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SOME RESULTS ON COMMUTATORS IN MULTIGROUP FRAMEWORK

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Abstract: Multigroup is a group-like algebraic structure drawn from multiset whose underlying set is a group. The concept of commutators in multigroup context has been hitherto introduced in literature. The purpose of this paper is to further explore the idea of commutators in the light of multigroups. A number of some related results are obtained and characterized. The idea of admissible submultigroups A and B of $C \in MG(X)$ under an operator domain $\mathcal{D}$ is explicated, and it is shown that (A, B) and $[A, B]$ are $\mathcal{D}$-admissible.

Keywords and Phrases: Commutator, Multiset, Multigroup, Submultigroup.

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1. Introduction

The idea of multisets as noted in [20], was first suggested by N. G. de Bruijn (cf. [5]) in a private communication to D. E. Knuth, as an important generalisation of crisp set theory, by violating a basic property of crisp sets that an element can belong to a set only once. The notion of multisets is a boost to the concept of multigroups via multisets, which generalises group theory. In [21], the concept of multigroups in multisets framework was proposed and a number of results were obtained. The notion is analogous to other non-classical groups (e.g., fuzzy groups [22], etc.). A complete survey on the concept of multigroups was carried out in [7, 17], and it was established that multigroup via multiset is a generalisation of group theory.

The notion of multigroups as an application of multisets has been extensively researched upon since inception. A number of algebraic properties of order of an element in a multigroup were considered in [3] and some results on multigroups which cut across some homomorphic properties were explored in [4]. The notions of upper and lower cuts of multigroups were proposed and discussed in details with some results in [6], and the notions were extended to homomorphic sense and a number of results were explored [10]. The ideas of submultigroups of multigroups and abelian multigroups were explicated with some results in [15]. In continuation, the concepts of normal submultigroups and characteristic submultigroups of multigroups were proposed in [13, 18] with some results, and some homomorphic properties of multigroups were studied in [12]. Some other group analogous theoretic concepts were established in [1], [2], [9], [11], [14], [16], [19], [23].

The concept of commutators in multigroup setting was proposed in [14]. However, some relevant properties of commutator submultigroups of multigroups are yet to be exploited. This constitutes the motivation of this article. The aim of this paper is to explore the concept of commutators in the light of multigroups and present a number of some related results. Also, the idea of admissible submultigroups under an operator domain is proposed and characterised. The remaining part of this article is thus presented: Section 2 provides some preliminaries on multisets and multigroups. In Section 3, the idea of commutators of multigroups is revisited and some of its properties are discussed. Also, admissible submultigroups under an operator domain is proposed and characterised. Section 4 concludes the paper and provides direction for future studies.

2. Preliminaries

In this section, we present some existing definitions and results to be used in the sequel.

Definition 2.1. [24] *Let X be a non-empty set. A multiset A over X is of the form*

$$A = \{\langle \frac{C_A(x)}{x} \rangle \mid x \in X\},$$

where C_A is a function $C_A : X \to \mathcal{N}$ and $\mathcal{N} = \{0\} \cup \mathbb{N}$.
The set X is called the ground or generic set of the class of all multisets containing objects from X.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a^2, b^2, c^3]$. Other forms of multiset representations can be found in literature. We denote the set of all multisets over X by $MS(X)$.

Definition 2.2. [8] *Let A and B be multisets of X. Then* *(i)* $A \subseteq B \Leftrightarrow C_A(x) \leq C_B(x)\, \forall x \in X$.

(ii) $A = B \Leftrightarrow C_A(x) = C_B(x)\, \forall x \in X$.

(iii) $A \subset B \Leftrightarrow A \subseteq B$ *and* $A \neq B$.

(iv) $A \cap B \Rightarrow C_A(x) \wedge C_B(x)\, \forall x \in X$.

(v) $A \cup B \Rightarrow C_A(x) \vee C_B(x)\, \forall x \in X$.

Note that $\wedge$ and $\vee$ denote minimum and maximum operations.

Definition 2.3. [21] *Let X be a group. A multiset A of X is said to be a multigroup of X if it satisfies the following two conditions:*

(i) $C_A(xy) \geq C_A(x) \wedge C_A(y)\ \forall x, y \in X$,

(ii) $C_A(x^{-1}) \geq C_A(x)\, \forall x \in X$,

where C_A denotes count function of A from X into a natural number $\mathbb{N}$.
It can be easily verified that if A is a multigroup of X, then

$$C_A(e) = \bigvee_{x \in X} C_A(x)\, \forall x \in X,$$

that is, $C_A(e)$ is the tip of A, where e is the identity element of X. Also, $CM_A(x^{-1}) = CM_A(x)\, \forall x \in X$, since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

The set of all multigroups of X is denoted by $MG(X)$.

Definition 2.4. [15] *Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \subseteq A$ if B is a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \subset A$, if $B \subseteq A$ and $A \neq B$.*

Definition 2.5. [15] *Let $A, B \in MG(X)$. Then, the product $A \circ B$ is defined to be a multiset of X as follows:*

$$C_{A \circ B}(x) = \begin{cases} \bigvee_{x=yz}[C_A(y) \wedge C_B(z)], & \text{if} \, \exists\, y, z \in X \text{ such that} \, x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.6. [21] *Let $A \in MS(X)$. Then $A \in MG(X)$ if and only if $A \circ A = A$.*

Proposition 2.7. [21] *Let $A \in MG(X)$. Then, the sets A_* and A^* defined by*

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}$$

are subgroups of X.

Proposition 2.8. [6] *Let $A \in MG(X)$. Then, the set $A_{[n]}$ defined by*

$$A_{[n]} = \{x \in X \mid C_A(x) \geq n, n \in \mathbb{N}\}$$

is a subgroup of X for $n \leq C_A(e)$ and $A^{[n]}$ defined by

$$A^{[n]} = \{x \in X \mid C_A(x) > n, n \in \mathbb{N}\}$$

is a subgroup of X for $n \geq C_A(e)$.

Definition 2.9. [13] *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then, A is called a normal submultigroup of B if for all $x, y \in X$,*

$$C_A(xyx^{-1}) = C_A(y).$$

Definition 2.10. [21]*Let $A \in MG(X)$. Then, A is said to be commutative if for all $x, y \in X$,*

$$C_A(xy) = C_A(yx).$$

Definition 2.11. [7] *Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then, f induces a homomorphism from A to B which satisfies*

(i) $C_{f(A)}(y_1y_2) \geq C_{f(A)}(y_1) \wedge C_{f(A)}(y_2)\ \forall y_1, y_2 \in Y$,

(ii) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2))\ \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x))\ \forall x \in X.$$

Definition 2.12. [18] *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then, A is called a characteristic submultigroup of B if*

$$C_{A^{\theta}}(x) = C_A(x)\ \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Proposition 2.13. [18] *Let $B \in MG(X)$. Then, every characteristic submultigroup of B is a normal submultigroup of B.*

3. Results on commutator in multigroups context

Recall that the commutator of two elements x and y of a group X is the element $[x, y] = x^{-1}y^{-1}xy \in X$. If H and K are subgroups of X, then the commutator subgroup or derived subgroup $[H, K]$ of X is generated by $\{[x, y] | x \in H,\ y \in K\}$.

Now, we recall the definition of commutator in multigroup context as thus [14]:

Definition 3.1. *Suppose A and B are submultigroups of $C \in MG(X)$. Then, (A, B) is a multiset of X defined as follows: $\forall x \in X$*

$$C_{(A,B)}(x) = \begin{cases} \bigvee_{x=[a,b]}[C_A(a) \wedge C_B(b)], & \text{if } x \text{ is a commutator in } X \\ 0, & \text{otherwise.} \end{cases}$$

The commutator of A and B is a multigroup $[A, B]$ of X generated by (A, B).

Now, we present some properties of commutator in multigroup as follow:

Proposition 3.2. *Let $A \in MG(X)$. If A is commutative, then $C_A([x, y]^{-1}) = C_A([y, x])\ \forall x, y \in X$.*

Proof. Suppose A is commutative. Then, we get

$$\begin{aligned} C_A([x, y]^{-1}) \geq C_A([x, y]) &= C_A(x^{-1}y^{-1}xy) \\ &= C_A(y^{-1}x^{-1}yx) \\ &= C_A([y, x]), \end{aligned}$$

$\Rightarrow C_A([x, y]^{-1}) \geq C_A([y, x])$. Similarly,

$$\begin{aligned} C_A([y, x]) = C_A(([y, x]^{-1})^{-1}) &\geq C_A([y, x]^{-1}) \\ &= C_A((y^{-1}x^{-1}yx)^{-1}) \\ &= C_A((x^{-1}y^{-1}xy)^{-1}) \\ &= C_A([x, y]^{-1}), \end{aligned}$$

$\Rightarrow C_A([y,x]) \geq C_A([x,y]^{-1})$. Hence $C_A([x,y]^{-1}) = C_A([y,x])\ \forall x, y \in X$.

Proposition 3.3. *Suppose $A \in MG(X)$ and $[x,y]$ is a commutator of x and y in A. Then $C_A([x,y]) = C_A(e)$ if and only if A is commutative.*

Proof. Assume $C_A([x,y]) = C_A(e)$, where e is the identity element in A. Then $C_A(x^{-1}y^{-1}xy) = C_A(e) \Rightarrow C_A((yx)^{-1}xy) = C_A(e) \Rightarrow C_A(xy) = C_A(yx)$. Thus A is commutative.

Conversely, if $C_A(xy) = C_A(yx)$. Then, we have

$$\begin{aligned} C_A((yx)^{-1}xy) &= C_A((yx)^{-1}yx) \\ &= C_A((x^{-1}x)(y^{-1}y)) \\ &\geq C_A(x^{-1}x) \wedge C_A(y^{-1}y) \\ &= C_A(e), \end{aligned}$$

Thus $C_A(x^{-1}y^{-1}xy) = C_A([x,y]) \geq C_A(e)$. Again we have $C_A(e) = C_A((yx)^{-1}yx) \geq C_A([x,y])$, since $C_A(e) \geq C_A(x) \forall x \in X$. Thus $C_A([x,y]) = C_A(e)$. Hence the result follows.

Remark 3.4. *If a multigroup A of X is commutative, then it is easy to see that $C_A(z[x,y]z^{-1}) = C_A([zxz^{-1}, zyz^{-1}])\ \forall x, y, z \in X$.*

Proposition 3.5. *Let $x, y, z \in X$ and A be a commutative multigroup of X. Then*

(i) $C_A([x,yz]) = C_A([x,z][x,y]^z)$,

(ii) $C_A([xy,z]) = C_A([x,z]^y[y,z])$.

Proof. For $x, y, z \in X$, we have

(i)

$$\begin{aligned} C_A([x,yz]) &= C_A(x^{-1}(yz)^{-1}xyz) \\ &= C_A(x^{-1}z^{-1}y^{-1}xyz) \\ &= C_A([x,y]^z) \end{aligned}$$

and similarly, we have

$$\begin{aligned} C_A([x,z][x,y]^z) &= C_A(x^{-1}z^{-1}y^{-1}xyz) \\ &= C_A([x,y]^z). \end{aligned}$$

The result follows.

(ii)

$$\begin{aligned} C_A([xy,z]) &= C_A((xy)^{-1}z^{-1}xyz) \\ &= C_A(y^{-1}x^{-1}z^{-1}xyz) \\ &= C_A(y^{-1}x^{-1}z^{-1}xzy) \\ &= C_A([x,z]^y). \end{aligned}$$

Similarly,

$$\begin{aligned} C_A([x,y]^y[y,z]) &= C_A((y^{-1}x^{-1}y^{-1}xyy)[y,z]) \\ &= C_A([x,z]^y). \end{aligned}$$

Hence the result.

Lemma 3.6. *If $x, y, z \in X$ and A is a commutative multigroup of X, we have*

(i) $C_A([xy,z]) = C_A([x,z][x,z,y][y,z])$,

(ii) $C_A([x,yz]) = C_A([x,z][x,y][x,y,z])$.

Proof. Straightforward from Proposition 3.5.

Proposition 3.7. *Let $x, y, z \in X$, and A be a multigroup of X. Then, we have $C_A([x,y^{-1},z]^y[y,z^{-1},x]^z[z,x^{-1},y]^x) = C_A(e)$, where e is the identity of X.*
Proof. For $x, y, z \in X$, we have

$$\begin{aligned} C_A([x,y^{-1},z]^y) &= C_A(y^{-1}[x^{-1}yxy^{-1},z]y) \\ &= C_A(y^{-1}(yx^{-1}y^{-1}x)z^{-1}(x^{-1}yxy^{-1})zy) \\ &= C_A(x^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy). \end{aligned}$$

Setting $a = xzx^{-1}yx$, $b = yxy^{-1}zy$ and $c = zyz^{-1}xz$. By observation, b and c can be obtained by cyclic permutation of x, y, z. Furthermore, we see that $C_A([x,y^{-1},z]^y) = C_A(a^{-1}b)$. Deducibly, it follows that $C_A([y,z^{-1},x]^z) = C_A(b^{-1}c)$ and $C_A([z,x^{-1},y]^x) = C_A(c^{-1}a)$. Since

$$C_A((a^{-1}b)(b^{-1}c)(c^{-1}a)) = C_A(e),$$

the result follows.

Theorem 3.8. *Let $x, y \in X$, $z = [x,y]$ commutes with both x and y. If A is a multigroup of X, then $C_A([x^i,y^j]) = C_A(z^{ij})\ \forall i,j$.*

Proof. By a given hypothesis, $C_A(z) = C_A(x^{-1}y^{-1}xy)$, and so $C_A(y^{-1}xy) = C_A(xz)$, where

$$\begin{aligned} C_A(y^{-1}x^iy) = C_A((y^{-1}xy)^i) &= C_A((xz)^i) \\ &= C_A(x^iz^i) \end{aligned}$$

as x and z commute. Conjugating by y gives

$$\begin{aligned} C_A(y^{-2}x^iy^2) = C_A(y^{-1}x^iz^iy) &= C_A(y^{-1}x^iyz^i) \\ &= C_A((x^iz^i)z^i) \\ &= C_A(x^iz^{2i}) \end{aligned}$$

as y and z commute. Repeating this argument j times, we have $C_A(y^{-j}x^iy^j) = C_A(x^iz^{ij})$, hence $C_A([x^i, y^j]) = C_A(z^{ij})$.

Theorem 3.9. *Suppose $A \in MG(X)$ and $x, y, z \in X$. If y commutes with z in A and X is abelian, then $C_A([x, y, z]) = C_A([x, z, y)$.*

Proof. Firstly, we have

$$\begin{aligned} C_A([x, y, z]) &= C_A([[x, y], z]) \\ &= C_A([x, y]^{-1}z^{-1}[x, y]z) \\ &= C_A(y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}xyz) \\ &= C_A(x^{-1}(xy^{-1}x^{-1}y)(xz^{-1}x^{-1}z)z^{-1}y^{-1}xyz). \end{aligned}$$

We observe that $xy^{-1}x^{-1}y$ and $xz^{-1}x^{-1}z$ lies in X, and thus commute. It follows that

$$C_A([x, y, z]) = C_A(x^{-1}(xz^{-1}x^{-1}z)(xy^{-1}x^{-1}y)z^{-1}y^{-1}xyz).$$

Since y and z commute, then

$$\begin{aligned} C_A([x, y, z]) &= C_A(z^{-1}x^{-1}zxy^{-1}x^{-1}z^{-1}xzy) \\ &= C_A([x, z, y]). \end{aligned}$$

This completes the proof.

Theorem 3.10. *Let $A \in MG(X)$ and $x, y, z \in X$. If $[x, y]$ commutes with both x and y, then*

$$C_A([x, y]^{-1}) = C_A([x^{-1}, y]) = C_A([x, y^{-1}]).$$

Proof. By synthesizing Lemma 3.6., we have

$$C_A(e) = C_A([xx^{-1}, y]) = C_A([x, y][x, y, x^{-1}][x^{-1}, y]).$$

But $[x, y, x^{-1}] = [[x, y], x^{-1}]$ and $[x, y]$ commutes with x by hypothesis. It follows that $C_A([x, y, x^{-1}]) = C_A(e) \Rightarrow [x, y, x^{-1}] = e$, where e is the identity of X. Thus

$$C_A(e) = C_A([x, y][x^{-1}, y]) \ \Rightarrow \ C_A([x, y]^{-1}) = C_A([x^{-1}, y]).$$

Similarly, we have $C_A([x, y]^{-1}) = C_A([x, y^{-1}])$. Hence the equality holds.

Definition 3.11. *Let $A, B \in MG(X)$ such that $A \subseteq B$. Define $A^{(0)} = A$. For $n \in \mathbb{N}$ and suppose $A^{(n-1)}$ can be defined for $n \geq 1$. Then, $A^{(n)}$ can be defined by*

$$A^{(n)} = (A^{(n-1)}, A^{(n-1)}).$$

Theorem 3.12. *If A is a multigroup of X, then $A^{(n)} \subseteq A^{(n-1)}$ for all $n \in \mathbb{N}$.*

Proof. We show the proof by induction on n. Let $x \in X$. If x is not a commutator in X. Then

$$C_{(A,A)} = 0 \text{ and } C_A(x) \geq 0 \ \Rightarrow C_{(A,A)} = 0 \leq C_A(x).$$

Suppose x is a commutator in X. Then, for $a, b \in X$ we have

$$\begin{aligned} C_{(A,A)} &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_A(b)] \\ &= \bigvee_{x=[a,b]} [C_A(a^{-1}) \wedge C_A(b^{-1}) \wedge C_A(a) \wedge C_A(b)] \\ &\leq \bigvee_{x=[a,b]} [C_A(a^{-1}b^{-1}ab)] \\ &= C_A(x), \end{aligned}$$

and so $(A, A) \subseteq A$. Since $A \in MG(X)$, then

$$C_{A^{(1)}}(x) = C_{(A^{(0)}, A^{(0)})}(x) \leq C_A(x) = C_{A^{(0)}}(x),$$

and so the result follows for $k = 1$. Since $C_{A^{(n)}}(x) \leq C_{A^{(n-1)}}(x)$ for $n \in \mathbb{N}$, we have

$$\begin{aligned} C_{A^{(k+1)}}(x) &= C_{(A^{(k)}, A^{(k)})}(x) \\ &\leq C_{(A^{(k-1)}, A^{(k-1)})}(x) \\ &= C_{A^k}(x). \end{aligned}$$

Thus $A^{(k+1)} \subseteq A^{(k)} \Rightarrow A^{(k)} \subseteq A^{(k-1)}$. Hence the result follows for $n = k$.

Theorem 3.13. *Suppose A and B are submultigroups of $C \in MG(X)$. Then $[A, B] = [B, A]$.*

Proof. Let $x \in X$. Since $[A, B]$ is a multigroup generated by (A, B), if we prove that $(A, B) = (B, A)$ we are done. Assume x is not a commutator in X, then x^{-1} is not a commutator and consequently,

$$C_{(A,B)}(x) = 0 = C_{(B,A)}(x^{-1}).$$

Suppose $x = [a, b]$ for some $a, b \in X$. Then

$$\begin{aligned} C_{(A,B)}(x) &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_B(b)] \\ &= \bigvee_{x^{-1}=[b,a]} [C_B(b) \wedge C_A(a)] \\ &= C_{(B,A)}(x^{-1}). \end{aligned}$$

Hence $(A, B) = (B, A)$ implies $[A, B] = [B, A]$.

Corollary 3.14. *Suppose A and B are submultisets of $C \in MG(X)$. Then*

(i) $[A, B]_* = [B, A]_*$,

(ii) $[A, B]^* = [B, A]^*$.

Proof. Using Proposition 2.7 and Theorem 3.13, the results follow.

Corollary 3.15. *If A and B are submultisets of $C \in MG(X)$. Then, for $n \in \mathbb{N}$,*

(i) $[A, B]_{[n]} = [B, A]_{[n]}$,

(ii) $[A, B]^{[n]} = [B, A]^{[n]}$.

Proof. Combining Proposition 2.8 and Theorem 3.13, the results follow.

Theorem 3.16. *Suppose $A \in MG(X)$, then $[A, A] \subseteq A$.*
Proof. If x is not a commutator in X, then $C_{(A,A)}(x) = 0$. Suppose x is a commutator, then

$$\begin{aligned} C_{(A,A)}(x) &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_A(b)] \text{ for some } a, b \in X \\ &= \bigvee_{x=[a,b]} [C_A(a^{-1}) \wedge C_A(b^{-1}) \wedge C_A(a) \wedge C_A(b)] \\ &\leq \bigvee_{x=[a,b]} [C_A(a^{-1}b^{-1}ab)] \\ &= C_A(x). \end{aligned}$$

Thus $(A, A) \subseteq A \Rightarrow [A, A] \subseteq A$.

Theorem 3.17. *Suppose $A, B, C, D \in MG(X)$ such that $A \subseteq B$ and $C \subseteq D$, then $[A, C] \subseteq [B, D]$.*
Proof. Recall that $[A, C] = \langle (A, C) \rangle$. If x is not a commutator in X, then $C_{(A,B)}(x) = 0$ and therefore there is nothing to prove. Suppose x is a commutator, then

$$\begin{aligned} C_{(A,C)}(x) &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_C(b)] \text{ for some } a, b \in X \\ &\leq \bigvee_{x=[a,b]} [C_B(a) \wedge C_D(b)] \\ &= C_{(B,D)}(x). \end{aligned}$$

Thus $[A, C] = \langle (A, C) \rangle \subseteq \langle (B, D) \rangle = [B, D]$. Hence $[A, C] \subseteq [B, D]$.

Corollary 3.18. *Let $A, B, C, D, E, F \in MG(X)$. If $[C, D] \subseteq [A, B]$ and $[E, F] \subseteq [A, B]$, then $[[C, D], [E, F]] \subseteq [A, B]$.*
Proof. Suppose $[C, D] \subseteq [A, B]$ and $[E, F] \subseteq [A, B]$. Then $[[C, D], [E, F]] \subseteq [[A, B], [A, B]] \subseteq [A, B]$. Hence $[[C, D], [E, F]] \subseteq [A, B]$.

Proposition 3.19. *Let $A, B \in MG(X)$. Then $[A, B] \circ [B, A] = [A, B]$.*
Proof. Combining Proposition 2.6 and Theorem 3.13, we get $[A, B] \circ [B, A] \subseteq [A, B]$. Again, if $x = 0$, we have $C_{[A,B]\circ[B,A]}(x) = 0$. Otherwise,

$$\begin{aligned} C_{[A,B]\circ[B,A]}(x) &= \bigvee_{x=ab} [C_{[A,B]}(a) \wedge C_{[B,A]}(b)] \\ &\geq C_{[A,B]}(x) \wedge C_{[A,B]}(e) \\ &= C_{[A,B]}(x), \end{aligned}$$

and so, $[A, B] \subseteq [A, B] \circ [B, A]$. Hence $[A, B] \circ [B, A] = [A, B]$.

Proposition 3.20. *Suppose $A, B, C, D, E, F \in MG(X)$ such that $[C, D] \subseteq [A, B]$ and $[E, F] \subseteq [A, B]$. Then $[C, D] \circ [E, F] \subseteq [A, B]$.*
Proof. Let $x \in X$. If $x = 0$, we have $C_{[C,D]\circ[E,F]}(x) = 0$. Otherwise,

$$\begin{aligned} C_{[C,D]\circ[E,F]}(x) &= \bigvee_{x=ab} [C_{[C,D]}(a) \wedge C_{[E,F]}(b)] \\ &\leq \bigvee_{x=ab} [C_{[A,B]}(a) \wedge C_{[A,B]}(b)] \\ &= C_{[A,B]\circ[A,B]}(x) \\ &= C_{[A,B]}(x). \end{aligned}$$

Hence $[C, D] \circ [E, F] \subseteq [A, B]$.

Lemma 3.21. *Suppose A and B are submultigroups of $D \in MG(X)$ such that $A \subseteq B$, then $[A, C] \subseteq [B, C]$ for any submultigroup C of D.*
Proof. Given $A \subseteq B$, then $C_A(x) \leq C_B(x)\ \forall x \in X$. Let C be any submultigroup of D. If x is not a commutator in X, then

$$C_{(A,C)}(x) = 0 = C_{(B,C)}(x).$$

Suppose $x = [a, b]$ for some $a, b \in X$. Then we have

$$\begin{aligned} C_{(A,C)}(x) &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_C(b)] \\ &\leq \bigvee_{x=[a,b]} [C_B(a) \wedge C_C(b)] \\ &= C_{(B,C)}(x), \end{aligned}$$

implies that $(A, C) \subseteq (B, C)$ and so $[A, C] \subseteq [B, C]$.

Theorem 3.22. *Let A and B be normal submultigroups of $D \in MG(X)$ and let C be any submultigroup of D. Then $[A \circ C, B] \subseteq [A, B] \circ [C, B]$ with equality holding if $CM_A(e) = CM_C(e)$, where e is the identity of X.*
Proof. Firstly, we show that $(A \circ C, B) \subseteq [A, B] \circ [C, B]$. Let $x \in X$. If x is not a commutator in X, then $C_{(A\circ C,B)}(x) = 0$ and the result is trivial. Suppose x is a commutator in X. Then

$$\begin{aligned} C_{(A\circ C,B)}(x) &= \bigvee\{C_{A\circ C}(a) \wedge C_B(b) | x = [a, b],\ a, b \in X\} \\ &= \bigvee\{\bigvee\{C_A(u) \wedge C_C(v) | a = uv,\ u, v \in X\} \wedge C_B(b) | x = [a, b],\ a, b \in X\} \\ &= \bigvee\{\bigvee\{[C_A(u) \wedge C_C(v)] \wedge C_B(b) | a = uv,\ u, v \in X\} | x = [a, b],\ a, b \in X\} \\ &= \bigvee\{\bigvee\{[C_A(u) \wedge C_B(b)] \wedge [C_C(v) \wedge C_B(b) | a = uv,\ u, v \in X\} | x = [a, b],\ a, b \in X\} \\ &= \bigvee\{[C_A(u) \wedge C_B(b)] \wedge [C_C(v) \wedge C_B(b)] | x = [uv, b],\ uv, b \in X\} \\ &\leq \bigvee\{C_{[A,B]}([u, b]) \wedge C_{[C,B]}([v, b]) | x = [uv, b],\ uv, b \in X\} \\ &= \bigvee\{C_{[A,B]}([u, b]^v) \wedge C_{[C,B]}([v, b]) | x = [uv, b],\ uv, b \in X\} \\ &\leq \bigvee\{C_{[A,B]}(y) \wedge C_{[C,B]}(z) | x = yz,\ y, z \in X\} \\ &= C_{[A,B]\circ[C,B]}(x), \end{aligned}$$

where $y = [u, b]^v$, $z = [v, b]$ and $x = yz = [uv, b] = [u, b]^v[v, b]$. Since $[A, B]$ is normal, $[A, B] \circ [C, B]$ is a multigroup of X. Hence $[A \circ C, B] \subseteq [A, B] \circ [C, B]$.

Finally, suppose $C_A(e) = C_C(e)$. Then, $A \subseteq A \circ C$ and $C \subseteq A \circ C$. Thus, $[A, B] \subseteq [A \circ C, B]$ and $[C, B] \subseteq [A \circ C, B]$ by Lemma 3.21. Hence, $[A, B] \circ [C, B] \subseteq$

$[A \circ C, B]$ since $[A \circ C, B]$ is a multigroup of X. Consequently, the desired equality holds.

Definition 3.23. *A non-empty collection $\mathcal{D}$ of endomorphisms of a group X is an operator domain on X. A multiset A of X is admissible under $\mathcal{D}$ or $\mathcal{D}$-admissible if for every $f \in \mathcal{D}$, $f(A) \subseteq A$, where f is a function. Thus $f(A) \subseteq A$ if and only if $A \subseteq f^{-1}(A)$.*

Theorem 3.24. *Suppose A and B are admissible submultisets of $C \in MG(X)$ under an operator domain $\mathcal{D}$, then (A, B) and $[A, B]$ are $\mathcal{D}$-admissible.*

Proof. The fact that A and B are $\mathcal{D}$-admissible, we have

$$C_{f^{-1}(A)}(x) \geq C_A(x) \text{ and } C_{f^{-1}(B)}(x) \geq C_B(x)\, \forall f \in \mathcal{D}.$$

Let $f \in \mathcal{D}$ and $x \in X$. If x is not a commutator in X, then

$$C_{(A,B)}(x) = 0 \leq C_{(A,B)}(f(x)).$$

Suppose $x = [a, b]$ for some $a, b \in X$. Then

$$\begin{aligned} C_{(A,B)}(x) &= \bigvee_{x=[a,b]} [C_A(a) \wedge C_B(b)] \\ &\leq \bigvee_{x=[a,b]} [C_A(f(a)) \wedge C_B(f(b))] \\ &= \bigvee_{f(x)=[f(a),f(b)]} [C_A(f(a)) \wedge C_B(f(b))] \\ &\leq \bigvee_{f(x)=[c,d]} [C_A(c) \wedge C_B(d)],\ c, d \in X \\ &= C_{(A,B)}(f(x)). \end{aligned}$$

Thus $(A, B) \subseteq f^{-1}((A, B))$. Hence (A, B) is $\mathcal{D}$-admissible. Since $[A, B]$ is generated by (A, B), it follows that $(A, B) \subseteq f^{-1}([A, B])$. Thus $[A, B] \subseteq f^{-1}([A, B])$. Hence $[A, B]$ is $\mathcal{D}$-admissible.

4. Conclusion

In this article, the ideas of commutator of multigroups and commutator submultigroups of multigroups have been proposed and characterised with a number of some related results. Some homomorphic images and preimages of commutator of multigroups were considered with some results. The notion of admissible submultisets A and B of $C \in MG(X)$ under an operator domain $\mathcal{D}$ was explained, and it was shown that (A, B) and $[A, B]$ are $\mathcal{D}$-admissible. However, more properties of commutators in multigroup setting could be exploited in future research.

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