J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 7, No. 2 (2020), pp. 99-108

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

ON THE OUTLINES OF PLANE CURVES OF THE FORM $(ax)^\alpha + (by)^\alpha = r^\alpha$ WITH $\alpha > 0$

Norihiro Someyama

Shin-yo-ji Buddhist Temple,
5-44-4 Minamisenju, Arakawa-ku, Tokyo 116-0003 JAPAN

E-mail : philomatics@outlook.jp
ORCID iD: https://orcid.org/0000-0001-7579-5352

(**Received:** *Mar. 03, 2020* **Accepted:** *Apr. 08, 2020* **Published:** *Jun. 30, 2020)*

Abstract: We consider plane curves of the form $(ax)^\alpha + (by)^\alpha = r^\alpha$ defined on the first quadrant of $\mathbb{R}^2$, where $\alpha > 0$ and $a, b, r > 0$. We summarize the outlines of them by using elementary differential calculus. We will in this note understand that they are classified into three types of curves, convex, straight and concave, depending on α.

Keywords and Phrases: Plane curve, implicit function, orthogonal representation, polar representation, differential calculus, area, integral.

2010 Mathematics Subject Classification: 14H50, 26B10.

1. Introduction

There are many famous plane curves in mathematics. We take up, in this note, plane curves represented by implicit functions of the form

$$F(x, y) = x^\alpha + y^\alpha - r^\alpha = 0$$

where $\alpha, r > 0$. They include the famous plane curves as follows:

- If $\alpha = 2$, the curve $x^2 + y^2 - r^2 = 0$ is the circle that its center is the origin and the length of its radius is r. (See Figure 1 for the outline.)

- If $\alpha = 1$, the curve $x + y - r = 0$ is the straight line whose y-intercept is r. (See Figure 2 for the outline.)

- If $\alpha = 2/3$, the curve $x^{2/3} + y^{2/3} - r^{2/3} = 0$ is the asteroid whose x and y-intercept are both r. (See Figure 3 for the outline.)

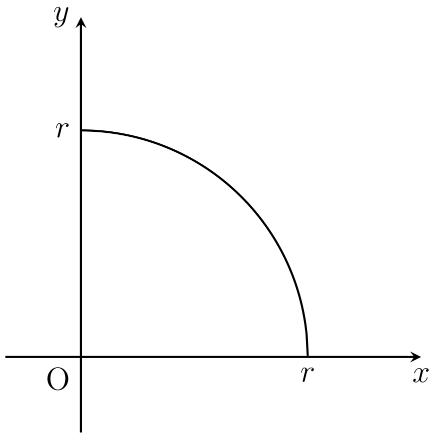

Figure 1: $\alpha = 2$; Circle

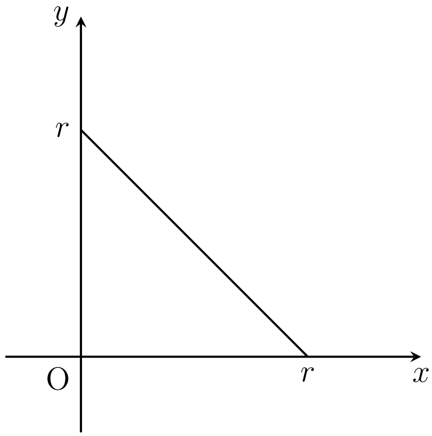

Figure 2: $\alpha = 1$; Straight line

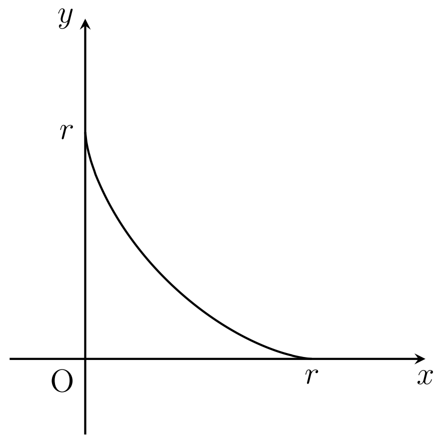

Figure 3: $\alpha = 2/3$; Asteroid

As we can see from Figures 1-3, it is expected that, for $C : x^\alpha + y^\alpha = r^\alpha$, (i) if $\alpha > 1$, then C is a convex curve; (ii) if $\alpha = 1$, then C is a straight line; (iii) if $0 < \alpha < 1$, then C is a concave curve. Let us prove that which is more generalized in the next section.

2. Orthogonal Representation of Plane Curves

Theorem 2.1. *We consider a curve*

$$C_1 : (ax)^\alpha + (by)^\alpha = r^\alpha \tag{2.1}$$

on $x \geq 0$ and $y \geq 0$, where α is a positive real number and a, b, r positive constants. Then, the following facts hold:

(i) If $\alpha > 1$, then C_1 is a convex curve (Figure 4);

(ii) If $\alpha = 1$, then C_1 is a straight line (Figure 5);

(iii) If $0 < \alpha < 1$, then C_1 is a concave curve (Figure 6).

Proof. Remark that $0 \leq x \leq r/a$ in any case, since

$$r^\alpha = (ax)^\alpha + (by)^\alpha \geq (ax)^\alpha$$

for all $\alpha > 0$. The case (ii) is obvious, so let us prove the cases (i) and (iii). We consider a function on $[0, r/a]$:

$$F(x) := y = \frac{1}{b}\left\{-(ax)^\alpha + r^\alpha\right\}^{1/\alpha}.$$

We notice that $F(x) > 0$ for all $x \in (0, r/a)$. Then, we have

$$F'(x) = -\frac{a^\alpha}{b} x^{\alpha-1} \{-(ax)^\alpha + r^\alpha\}^{1/\alpha-1},$$

$$F''(x) = -\frac{a^\alpha}{b}(\alpha-1)x^{\alpha-1} \{-(ax)^\alpha + r^\alpha\}^{1/\alpha-2} \{-(ax)^\alpha + r^\alpha + a^\alpha x^{\alpha-1}\}.$$

Since $-(ax)^\alpha + r^\alpha + a^\alpha x^{\alpha-1} > 0$ on $(0, r/a)$, we obtain the following tables on the increase and decrease of F and the outlines of C_1:

(i) In case of $\alpha > 1$:

x	0	$\cdots$	r/a
$F'(x)$	0	$-$	0
$F''(x)$	0	$-$	0
$F(x)$	r/b	$\frown$	0

This represents that C_1 is the convex curve on $[0, r/a]$.

(iii) In case of $\alpha < 1$:

x	0	$\cdots$	r/a
$F'(x)$	0	$-$	0
$F''(x)$	0	$+$	0
$F(x)$	r/b	$\smile$	0

This represents that C_1 is the concave curve on $[0, r/a]$.

This completes the proof.

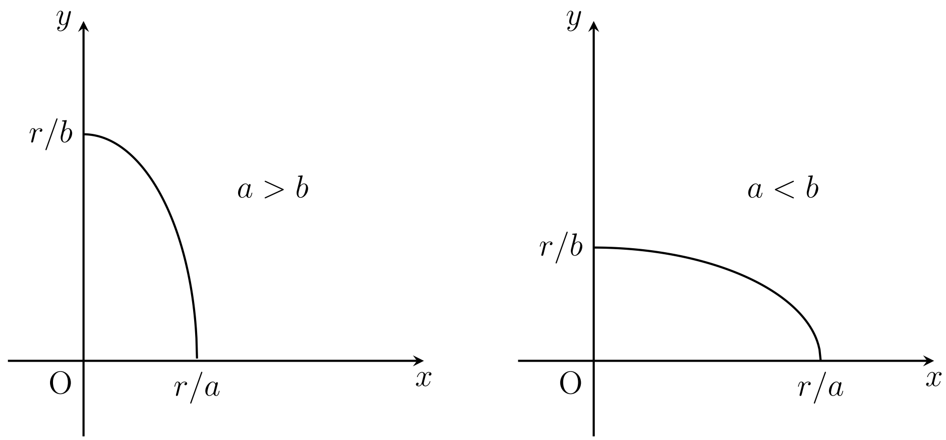

Figure 4: The case of $\alpha > 1$

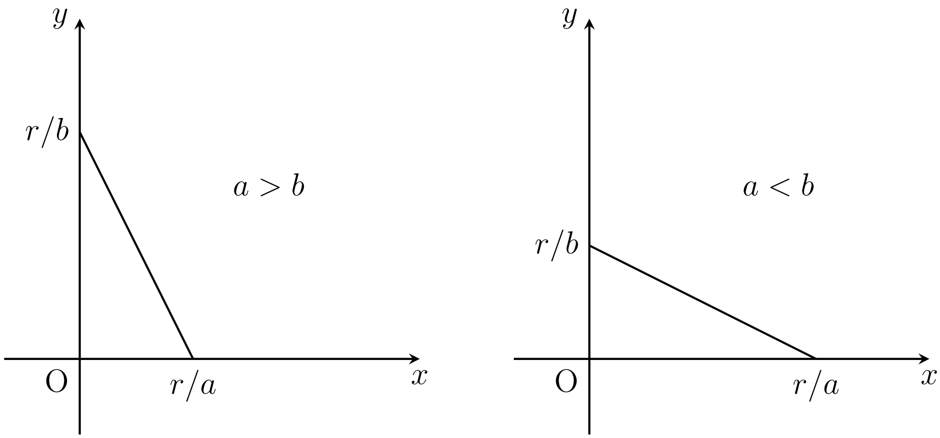

Figure 5: The case of $\alpha = 1$

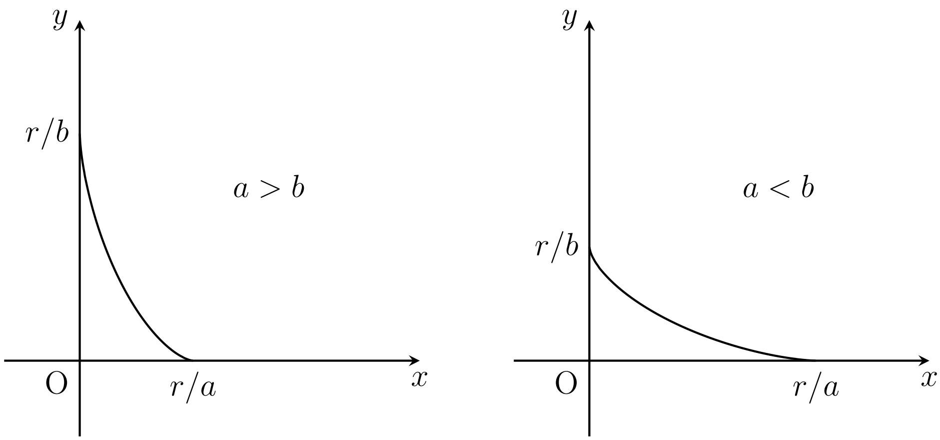

Figure 6: The case of $0 < \alpha < 1$

Like this, we can draw the graphs of curves of the form C_1. It is interesting to know areas of the graphs as the information of curves. We investigate the formula of the area of the hypograph of C_1.

Proposition 2.2. *The area* $\mathrm{A}(C_1; [0, r/a])$ *of the hypograph of* C_1*, (2.1), is given by*

$$\mathrm{A}(C_1; [0, r/a]) = \frac{r^2}{ab} \int_0^{\pi/2} (1 - \cos^\alpha \theta)^{1/\alpha} \sin\theta \, d\theta. \tag{2.2}$$

Proof. Recall that C_1 is rewritten as $y = (1/b)\{-(ax)^\alpha + r^\alpha\}^{1/\alpha} \geq 0$ if $x, y \geq 0$.

We shall calculate the integral

$$\mathrm{A}(C_1;[0,r/a]) = \int_0^{r/a} \frac{1}{b}\left\{-(ax)^\alpha + r^\alpha\right\}^{1/\alpha} dx,$$

but we consider a change of variables $x \mapsto (r/a)\cos\theta$. Then, it follows that

$$\frac{dx}{d\theta} = -\frac{r}{a}\sin\theta \qquad \begin{array}{c||ccc} x & 0 & \to & r/a \\ \hline \theta & \pi/2 & \to & 0 \end{array}$$

Hence, we have obtained the desired result by organizing.

Example 2.1. If $a = b = 1$ and $\alpha = 2$, C_1 is the quadrant whose center is the origin and whose length of its radius is r. Hence the area of the hypograph of C_1 is $\pi r^2/4$ since it is a quarter of the circle whose length of its radius is r, but we have the same result from (2.2):

$$\begin{aligned} \mathrm{A}(C_1;[0,r]) &= r^2 \int_0^{\pi/2} (1-\cos^2\theta)^{1/2} \sin\theta\, d\theta \\ &= r^2 \int_0^{\pi/2} \sin^2\theta\, d\theta \\ &= \frac{\pi r^2}{4}. \end{aligned}$$

3. Polar Representation of Plane Curves

Plane coordinate systems have the polar coordinate $(x, y) \mapsto (r\cos\theta, r\sin\theta)$ with $r > 0$ and $0 \le \theta < 2\pi$ in addition to the orthogonal coordinate. We mention the outline of a curve represented by the polar coordinate.

Theorem 3.1. *We consider a curve*

$$C_2 : \begin{cases} x = r\cos^k\theta, \\ y = r\sin^k\theta \end{cases} \tag{3.1}$$

for $\theta \in [0, \pi/2]$, where k is a positive real number and r a positive constant. Then, the following facts hold:

(i) If $k > 2$, then C_2 is a concave curve;

(ii) If $k = 2$, then C_2 is a straight line;

(iii) If $0 < k < 2$, then C_2 is a convex curve.

Proof. C_2 can be represented as the orthogonal form

$$x^{2/k} + y^{2/k} = r^{2/k}, \tag{3.2}$$

so there is no difference between the proof of this theorem and that of Theorem 2.1. In fact, this situation is the case that $a = b = 1$ and $\alpha = 2/k$ in Theorem 2.1.

Example 3.1. On the hand, setting $k = 3$ in (3.1), we have

$$x^{2/3} + y^{2/3} = r^{2/3}$$

by (3.2). This represents the asteroid and thus C is the concave curve. On the other hand, setting $k = 1$ in (3.1), we have

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

since $(x, y) = (r\cos\theta, r\sin\theta)$. This represents the circle and thus C is the convex curve.

Example 3.2. If $a = b = 1$ and $\alpha = 1/2$ in Theorem 3.1, C_1 represents the parabola. C_1 is actually the parabola obtained by rotating

$$y = \frac{1}{\sqrt{2}r}x^2 + \frac{r}{2\sqrt{2}}$$

by $-\pi/4$ around the origin. This thing also declares that the parabola which has the axis $y = x$ represents by the polar coordinate

$$\begin{cases} x = r\cos^4\theta, \\ y = r\sin^4\theta \end{cases}$$

since $\sqrt{x} + \sqrt{y} = \sqrt{r}$ is the curve in the case of $k = 4$ in Theorem 3.1.

4. Symmetry of Curves of the Form C_1 or C_2

We finally investigate the symmetry in the real II-IV quadrants of the form C_1 or C_2 defined in the real I-quadrant. We discuss only C_1 hereafter, because C_2 is the special curve of C_1. This note introduces the following terminology for convenience.

Definition 4.1. *Let C be a curve defined in the real I-quadrant. We say that C is kaleidoscope-type symmetric, if C is symmetric with respect to the x-axis, y-axis and origin.*

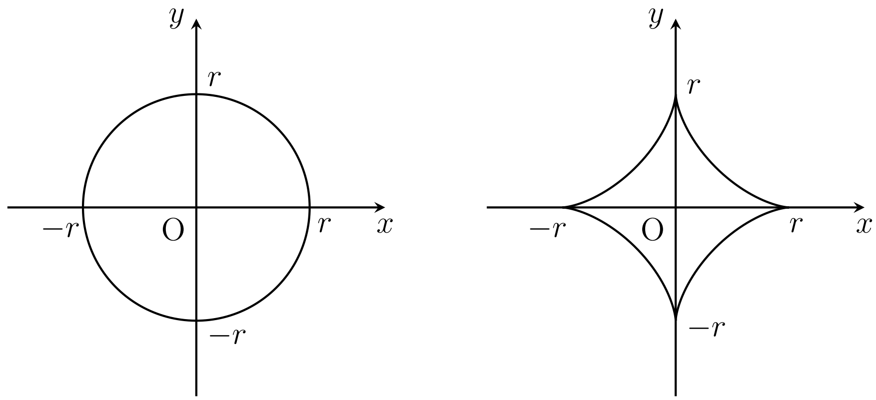

Figure 7: Circle (Left), Asteroid (Right)

The circle ($\alpha = 2$) and the asteroid ($\alpha = 2/3$) are kaleidoscope-type symmetric (See Figure 7.), but the straight line ($\alpha = 1$) is not so. We can thus see that not every curve C_1 is kaleidoscope-type symmetric. So what is the value of $\alpha > 0$ and what makes kaleidoscope-type symmetric?

Proposition 4.2. *C_1 is kaleidoscope-type symmetric if and only if α is not 1 and is a real number factored by 2: $\alpha \neq 1$ and there exists $\widetilde{\alpha} \in \mathbb{R}$ such that $\alpha = 2\widetilde{\alpha}$.*
Proof. It is trivial that $\alpha \neq 1$. We put

$$F_\alpha(x, y) := (ax)^\alpha + (by)^\alpha - r^\alpha = 0.$$

For the proof, it is sufficient to verify that $\alpha \in (0, 1) \cup (1, \infty)$ such that

$$F_\alpha(x, -y) = F_\alpha(x, y), \quad F_\alpha(-x, y) = F_\alpha(x, y) \quad \text{and} \quad F_\alpha(-x, -y) = F_\alpha(x, y) \tag{4.1}$$

is represented as $\alpha = 2\widetilde{\alpha}$ where $\widetilde{\alpha} \in \mathbb{R}$, since it is easy to see the converse. We write $\mathbb{Q}_+$ for the set of positive rational numbers. To beginning with, we prove (4.1) if $\alpha \in \mathbb{Q}_+ \setminus \{1\}$. For that, we should prove that the necessary and sufficient condition for $(-1)^m = 1$, $m \in \mathbb{N}$, is that m is even. It is however obvious by virtue of the theory of complex numbers. Now recall that, for any real number, there exists a certain sequence which converges to that real number. Then, we consider a sequence $\{q_n := 2\widetilde{q}_n\} \subset \mathbb{Q}_+ \setminus \{1\}$ such that

$$q_n = 2\widetilde{q}_n \xrightarrow[n\to\infty]{} 2\widetilde{\alpha} = \alpha \in (0, 1) \cup (1, \infty).$$

Since it holds that

$$\lim_{n\to\infty} F_{q_n}(x,y) = F_\alpha(x,y)$$

because of the continuity of exponential functions, three expressions in (4.1) hold. Hence, this completes the proof.

5. Comments

This note gives us the results (Theorem 2.1, 3.1) on the outlines of plane curves of the forms C_1 and C_2 in the first quadrant of $\mathbb{R}^2$. Moreover, it mentions the integral-formula (Proposition 2.2) of the area of the closed region made by the plane curve, x-axis and y-axis. Finally, the kaleidoscope-type symmetry (Proposition 4.1) of C_1 and C_2 was revealed.

One of purposes to know outlines of curves is to find areas (or volumes) of the graphs. In case of C_1, we can fortunately calculate the area of the hypograph of it by integrals without drawing of the graphs, because

$$y = \frac{1}{b}\left\{-(ax)^\alpha + r^\alpha\right\}^{1/\alpha} \geq 0$$

on $[0, r/a]$. (It is also so for C_2.) That is however generally rare, and it is often necessary to draw the graphs. We should thus know the outlines of several fundamental curves in order to calculate those areas. Furthermore, it is useful to know the kaleidoscope-type symmetry of the graphs when we draw them.

We hope that fundamental results obtained in this note will be common sense not only among scholars but also among students studying mathematics.

References

[1] Brieskorn, E. and Knörrer, H. (trans. Stillwell, J.), Plane Algebraic Curves, Birkhaeuser (2012).

[2] Friedman, A., Advanced Calculus, Dover Publications (2007).