 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}} \left\{ \frac{\alpha(1-\alpha)}{q} \right\}^{1/24}, \quad f(q^5) = \frac{\sqrt{z_5}}{\sqrt[6]{2}} \left\{ \frac{\beta(1-\beta)}{q^5} \right\}^{1/24}. \tag{2.6}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^5) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\beta(1-\beta)}{q^5}}}. \tag{2.7}$$

Let us assume that

$$P = \frac{f(q)}{q^{1/6} f(q^5)} = \sqrt{m} \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.8}$$

and

$$Q = \frac{\chi(q^5)}{q^{1/6} \chi(q)} = \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.9}$$

Thus we have

$$\frac{P}{Q} = \sqrt{m} \tag{2.10}$$

Now, eliminating α, β and m from (1.11) and (1.12) by making use of (2.9) and (2.10) we get the modular identity,

$$P^4 Q^4 + P^8 = P^2 Q^6 + 5Q^2. \tag{2.11}$$

(iii) In the modular equations (1.13) and (1.14), β is degree 7 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}} \left\{ \frac{\alpha(1-\alpha)}{q} \right\}^{1/24}, \quad f(q^7) = \frac{\sqrt{z_7}}{\sqrt[6]{2}} \left\{ \frac{\beta(1-\beta)}{q^7} \right\}^{1/24}. \tag{2.12}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^7) = \frac{\sqrt[6]{2}}{\sqrt[24]{\frac{\beta(1-\beta)}{q^7}}}. \tag{2.13}$$

Let us assume that

$$P = \frac{f(q)}{q^{1/4} f(q^7)} = \sqrt{m} \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.14}$$

and

$$Q = \frac{\chi(q^7)}{q^{1/4} \chi(q)} = \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.15}$$

Thus we have

$$\frac{P}{Q} = \sqrt{m} \tag{2.16}$$

Now, eliminating α, β and m from (1.13) and (1.14) by making use of (2.15) and (2.16) we get the modular identity,

$$P^8 Q^8 + P^4 + 8P^4 Q^4 = 49Q^4 + Q^{12} P^4 + 8P^4 Q^8. \tag{2.17}$$

(iv) In the modular equations (1.15) and (1.16), β is degree 9 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}} \left\{ \frac{\alpha(1-\alpha)}{q} \right\}^{1/24}, \quad f(q^9) = \frac{\sqrt{z_9}}{\sqrt[6]{2}} \left\{ \frac{\beta(1-\beta)}{q^9} \right\}^{1/24}. \tag{2.18}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^9) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\beta(1-\beta)}{q^9}}}. \tag{2.19}$$

Let us assume that

$$P = \frac{f(q)}{q^{1/3} f(q^9)} = \sqrt{m} \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.20}$$

and

$$Q = \frac{\chi(q^9)}{q^{1/3} \chi(q)} = \left\{ \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right\}^{1/24}. \tag{2.21}$$

Thus we have

$$\frac{P}{Q} = \sqrt{m} \tag{2.22}$$

Now, eliminating α, β and m from (1.13) and (1.14) by making use of (2.21) and (2.22) we get the modular identity,

$$P^2 Q^2 + P = PQ^2 + 3Q. \tag{2.23}$$

(v) In the modular equations (1.17) and (1.18), β is degree 13 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}}\left\{\frac{\alpha(1-\alpha)}{q}\right\}^{1/24}, \quad f(q^{13}) = \frac{\sqrt{z_{13}}}{\sqrt[6]{2}}\left\{\frac{\beta(1-\beta)}{q^{13}}\right\}^{1/24}. \tag{2.24}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^{13}) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\beta(1-\beta)}{q^{13}}}}. \tag{2.25}$$

Let us assume that

$$P = \frac{f(q)}{q^{1/2}f(q^{13})} = \sqrt{m}\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{2.26}$$

and

$$Q = \frac{\chi(q^{13})}{q^{1/2}\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{2.27}$$

Thus we have

$$\frac{P}{Q} = \sqrt{m} \tag{2.28}$$

Now, eliminating α, β and m from (1.17) and (1.18) by making use of (2.27) and (2.28) we get the modular identity,

$$13Q^2 + P^2Q^6 + 4P^2Q^4 = P^2 + 4Q^2 + P^4Q^4. \tag{2.29}$$

(vi) In the modular equations (1.19) and (1.20), β is degree 25 over α and m is the multiplier associated with α and β so, from (1.21) and (1.22), we have

$$f(q) = \frac{\sqrt{z_1}}{\sqrt[6]{2}}\left\{\frac{\alpha(1-\alpha)}{q}\right\}^{1/24}, \quad f(q^{25}) = \frac{\sqrt{z_{25}}}{\sqrt[6]{2}}\left\{\frac{\beta(1-\beta)}{q^{25}}\right\}^{1/24}. \tag{2.30}$$

$$\chi(q) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\alpha(1-\alpha)}{q}}}, \quad \chi(q^{25}) = \frac{\sqrt[6]{2}}{\sqrt[24]{\dfrac{\beta(1-\beta)}{q^{25}}}}. \tag{2.31}$$

Let us assume that

$$P = \frac{f(q)}{qf(q^{25})} = \sqrt{m}\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{2.32}$$

and

$$Q = \frac{\chi(q^{25})}{q\chi(q)} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/24}. \tag{2.33}$$

Thus we have

$$\frac{P}{Q} = \sqrt{m} \tag{2.34}$$

Now, eliminating α, β and m from (1.19) and (1.20) by making use of (2.33) and (2.34) we get the modular identity,

$$P^2Q^2 + P + 2PQ = 5Q + PQ^3 + 2PQ^2. \tag{2.35}$$

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