J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 7, No. 2 (2020), pp. 137-144

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

ADDENDUM TO HIERARCHIES OF PALINDROMIC SEQUENCES IN THE SYMMETRIC GROUP S_n

T. D. Krishnamachari and K. Srinivasa Rao*

Adviser (Retd.), Lucas Indian Service (Lucas TVS),
Chennai - 600002, INDIA.

E-mail : tdk@wipeindia.com

*Senior Professor (Retd.),
The Institute of Mathematical Sciences, Chennai - 600113, INDIA
& Director (Hon.), Srinivasa Ramanujan Academy of Maths Talent,
90 / 1, Second Main Road, Gandhi Nagar, Adyar, Chennai-600020, INDIA

E-mail : ksrao18@gmail.com

(**Received:** *May 10, 2020* **Accepted:** *May 19, 2020* **Published:** *Jun. 30, 2020)*

Abstract: In this note we provide important additional observations on the Hierarchies of Palindromic sequences in the Symmetric Group S_n.

Keywords and Phrases: Sequences, sets, palindromic numbers.

2010 Mathematics Subject Classification: 11A99, 11B99, 97A20, 97F30.

1. Introduction, Notations and Definitions

A palindrome is a word, phrase or a number or sequence of words that reads the same forwards or backwards. A few examples of palindromic words are:

BOB, EVE, ANNA, HANNAH, MADAM, MALAYALAM, $\cdots$

"The longest palindromic word in the Oxford English Dictionary is:

tattarrattat$\Leftrightarrow$ tat tar rat tat

coined by James Joyce in Ulysses (1922) for a knock on the door. The Guinness Book of Records gives[1] the title to 'detartrated'..., meaning 'to remove tartrates', in Chemical Engineering."

[1] From the ubiquitous Wikepedia.

The simple palindromic sentence:

ABLE WAS I ERE I SAW ELBA

is an example where the punctuation and the spacing between the words are not different. But in general, the punctuation and spacing between the words are allowed to be different, an example of which is:

A MAN, A PLAN, A CANAL: P ANAMA (read forwards) $\Leftrightarrow$
(read backwards)AMANA P : LANAC A ,NALP A ,NAM A.

or

NO X IN NIXON (forward) $\Leftrightarrow$ NOXIN NI X ON (backward)

or

A SANTA AT NASA (forward) $\Leftrightarrow$ ASAN AT ATNAS A (backward)

The following is a strange, if not bizarre and amusing construction[2]:

"Are we not pure? 'No sir!' Panama's moody Noriega brags.
'It is garbage!' Irony dooms a man; a prisoner up to new era."

– of course, ignoring punctuations, spacings, just for the heck of it!

In the realm of numbers, if i, j, k, ℓ, m, n are indices for integers, then an example for a 12-digit, non-trivial, palindromic number is:

$$ijk\ell mnnm\ell lkji, \qquad \text{for} \qquad i \neq j \neq k \neq \ell \neq m \neq n. \tag{1}$$

In the realm of recreational arithmetic, the palindromic prime numbers are:

$$11, 101, 111, 121, 131, \cdots 191, 1001, 1111, 1221, \cdots 1991, \cdots, to\ \infty \tag{2}$$

Similarly, one can list the palindromic square numbers as the sequence:

$$121, 484, 676, \cdots \tag{3}$$

The following well-known simple multiplication table:

$$\begin{aligned}
1 \times 11 &= 11\\
11 \times 11 &= 121\\
111 \times 111 &= 12321\\
1111 \times 1111 &= 1234321\\
11111 \times 11111 &= 123454321\\
111111 \times 111111 &= 12345654321\\
1111111 \times 1111111 &= 1234567654321\\
11111111 \times 11111111 &= 123456787654321\\
111111111 \times 111111111 &= 12345678987654321
\end{aligned} \tag{4}$$

shows an interesting way to generate a pyramid of 2,3,5,7,9,11,13,15,17-digit palindromic integers. Besides 11^2, one can observe that 11^3 and 11^4 also give rise to 3,4,5-digit palindromic numbers:

$$\begin{aligned} 11^2 = 11 \times 11 &= 121 \\ 11^3 = 11 \times 11 \times 11 &= 1331 \\ 11^4 = 11 \times 11 \times 11 \times 11 &= 14641 \end{aligned} \tag{5}$$

The Pascal triangle gives rise to one of the most interesting number patterns, due to the coefficients in the Binomial expansion:

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \; a^r \; b^{n-r}, \tag{6}$$

where the Binomial coefficients are defined as:

$${}^nC_r = \frac{n!}{r!\,(n-r)!}, \quad \text{for} \quad n \quad \text{integer and} \quad r \leq n. \tag{7}$$

When these coefficients are arranged in their increasing order one obtains an interesting pyramidal arrangement of coefficients on the l.h.s and their corresponding numerical values are on the r.h.s of the equation:

$$\begin{array}{c} {}^0C_0 \\ {}^1C_0 \quad {}^1C_1 \\ {}^2C_0 \quad {}^2C_1 \quad {}^2C_2 \\ {}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3 \\ {}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4 \\ {}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5 \\ {}^6C_0 \quad {}^6C_1 \quad {}^6C_2 \quad {}^6C_3 \quad {}^6C_4 \quad {}^6C_5 \quad {}^6C_6 \\ {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3 \quad {}^7C_4 \quad {}^7C_5 \quad {}^7C_6 \quad {}^7C_7 \end{array} \quad = \quad \begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \\ 1\ 6\ 15\ 20\ 15\ 6\ 1 \\ 1\ 7\ 21\ 35\ 35\ 21\ 7\ 1 \end{array} \tag{8}$$

The binomial coefficients give rise to palindromic sequences of numbers (in the rows on the r.h.s), due to their symmetry property:

$${}^nC_r \; = \; {}^nC_{n-r}. \tag{9}$$

The sum of the binomial coefficients, for a given n is:

$$\sum_{r=0}^{n} {}^nC_r \; = \; 2^{n-1}. \tag{10}$$

The r.h.s is called the Pascal triangle.

Note also that if all the numbers in the triangle are 1:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array} \tag{11}$$

then the successive additions of numbers in the rows give rise to the triangular number sequence:

$$1, 3, 6, 10, 15, 21, 28, ..., n(n+1)/2. \tag{12}$$

In a recent publication of Srinivasa Rao and Pankaj Pundir[1], the consequences of assigning a place value to Permutations, enabled the permutations of the elements of the group S_n to be ordered and surprisingly, the differences between successive ordered elements gave rise to Palindromic sequences of $n! - 1$ elements. The properties of these sequences were discussed in the publication referred to.

Here, we observe that the concept of palidromicity can be applied to the Natural numbers in any system of numeration, such as binary, octal, hexadecimal systems. Consider a number $n > 0$, with $k+1$ digits a_i and base $b \geq 2$, in standard notation, as:

$$n = \sum_{i=0}^{k} a_i b^i, \quad \text{with} \quad 0 \leq a_i < b, \quad \forall \quad i, \quad \text{and} \quad a_k \neq 0. \tag{13}$$

Then n is palindromic iff

$$a_i = a_{k-i}, \qquad \forall \quad i. \tag{14}$$

The meaning of palindromic numbers in the context of genetics is different from that used for words, sentences, numbers and number sequences. For instance, palindromic sequences play an important role in Molecular Biology.

2. Main Results

Definition 2.1. *Let*

$$f_B(\sigma) = \sum_{i=1}^{n} \sigma_i \ B^{n-i}, \quad \text{for some} \quad B \ \varepsilon \ N, \qquad B > 2, \tag{15}$$

where f_B is a place-valued function.

The definition enables arithmetic operations to be made on the permutations. Differences between adjacent place-value ordered permutations lead to a sequence of numbers.

Theorem 2.2. *The palindromic sequence of numbers is the same as that for 1234, also for the following 4-digit numbers: 0123, 2345, 3456, 4567, 5678* and *6789.*

Proof. For example, write down explicitly in increasing order the 24 numbers from 3456 up to 6543:

$$3456, 3465, 3546, 3564, 3645, 3654, 4356, 4365, 4536, 4563, 5346, 5364,$$

$$5436, 5463, 5634, 5643, 6534, 5643, 6345, 6354, 6435, 6453, 6534, 6543 \tag{16}$$

The 23 differences of successive numbers will be, the sequence:

$$9, 81, 18, 81, 9, 702, 9, 171, 27, 72, 18, 693,$$

$$18, 72, 27, 171, 9, 702, 9, 81, 18, 81, 9 \tag{17}$$

This is the same as the palindromic sequence of length 23 obtained for the 24-permutations of 1234, in ref. 1.

Since each one of these numbers is divisible by 9, it follows that the irreducible palindromic sequence is:

$$1, 9, 2, 9, 1, 78, 1, 19, 3, 8, 2, 77, 2, 8, 3, 19, 1, 78, 1, 9, 2, 9, 1. \tag{18}$$

A similar proof is true for: 0123, 2345, 4567, 5678, 6789.

Theorem 2.3. *Only an ordered sequence of consecutive integers give rise to Palindromic sequences.*

Proof. The proof by contradiction is simple and straightforward.

(i) First, consider a three digit number such as 137 and write down the ordered sequence of permutations of 137 as:

$$137, 173, 317, 371, 713, 731 \tag{19}$$

The differences between consecutive numbers is:

$$36, 144, 54, 342, 18 \qquad \Leftrightarrow \qquad 2, 8, 3, 19, 1 \tag{20}$$

The reducible and irreducible sequences are not palindromic!

(ii) The renowned 4-digit Kaprekar number 6174, is obtained from a given four digit number, whose four digits are distinctly different, with no repetitions. Take for example the number 9218:

Step 1. Write down its permutation with its digits in decreasing order: 9821.

Step 2. Subtract from it the number obtained by reversing its digits:

$$9821 - 1289 = 8532 \tag{21}$$

Step 3. Reverse the digits of the obtained number and subtract it from the number obtained in Step 2:

$$8532 - 2358 = 6174 = 2 \times 3^2 \times 7^3, \tag{22}$$

the resultant number (after a few iterations) will always be 6174, which is called the Kaprekar Number, its discoverer.

For the Kaprekar number, the 24 permutations in ascending order are:

$$1467, 1476, 1647, 1674, 1746, 1764, 4167, 4176, 4617, 4671, 4716, 4761,$$

$$6147, 6174, 6417, 6471, 6714, 6741, 7146, 7164, 7416, 7461, 7614, 7641. \tag{23}$$

The 23-differences of the successive numbers are:

$$9, 171, 27, 72, 18, 2403, 9, 441, 54, 45, 45, 1386,$$

$$27, 243, 54, 243, 27, 405, 18, 252, 45, 153, 27. \tag{24}$$

Equivalently, the irreducible sequence of differences is:

$$1, 19, 3, 8, 2, 267, 1, 38, 6, 5, 5, 154,$$

$$3, 17, 6, 17, 3, 47, 2, 38, 5, 17, 3. \tag{25}$$

The 4-digit Kaprekar number, 6174, does not give rise to a palindromic sequence, because the four digits of this number are not consecutive digits! (iii) Another famous four digit number is the Ramanujan (Taxi cab) Number 1729. In the ordered sequence of permutations of the four digits, which starts with 1279 and ends with 9721, 1729 is the third in the sequence:

$$1279, 1297, 1729, 1792, 1927, 1972, 2179, 2197, 1719, 2791, 2917, 2971,$$

$$7129, 7192, 7219, 7291, 7912, 7921, 9127, 9172, 9217, 9271, 9712, 9721. \tag{26}$$

The corresponding sequence of 23-differences are:

$$18, 432, 63, 135, 45, 207, 18, 522, 72, 126, 54, 4158,$$
$$63, 27, 72, 621, 9, 1206, 45, 45, 54, 441, 9. \tag{27}$$

Since all the numbers have 9 as a factor in them, the corresponding irreducible sequence of 23-numbers are:

$$2, 37, 7, 15, 5, 23, 2, 58, 8, 14, 6, 462, 7, 3, 8, 69, 1, 134, 5, 5, 6, 49, 1. \tag{28}$$

This sequence is also not palindromic.

From the above specific examples, it is possible to arrive at the following:

Corollary 2.4. *The sequence of differences which arise from the ordered permutations of a given 4-digit number is not palindromic if the digits are not consecutive numbers.*

Therefore, the necessary and sufficient condition for generating a palindromic sequence of numbers from a given 4-digit number is that the digits in the 4-digit number must be consecutive numbers. Or, in other words, the only four-digit numbers which will give rise to palindromic sequences of numbers are:

$$0123, 1234, 2345, 3456, 4567, 5678, 6789. \tag{29}$$

In all these seven instances, the sum of the sequence of 23 reducible numbers, Eq. (17), is 3087 and the corresponding sum of the sequence of irreducible numbers, Eq. (18), is 343 (a palindromic number!).

These observations supplement the results presented, in this journal, earlier by Srinivasa Rao and Pankaj Pundir[2]

3. Acknowledgements

The authors wish to place on record their thanks for a careful reading of the manuscript by Professor S.N. Singh.

References

[1] K. Srinivasa Rao and Pankaj Pundir, Hierarchies of Palindromic Sequences in the Symmetric Group S_n, J. Ramanujan Soc. of Math. and Math. Sci., Vol. 6, No. 1, (2017) 53 - 62.

[2] This quotation is from the ubiquitous Wikepedia.

[2] K. Srinivasa Rao and Pankaj Pundir, *Hierarchies of Palindromic Sequences in the Symmetric Group S_n*, J. Ramanujan Soc. of Math. and Math. Sci., Vol. 6, No. 1, (2017) 53 - 62.