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SOME APPLICATIONS OF EXTENSION k-GAMMA, k-BETA FUNCTIONS AND k-BETA DISTRIBUTION

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Abstract: By Applying extension of k-Gamma and k-Beta functions we derive the interesting results with help k- Gamma and k-Beta functions involving elementary functions. Using extension of k-Beta distribution we will discuss the maximum likelihood estimators, central moments and some properties based on expectation. For real life application will be computed the hazard rate function, mean residue life function and entropy.

Keywords and Phrases: k-Gamma function, extension k-gamma function, k-Beta function, k-beta function central moment, Hazard Rate Function.

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1. Introduction

Many researchers are developing the fractional calculus theory and its applications with the help some special functions like Gamma, Beta, extended Gamma, extended Beta , k-Gamma, k-Beta, extended k-Gamma and extended k-Beta functions (Mathai et al [9] , Chaudhry et al [3], Daiz et al [4] Mubeen et al [12]) . Many problems of science and engineering can be evaluated by help these functions (Bapna at al [1], Krishanamoorthy [8]) Presently more researchers are working on k-Beta function and extension of k-beta function and these functions have more applications in mathematical analysis and pure statistics. The aim of this paper is to develop extended k-Beta and extended k-Gamma functions involving fractional calculus theory and statistical distribution theory. We are giving the following basic definitions based on our main results related to k-Beta function and extension of k-beta function.

2. Gamma and Beta functions

The Gamma function Γz is introduced by Euler (1707-1783). Many forms of Gamma function are given by Euler, Carl Friendrich Gauss, Karl Weierstrass and Egan which are useful in various scientific and real life applications(Mathai et al. [9], Bapna et al.[1], Vyas [16],and Walac [17]) The Gamma is defined by the formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{(z)_n} \tag{1}$$

Its integral representation is also given as

$$\Gamma z = \int_0^\infty x^{z-1} e^{-x} dx; \qquad \mathrm{Re}(z) > 0 \tag{2}$$

And

$$\Gamma(z+1) = z\Gamma z \tag{3}$$

Beta function $B(m,n)$ is defined as

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \;\; ; \quad \mathrm{Re}(m) > 0, R(n) > 0 \tag{4}$$

The Beta function in term of Gamma function is given by

$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} \tag{5}$$

Let n be positive integer. Then Pochhammer symbol $(a)_n$ defined as

$$(a)_n = a(a+1)(a+2)(a+3)\ldots(a+(n-1)) \tag{6}$$

$$(a)_0 = 1 \tag{7}$$

1.2. k-Gamma and k-Beta functions

In 2007 Diaz and Pariguan (see Diaz et al.[4], Diaz et al.[5]) have introduced the following Pochhammer k-symbol and k-Gamma function .The Pochhammer k-symbol is $(a)_{n,k}$ defined as

$$(a)_{n,k} = \begin{Bmatrix} a\left(a+k\right)\left(a+2k\right)\ldots\left(a+\left(n-1\right)k\right) & ;n \geq 1, k > 0 \\ 1 & ;n = 0 \end{Bmatrix} \tag{8}$$

For $k > 0 Re(z) > 0$ the k-Gamma function is defined as

$$\Gamma_k\left(z\right) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{\left(z\right)n,k} \tag{9}$$

$$\left(a\right)_{n,k} = \frac{\Gamma_k\left(z+nk\right)}{\Gamma_k\left(z\right)} \tag{10}$$

And its integral form as

$$\Gamma_k\left(z\right) = \int\limits_0^\infty x^{z-1}e^{-\frac{x^k}{k}}dx \tag{11}$$

$$\Gamma_k\left(z+k\right) = z\Gamma_k\left(z\right) \tag{12}$$

The k- beta function for $k > 0$ and $\mathrm{Re}\left(m\right) > 0, \mathrm{Re}\left(n\right) > 0$ is given by

$$B_k\left(m,n\right) = \int\limits_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}dx \tag{13}$$

The relation between k-Beta and k-Gamma function are given as

$$B_k\left(m,n\right) = \frac{\Gamma_k\left(m\right)\Gamma_k\left(n\right)}{\Gamma_k\left(m+n\right)} \tag{14}$$

Many researchers (see more detail Kokologiannak et al [6], Kokologiannak et al [7], Mubeen et al. [11], Rehman et al [15], Merovci [14] and Wang [18]) have investigated some properties of k-Gamma and k-Beta functions.

1.3. Extension of Gamma and Beta functions

Extension of gamma function $\Gamma_b\left(z\right)$ is investigated by Chaudhry and Zubair [3]. It is defined as

$$\Gamma_b\left(z\right) = \int\limits_0^\infty x^{z-1}e^{-x-bx^{-1}}dx; \quad \mathrm{Re}\left(z\right) > 0, b \geq 0 \tag{15}$$

If $b = 0$ then (15) tend to (2) and extension of Beta function is given as

$$B\left(m,n;b\right)=\int\limits_{0}^{1}t^{m}(1-t)^{n}e^{-\frac{b}{t(1-t)}}dt \tag{16}$$

Where $\mathrm{Re}\left(m\right)>0,\mathrm{Re}\left(n\right)>0,\mathrm{Re}\left(b\right)>0$ If $b=0$ then (16) tend to (4) (see Chaudhry et al. [2], Chaudhry et al [3]).

1.4. Extension of k-Gamma and k-Beta functions

In 2016, Mubeen et al. have introduced the following extension of k-Gamma function

$$\Gamma_{b,k}\left(z\right)=\int\limits_{0}^{\infty}x^{z-1}e^{-\frac{x^{k}}{k}-\frac{b^{k}x^{-k}}{k}}dx \tag{17}$$

Where $k>0,\mathrm{Re}\left(m\right)>0,\mathrm{Re}\left(n\right),\mathrm{Re}\left(b\right)>0$
When (i) put $b=0$, (17) tend to (11) (ii) put $k=1$, (17) tend to (15) (iii) put both $k=1$ and $b=0$, (17) tend to (2)
The extension of k-Beta function of two variables m and n is denoted

$$B_{k}\left(m,n;b\right)=\frac{1}{k}\int\limits_{0}^{1}x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^{k}}{kx(1-x)}}dx \tag{18}$$

Where $\mathrm{Re}\left(m\right)>0,\mathrm{Re}\left(n\right)>0,k>0,\mathrm{Re}\left(b\right)>0$
If (i) put $k=1$ (18) tend to (16) (ii) put $b=0$ (18) tend to (13) (iii) put $k=1$ and b=0 (18) tend (4)
Trigonometry representation of extended k-Beta function is given as (Mubeen et al. [11])

$$B_{k}\left(m,n;b\right)=\frac{2}{k}\int\limits_{0}^{\frac{\pi}{2}}\left(\cos\theta\right)^{\frac{2m}{k}-1}\left(\sin\theta\right)^{\frac{2n}{k}-1}e^{-\frac{b^{k}}{k}\sec^{2}\theta\cos ec^{2}\theta}d\theta \tag{19}$$

Integral representation of extended k- Beta function is given as (Mubeen et al. [11])

$$\int\limits_{0}^{\infty}b^{s-1}B_{k}\left(m,n;b\right)dx=\Gamma_{k}\left(s\right)B_{k}\left(m+s,n+s\right) \tag{20}$$

Let $\Gamma_{b,k}(m)$ denotes extended k-Gamma function then (Mubeen et al. [11])

$$\Gamma_{b,k}(m)\,\Gamma_{b,k}(n)=\frac{2}{k}\int_{0}^{\infty}r^{2\frac{m+n}{k}-1}e^{-\frac{r^2}{k}}\left(\frac{2}{k}\int_{0}^{\frac{\pi}{2}}(\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1}e^{-\frac{b^k}{kr^2\sin^2\theta\cos^2\theta}}d\theta\right)dr \tag{21}$$

2. Main result some properties of extended k-beta function

Theorem 2.1. *For* $k>0, \operatorname{Re}(m)>0, \operatorname{Re}(n)>0, \operatorname{Re}(r)>0$ *and* $\operatorname{Re}(s)>0$ *then following integral representation holds true*

$$(i)\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}(\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)\,B_k(m+s,n+s) \tag{22}$$

$$(ii)\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}(\cos\theta)^{\frac{2m}{k}-1}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)\,B_k\left(m+s,s+\frac{k}{2}\right) \tag{23}$$

$$(iii)\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}(\sin\theta)^{\frac{2n}{k}-1}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)\,B_k\left(s+\frac{k}{2},s+n\right) \tag{24}$$

$$(iv)\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)\,B_k\left(s+\frac{k}{2},s+\frac{k}{2}\right) \tag{25}$$

$$(v)\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}(\cos\theta)^{p}(\sin\theta)^{q}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)B_k\left(s+\frac{k(p+1)}{2},s+\frac{k(q+1)}{2}\right) \tag{26}$$

Proof. (i) we have by (20) $\int_{0}^{\infty}b^{s-1}B_k(m,n;b)\,dx=\Gamma_k(s)B_k(m+s,n+s)$ Using in L.H.S. of equation (19) and replace $b=r$

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{\infty}r^{s-1}(\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1}e^{-\frac{b^k}{k}\sec^2\theta\cos ec^2\theta}d\theta dr=\frac{k}{2}\Gamma_k(s)\,B_k(m+s,n+s)$$

Which is completes proof of equation (22)
(ii)When in equation (22) put $n = \frac{k}{2}$ then equation (23) will be
(iii) When in equation (22) put $m = \frac{k}{2}$ then equation (24) will be
(iv)When in equation (22) put $m = \frac{k}{2}$ and $n = \frac{k}{2}$ then equation (25) will be
(v)When in equation (22) put $\frac{2m}{k} - 1 = p$ and $\frac{2n}{k} - 1 = q$ then equation (26) will be

Theorem 2.2. *Let* $k > 0; \mathrm{Re}\,(m) > 0, \mathrm{Re}\,(n) > 0, \mathrm{Re}\,(r) > 0$ *and* $\mathrm{Re}\,(b) > 0$ *then extended k-Gamma, k- Gamma and k-Beta function have relation*

$$\Gamma_{b,k}\,(m)\,\Gamma_{b,k}\,(n) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{b^k}{k}\right)^s B_k\,(m - sk, n - sk)\,\Gamma_k\,(m + n - sk)$$

Where

$$\mathrm{Re}\,(m - sk) > 0, \mathrm{Re}\,(n - sk) > 0 \tag{27}$$

Proof. Using the equation (21)

$$\Gamma_{b,k}\,(m)\,\Gamma_{b,k}\,(n) = \frac{2}{k}\int_0^{\infty} r^{2\frac{m+n}{k}-1} e^{-\frac{r^2}{k}} \left(\frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2m}{k}-1} (\sin\theta)^{\frac{2n}{k}-1} e^{-\frac{b^k}{kr^2\sin^2\theta\cos^2\theta}} d\theta\right) dr$$

$$= \frac{2}{k}\int_0^{\infty} r^{2\frac{m+n}{k}-1} e^{-\frac{r^2}{k}} \left(\frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2m}{k}-1} (\sin\theta)^{\frac{2n}{k}-1} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{b^k}{kr^2\sin^2\theta\cos^2\theta}\right)^s\right) dr$$

$$= \frac{2}{k}\int_0^{\infty} r^{2\frac{m+n}{k}-1} e^{-\frac{r^2}{k}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{b^k}{kr^2}\right)^s \left(\frac{2}{k}\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2(m-sk)}{k}-1} (\sin\theta)^{\frac{2(n-sk)}{k}-1} d\theta\right) dr$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{b^k}{k}\right)^s B_k\,(m - sk, n - sk) \frac{2}{k}\int_0^{\infty} r^{2\frac{m+n-sk}{k}-1} e^{-\frac{r^2}{k}} dr$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{b^k}{k}\right)^s B_k\,(m - sk, n - sk)\,\Gamma_k\,(m + n - sk)$$

Thus equation (27) has proved.

Theorem 2.3. *Let* $k > 0; \operatorname{Re}(m) > 0, \operatorname{Re}(n) > 0$ *and* $\operatorname{Re}(b) > 0$ *then extended k-Gamma functions have property*

$$\Gamma_{b,k}(m)\,\Gamma_{b,k}(n) = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{b^k}{k}\right)^{s+r}\frac{(-1)^{s+r}}{s!r!}\Gamma_k(m-rk)\,\Gamma_k(n-sk) \tag{28}$$

Proof.

$$\Gamma_{b,k}(m)\,\Gamma_{b,k}(n) = \int_0^\infty x^{m-1}e^{-\frac{x^k}{k}-\frac{b^k}{kx^k}}dx\int_0^\infty y^{m-1}e^{-\frac{y^k}{k}-\frac{b^k}{ky^k}}dy$$

$$= \sum_{r=0}^{\infty}\left(\frac{b^k}{k}\right)^r\frac{(-1)^r}{r!}\int_0^\infty x^{m-rk-1}e^{-\frac{x^k}{k}}dx\sum_{s=0}^{\infty}\left(\frac{b^k}{k}\right)^s\frac{(-1)^s}{s!}\int_0^\infty y^{n-sk}e^{-\frac{y^k}{k}}dy$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{b^k}{k}\right)^{s+r}\frac{(-1)^{s+r}}{s!r!}\Gamma_k(m-rk)\,\Gamma_k(n-sk)$$

Theorem 2.4. *Let* $\Gamma_{b,k}(m)$ *denotes extended k-Gamma function then prove that*

$$\int_0^\infty e^{-bs}\Gamma_{b,k}(m)\,\Gamma_{b,k}(n)\,db = \sum_{r=0}^{\infty}\left(\frac{-1}{k}\right)^r\frac{1}{r!}B_k(m-rk,n-rk)\,\Gamma_k(m+n-rk)\,\frac{\Gamma_k(rk+1)}{p^{rk+1}} \tag{29}$$

Proof. Using the equation (27)

$$\int_0^\infty e^{-bs}\Gamma_{b,k}(m)\,\Gamma_{b,k}(n)\,db = \int_0^\infty e^{-sb}\sum_{s=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{b^k}{k}\right)^r B_k(m-rk,n-rk)\,\Gamma_k(m+n-rk)\,db$$

$$= \sum_{r=0}^{\infty}\left(\frac{-1}{k}\right)^r\frac{1}{r!}B_k(m-rk,n-rk)\,\Gamma_k(m+n-rk)\int_0^\infty e^{-bs}b^{rk}db$$

$$= \sum_{r=0}^{\infty}\left(\frac{-1}{k}\right)^r\frac{1}{r!}B_k(m-rk,n-rk)\,\Gamma_k(m+n-rk)\,\frac{\Gamma_k(rk+1)}{s^{rk+1}}$$

3. Extended k- Beta distribution

Let a continuous random variable X is said to have extension of k-Beta distribution if its probability density function is given as (Rehman et al. [13], Mubeen et al. [11])

$$\begin{Bmatrix} \frac{1}{B_k(m,n;b)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} \\ 0 \end{Bmatrix} \begin{matrix} ;0<x<1 \\ ;Otherwise \end{matrix} \tag{30}$$

And cumulative density function of extended k-Beta function is given as

$$F_x\left(X \le x\right) = \frac{1}{kB_k\left(m,n;b\right)} \int_0^x x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} dx = \frac{I_{x,k}\left(m,n;k\right)}{B_k\left(m,n;b\right)} \tag{31}$$

Mean of extended k-beta distribution is given as

$$E\left(x\right) = \frac{1}{kB_k\left(m,n;b\right)} \int_0^\infty x x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} dx = \frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n;b\right)} \tag{32}$$

Theorem 3.1. *Let X is extended k- beta Bk(m,n;b) random variable then nth moment about mean is random variable then given as*

$$\mu_n = \sum_{r=0}^{s} {}^sC_r(-1)^r \left(\frac{B_k\left(m+k,n;b\right)}{B\left(m,n;b\right)}\right)^{s-r} \frac{B_k\left(m+rk,n;b\right)}{B_k\left(m,n;b\right)} \tag{33}$$

Proof. Using (30)

$\mu_n = \int\limits_0^1 (x-A)^s \frac{1}{B_k(m,n;b)} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} dx$

$\because A = \frac{B_k(m+k,n;b)}{B(m,n;k)}$

$= \frac{1}{kB_k(m,n;b)} \int\limits_0^1 \sum\limits_{r=0}^{s} {}^sC_r(-1)^r x^r A^{s-r} x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} dx$

$= \sum\limits_{r=0}^{s} {}^sC_r(-1)^r A^{s-r} \frac{1}{B_k(m,n;b)} \int\limits_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{-\frac{b^k}{kx(1-x)}} dx$

$= \sum\limits_{r=0}^{s} {}^sC_r(-1)^r A^{s-r} \frac{B_k(m+rk,n;b)}{B_k(m,n;b)}$

$= \sum\limits_{r=0}^{s} {}^sC_r(-1)^r \left(\frac{B_k(m+k,n;b)}{B_k(m,n;b)}\right)^r \frac{B_k(m+rk,n;b)}{B(m,n;b)}$

Which is represent the nth moment about mean of extended k-Beta distribution When Put $s = 0$, 1,2,3,4 in equation (33) then first four central moment respectively

$$\mu_0 = 1 \tag{34}$$

$$\mu_1 = 0 \tag{35}$$

$$\mu_2 = \frac{B_k\left(m+2k,n;b\right)B_k\left(m,n;b\right) - B_k^2\left(m+k,n;b\right)}{B_k^2\left(m,n\right)} \tag{36}$$

$$\mu_3 = \left(\frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n\right)}\right)^3 - 3\left(\frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n;b\right)}\right)^2\frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n;b\right)}$$
$$+3\frac{B_k\left(m+k,n\right)}{B_k\left(m,n\right)}\frac{B_k\left(m+2k,n\right)}{B_k\left(m,n\right)} + \frac{B_k\left(m+3k,n;b\right)}{B_k\left(m,n\right)} \tag{37}$$

$$\mu_4 = \left(\frac{B_k\left(m+k,n\right)}{B_k\left(m,n\right)}\right)^4 - 4\left(\frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n;b\right)}\right)^3\frac{B_k\left(m+k,n;b\right)}{B_k\left(m,n;b\right)} +$$

$$6\left(\frac{B_k\left(m+k,n\right)}{B_k\left(m,n;b\right)}\right)^2\frac{B_k\left(m+2k,n\right)}{B_k\left(m,n;b\right)} + \frac{B_k\left(m+k,n\right)}{B\left(m,n\right)}\frac{B_k\left(m+3k,n\right)}{B_k\left(m,n;b\right)} + \frac{B_k\left(m+4k,n\right)}{B_k\left(m,n\right)} \tag{38}$$

Remark: In equations (34, (35), (36) and (37), (i) put b=0 then these will follow k-Beta distribution (see Rehman et al. [13]) (ii) put both $k = 1$ and $b = 0$ these will follow Beta Distribution.

Theorem 3.2. Maximum likelihood estimators
Let $X_1, X_2, X_3, \ldots\ldots., X_3$ be random variable of extended k-Beta distribution with parameter (α, β, b) if $\theta = (\alpha, \beta, b)$ then likelihood function of parameters is given by

$$L\left(\theta\right) = \prod_{i=0}^{n} \frac{1}{kB_k\left(\alpha,\beta;b\right)} x_i^{\frac{\alpha}{k}}\left(1-x_i\right)^{\frac{\beta}{k}-1} e^{-\frac{b^k}{kx_i(1-x_i)}} \tag{39}$$

Taking the natural logarithm of the above equation, we get

$$\log L\left(\theta\right) = \left(\tfrac{\alpha}{k}-1\right)\log\prod_{i=1}^{n} x_i + \left(\tfrac{\beta}{k}-1\right)\log\prod_{i=1}^{n}\left(1-x_i\right) - \tfrac{b^k}{k}\sum_{i=1}^{n}\tfrac{1}{x_i(1-x_i)}$$
$$-n\log k - n\log B\left(\alpha,\beta;b\right) \tag{40}$$

Equation (40) is differentiated with respect to α, β and b

$$\frac{\partial L\left(\theta\right)}{\partial\alpha} = \frac{1}{k}\log\prod_{i=1}^{n} x_i - \frac{n}{B_k\left(\alpha,\beta;b\right)}\frac{\partial B_k\left(\alpha,\beta;b\right)}{\partial\alpha} = 0 \tag{41}$$

$$\frac{\partial L(\theta)}{\partial \beta} = \frac{1}{k}\log\prod_{i=1}^{n}(1-x_i) - \frac{n}{B_k(\alpha,\beta;b)}\frac{\partial B_k(\alpha,\beta;b)}{\partial \beta} = 0 \tag{42}$$

$$\frac{\partial L(\theta)}{\partial b} = -b^{k-1}\sum_{i=1}^{n}\frac{1}{x_i(1-x_i)} - \frac{n}{B_k(\alpha,\beta;b)}\frac{\partial B_k(\alpha,\beta;b)}{\partial b} = 0 \tag{43}$$

Case –I If α and β known and b unknown then the maximum likelihood estimator of b is obtained by (43)
Case-II If β and b known and α unknown then the maximum likelihood estimator α of is obtained by (41)
Case-III If α and b known and β unknown then the maximum likelihood estimator of β is obtained by (42) We will solve the equation received in case-I, II, III with help of Newton-Raphson method.

Theorem 3.3. *Let X be extended k-Beta random variable with parameter m, n and b then extended k-Beta distribution satisfies following properties*

$$(i)E(\log x) = k\left[\frac{\partial \log B_k(m,n;b)}{\partial m}\right] \tag{44}$$

$$(ii)E(\log(1-x)) = k\left[\frac{\partial \log B_k(m,n;b)}{\partial n}\right] \tag{45}$$

$$(iii)E\left(\log^2 x\right) = k^2\left[\left(\frac{\partial \log B_k(m,n;b)}{\partial m}\right)^2 - \frac{\partial^2 \log B_k(m,n;b)}{\partial n}\right] \tag{46}$$

$$(iv)E\left(\log^2(1-x)\right) = k^2\left[\left(\frac{\partial \log B_k(m,n;b)}{\partial n}\right)^2 - \frac{\partial^2 \log B_k(m,n;b)}{\partial n}\right] \tag{47}$$

Proof. (i) $E(\log x) = \frac{1}{kB_k(m,n;b)}\int\limits_0^1 (\log x)\, x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}dx$

$= \frac{k}{kB_k(m,n;b)}\int\limits_0^1 \frac{\partial x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}}{\partial m}e^{-\frac{b^k}{kx(1-x)}}dx$

$= \frac{k}{B_k(m,n;b)}\frac{\partial B_k(m,n;b)}{\partial m}$

$= k\frac{\partial \log B_k(m,n;b)}{\partial m}$

(ii)This proof is similar to proof of (i)

$(iii)E\left(\log^2(x)\right) = \frac{1}{kB_k(m,n;b)}\int\limits_0^1 \left(\log^2 x\right) x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}dx$

$= \frac{k^2}{kB_k(m,n;b)}\int\limits_0^1 \frac{\partial^2}{\partial m^2}x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b}{kx(1-x)}}dx$

$= \frac{k^2}{B_k(m,n;b)}\frac{\partial^2 B_k(m,n;b)}{\partial m^2}$

$= k^2\left[\left(\frac{\partial B_k(m,n;b)}{\partial m}\right)^2 - \frac{\partial^2 \log B_k(m,n;b)}{\partial m^2}\right]$

Hence the required result is

(iv) Proof is similar to proof of (i).

Corollary 3.4. *Let X be the extended k- beta random variable with parameter m, n and b then*

$$(i)Var(\log x) = -k^2\left(\frac{\partial^2 \log B_k(m,n;b)}{\partial m^2}\right) \tag{48}$$

$$(ii)Var(\log(1-x)) = -k^2\left(\frac{\partial^2 \log B_k(m,n;b)}{\partial m^2}\right) \tag{49}$$

$$(iii)E\left(x^{\frac{1}{2}}\right) = \frac{B_k\left(m+\frac{k}{2},n;b\right)}{B_k(m,n;b)} \tag{50}$$

$$(iv)E\left(x^{(2r-1)\frac{1}{2}}\right) = \frac{B_k\left(m+(2r-1)\frac{k}{2},n;b\right)}{B_k(m,n;b)} \tag{51}$$

$$(v)E\left(\frac{1}{(1-x)^{\frac{1}{s}}}\right) = \sum_{r=0}^{\infty}\frac{\left(\frac{1}{s}\right)_r}{\Gamma(r+1)}\frac{B_k(m+rk,n;b)}{B_k(m,n;b)} \tag{52}$$

Theorem 3.5. *For $\beta < 1$ and X, $k > 0$, be extended k-beta $B_k(m,n;b)$ random variable then*

$$(i)f_x(x\,|0<X<\beta) = \frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{I_{\beta,k}(m,n;b)} \tag{53}$$

$$(ii)E(x\,|0<X<\beta) = \frac{I_{\beta,k}(m+k,n;b)}{I_{\beta,k}(m,n;b)} \tag{54}$$

$$(iii)E\left(x^2 \left|0 < X \leq \beta\right.\right) = \frac{I_{\beta,k}\left(m+2k,n;b\right)}{I_{\beta,k}\left(m,n;b\right)} \tag{55}$$

$$(iv)Var\left(x \left|0 < X \leq \beta\right.\right) = \frac{I_{\beta,k}\left(m+2k,n\right)}{I_{\beta,k}\left(m,n;b\right)} - \left[\frac{I_{\beta,k}\left(m+k,n\right)}{I_{\beta,k}\left(m,n;b\right)}\right]^2 \tag{56}$$

Where

$$I_{\beta,k}\left(m,n;b\right) = \frac{1}{k}\int_0^\beta x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}dx \tag{57}$$

is incomplete extended k-Beta function.

Proof. (i) Using equation (30) and (31)

$$f_x\left(x \left|0 < X < \beta\right.\right) = \frac{f_x\left(x\right)}{F\left(x \leq \beta\right)} = \frac{\frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{kB_k(m,n;b)}}{\frac{1}{kB_k(m,n;b)}\int_0^\beta x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}$$

$$= \frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{I_{\beta,k}\left(m,n;b\right)}$$

$$(ii)E\left(x \left|0 < x < \beta\right.\right) = \frac{1}{I_{\beta,k}(m,n;b)}\int_0^\beta x^{\frac{m}{k}}\left(1-x\right)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}dx = \frac{I_{\beta,k}(m+k,n;b)}{I_{\beta,k}(m,n;b)}$$

$$(iii)E\left(x^2 \left|0 < X < \beta\right.\right) = \frac{1}{I_{\beta,k}(m,n;b)}\int_0^\beta x^{\frac{m}{k}+1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}dx = \frac{I_{\beta,k}(m+2k,n;b)}{I_{\beta,k}(m,n;b)}$$

$$(iv)Var\left(x \left|0 < X < \beta\right.\right) = E\left(x^2 \left|0 < X < \beta\right.\right) - \left[E\left(x \left|0 < X < \beta\right.\right)\right]^2$$

$$= \frac{I_{\beta,k}\left(m+2k,n;b\right)}{I_{\beta,k}\left(m,n;b\right)} - \left[\frac{I_{\beta,k}\left(m+k,n;b\right)}{I_{\beta,k}\left(m,n;b\right)}\right]^2$$

Hence theorem (3.5) has been proved.

3.6 Applications in real life

3.6.1 Life Time of Component Let X is extended k-Beta $B_k\left(m,n;b\right)$ random variable then the probability of failure till time x is given by

$$F_x\left(x\right) = P\left(X \leq x\right) = \frac{I_{x,k}\left(m,n;b\right)}{B_k\left(m,n;b\right)} \tag{58}$$

Where $I_k(m,n;b)$ denotes incomplete k-Beta function
The probability that the component survives until time x is denoted $S_x(x)$ and can be expressed as

$$S_x(x) = F(X \geq x) = \int_x^1 f_x(x)\, dx = \frac{I^c_{x,k}(m,n;b)}{B_k(m,n;b)} \tag{59}$$

Where $I^c_{x,k}(m,n;b)$ denotes complement of incomplete extended k-Beta function which is used for many problem of mathematical analysis by mathematician.

3.6.2 The Hazard Rate Function $h(x)$
Let $f_x(x)$ be the failure density function of extended k-Beta $B_k(m,n;b)$ random variable then Hazard rate function is defined by

$$h(x) = \frac{f_x(x)}{S_x(x)} \tag{60}$$

Using (30) and (56) in (57)

$$h(x) = \frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} e^{\frac{b^k}{kx(1-x)}}}{I^c_{x,k}(m,n;b)} \tag{61}$$

The Mean Residue Life Time $\kappa(x)$
For extended k- Beta $B_k(m,n;b)$ random variable X the mean residue life function $\kappa(x)$ is defined as

$$\kappa(x) = E(X - x\,|X \geq x) = \frac{\int_x^\infty (t-x)\, f_t(t) dt}{S(x)} = \frac{\int_x^\infty t f_t(t)\, dt}{S(x)} - x \tag{62}$$

Here $\int_x^\infty t f_t(t) dt = \frac{I^c_{xk}(m+k,n;b)}{B_k(m,n;b)}$ thus mean residue life function will be as

$$\kappa(x) = \frac{I^c_{x,k}(m+k,n;b)}{I^c_{xk}(m,n;b)} - x \tag{63}$$

3.7 Quantity information (entropy)

3.7.1 Differential entropy

For extended k- Beta $B_k(m,n;b)$ random variable X, the differential entropy $h(x)$ is defined as

$$h(x) = E(-\log(f_x(x))) = \int_{-\infty}^{\infty} -f_x(x)\log(f_x(x))\,dx \tag{64}$$

Using equation (30) in (63)

$$h(x) = \int_0^1 -\frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{kB_k(m,n;b)}$$

$$\left[\left(\frac{m}{k}-1\right)\log x + \left(\frac{n}{k}-1\right) + \log\left(\frac{n}{k}-1\right) - \frac{b^k}{kx(1-x)} - \log kB_k(m,n;b)\right] dx$$

$$h(x) = \log kB_k(m,n;b) + \frac{b^k}{k}\frac{B_k(m-k,n-k;b)}{B_k(m,n;b)}$$

$$-\left(\frac{m}{k}-1\right)k\frac{\partial \log B_k(m,n;b)}{\partial m} - \left(\frac{n}{k}-1\right)k\frac{\partial B_k(m,n;b)}{\partial n} \tag{65}$$

Equation (64) is differential entropy for extended k-Beta distribution (measured in nuts).

3.7.2 Cross Entropy

For two extended k-Beta random variables such that $X_1\tilde{}k-B_k(m,n;b)$ and $X_1\tilde{}k-B_k(m',n';b)$ then the cross entropy is expressed as (measured in nats)

$$H(X_1,X_2) = \int_0^1 -f_{X_1}(x)\log f_{X_2}(x)dx$$

$$= \int_0^{\infty} \frac{x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{kB_k(m,n;b)} \log\frac{x^{\frac{m'}{k}-1}x^{\frac{n'}{k}-1}e^{-\frac{b^k}{kx(1-x)}}}{kB_k(m',n';b)}dx$$

$$= \log kB_k(m',n';b) + \frac{b^k}{k}\frac{B_k(m-k,n-k;b)}{B(m,n;b)}$$

$$-k\left(\frac{m'}{k}-1\right)\frac{\partial B_k(m,n;b)}{\partial m} - k\left(\frac{n'}{k}-1\right)\frac{\partial B_k(m,n;b)}{\partial n} \tag{66}$$

3.7.3 Kullback-Leibler Divergence (Relative Entropy) $D_{KL}\left(X_1 \| X_2\right)$

The relative entropy $D_{KL}\left(X_1 \| X_2\right)$ for two extended k-Beta random variables such that $X_1\tilde{}k - B_k\left(m, n; b\right)$ and $X_1\tilde{}k - B_k\left(m', n'; b\right)$ is defined by(measured in nats)

$$D_{KL}\left(X_1 \| X_2\right) = \int_0^1 f_{X_1}\left(x\right) \log\left(\frac{f_{X_1}\left(x\right)}{f_{X_2}\left(x\right)}\right) dx = -h\left(X_1\right) + H\left(X_1, X_2\right) \tag{67}$$

Using equation (64) and (65) in (66)

$$D_{KL}\left(X_1 \| X_2\right) = \left(m - m'\right) \frac{\partial \log B_k\left(m, n; b\right)}{\partial m} + \left(n - n'\right) \frac{\partial \log B_k\left(m, n; b\right)}{\partial n} + \log \frac{B_k\left(m', n'; b\right)}{B_k\left(m, n; b\right)} \tag{68}$$

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