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ON NEW FORMULAS FOR THE ROGERS-RAMANUJAN IDENTITIES

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Abstract: In a previous work we presented a correspondence between unrestricted partitions of n and the number of representations of m as a t-squared partition (defined as the frequency of m), for m in the interval $[1\,,\,n^2-1]$. Here we want to go a step further presenting also correspondences between the Rogers-Ramanujan identities and frequency of numbers also represented by t-squared partitions.

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1. Introduction

The relation between partitions and two-line matrices started at the beginning of the last century with Frobenius [4], and were further developed by Andrews [1]. A new approach was introduced in Mondek-Ribeiro-Santos [7], also relating partitions

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and two-line matrices, but now adding an important feature to this matricial representation, since the conjugate of the partition can also be read from the matrix representation. Other important references on this subject are Brietzke-Santos-Silva [2, 3] where generalizations involving mock theta functions are presented. The relation goes as follows:

Let $n, \beta \in \mathbb{N}$, with $\beta < n-1$, and $\delta \in \mathbb{N} \cup \{0\}$. Let us define $\mathbb{M}(n, \beta, \delta)$ to be the set of all two-line matrices

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1}$$

such that $c_j, d_j \in \mathbb{N} \cup \{0\}$ and

$$c_s = \beta, \quad c_j = c_{j+1} + d_{j+1} + \delta \quad \text{and} \quad \sum_{i=1}^{s} (c_i + d_i) = n. \tag{2}$$

The condition $\beta < n-1$ follows from the fact that

$$|\mathbb{M}(n, n, \delta)| \;=\; |\mathbb{M}(n, n-1, \delta)| \;=\; 1 \quad \text{and} \quad \mathbb{M}(n, n+t, \delta) = \emptyset, \;\; \forall t \in \mathbb{N}.$$

Given $M \in \mathbb{M}(n, \beta, \delta)$, written as (1), if we define $c_j + d_j = \mu_j$ we would have the partition of n

$$n \;=\; \mu_1 + \cdots + \mu_s,$$

with the least part $\mu_s \geq \beta$ and $\mu_j - \mu_{j-1} \geq \delta$. On the other hand, given a partition $n \;=\; \mu_1 + \cdots + \mu_s$, with $\mu_s \geq \beta$ and $\mu_{j-1} - \mu_j \geq \delta$, we can write

$$\begin{array}{lllllll} \mu_s & = & \beta + d_s, & & & & \\ \mu_{s-1} & = & \mu_s + \delta + d_{s-1} & = & c_{s-1} + d_{s-1}, & & \\ \mu_{s-2} & = & \mu_{s-1} + \delta + d_{s-2} & = & c_{s-1} + d_{s-1} + \delta + d_{s-2} & = & c_{s-2} + d_{s-2}, \end{array}$$

and continuing this process we obtain a matrix $M \in \mathbb{M}(n, \beta, \delta)$ (see (1)). This establish a bijection between the set $\mathbb{M}(n, \beta, \delta)$ and the set of all partitions of n with the smallest part being at least β and the minimum distance between parts being at least δ. In particular we have that (here $|A|$ means the cardinality of the set A)

(a) $|\mathbb{M}(n, 1, 1)|$ is equal to the number of partitions of n into distinct parts;

(b) $|\mathbb{M}(n, 1, 2)|$ is equal to the number of partitions of n where the difference between two parts is at least two (Rogers-Ramanujan of type I);

(c) $|\mathbb{M}(n, 2, 2)|$ is equal to the number of partitions of n where the difference between two parts is at least two and each part is greater than one (Rogers-Ramanujan of type II).

This theory was extended in Matte-Santos [8], where a correspondence (known as *Path Procedure*) is presented between these matrices, paths in the Cartesian plane and partitions into distinct odd parts all greater than one. In the same paper, Matte and Santos studied these partitions in detail and interesting properties are presented.

Motivated by the ideas presented in [8], we introduced in [5] a correspondence between unrestricted partitions of n and the number of representations of m as a t-squared partition (defined as the frequency of m), for m in the interval $[1\,,\, n^2 - 1]$. Here we want to go a step further presenting also correspondences between the Rogers-Ramanujan identities and frequency of numbers also represented as t-squared partitions. We start with a series of results designed to improve our understanding of numbers m admitting t-squared partitions and their frenquencies, proving for example that any $m \in \mathbb{N}$ admits a t-squared partition if, and only if, $m \equiv 0$ or $3 \pmod 4$, provided m is not one of the 12 exceptional values. We end this section mentioning that the peculiar shape of the t-squared partitions is due to its close relation to the partitions into distinct odd parts obtained by the *Path Procedure*.

2. t-Squared Partitions

We say that $m \in \mathbb{N}$ admits a *t-squared partition* if m can be written as

$$m = (c_1 + c_2 + \cdots + c_t)^2 \; + \; 2(c_1^2 + c_2^2 + \cdots + c_t^2). \tag{3}$$

For example, the numbers 107 and 144 can be written as

$$107 = (5 + 2)^2 \quad + \quad 2 \times (5^2 + 2^2)$$
$$144 = (3 + 3 + 1 + 1 + 1 + 1)^2 \quad + \quad 2 \times (3^2 + 3^2 + 1^2 + 1^2 + 1^2 + 1^2)$$

that is, 107 admits a 2-squared partition and 144 admits a 6-squared partition. In this section, we repeat a few lemmas proved in [5], with the intention of keeping this paper as self-contained as possible.

Lemma 2.1. *Let $m \in \mathbb{N}$ and suppose that m admits a t-squared partition. Then we can find $a, b \in \mathbb{N}$ such that $m = b^2 + 2a$ with*

$$a \equiv b \pmod 2 \quad \text{and} \quad ta \geq b^2 \geq a \geq b.$$

Proof. The fact that $m = b^2 + 2a$ follows from (3), and since $x^2 \equiv x \pmod 2$, for any integer x, we have that $a \equiv b \pmod 2$. Now we focus our attention in proving the inequalities. It is easy to see that a positive integer m admits a *t-squared partition* if m can be written as $m = b^2 + 2a$, and we can find a solution for the system

$$\begin{cases} b = x_1 + \cdots + x_t, \\ a = x_1^2 + \cdots + x_t^2, \end{cases} \tag{4}$$

with $x_1, \ldots, x_t \in \mathbb{N}$. Since these are all natural numbers it follows easily that $b^2 \geq a \geq b$. The last inequality follows from the Cauchy-Schwarz inequality since

$$b^2 = (\sum_{i=1}^{t} x_i)^2 = (\sum_{i=1}^{t} x_i \cdot 1)^2 \leq (\sum_{i=1}^{t} x_i^2)(\sum_{i=1}^{t} 1^2) = ta.$$

Lemma 2.2. *Let $m \in \mathbb{N}$. The integer m admits a t-squared partition only if $m \equiv 0$ or $3 \pmod 4$.*

Proof. If m admits a *t-squared partition* then it can be written as $m = b^2 + 2a$, with $a \equiv b \pmod 2$. From this congruence condition follows the result of this lemma.

For some special values of m, and also for small values of t is easy to obtain t-squared partitions, as can be seen in the next two results.

Lemma 2.3. *Let m be a positive integer. If $m + 1 = d^2$, for some $d \in \mathbb{N}$, then m admits a $(d-1)$-squared partition.*

Proof. Let us write $m = d^2 - 1 = (d-1)^2 + 2(d-1)$. Now take $x_1 = \cdots = x_{d-1} = 1$ as a solution for the system (4), with $t = d - 1$ and $a = b = d - 1$.

Lemma 2.4. *Let m be a positive integer written as $m = b^2 + 2a$. Then*

(a) m admits a 1-squared partition if, and only if, $a = b^2$.

(b) m admits a 2-squared partition if, and only if, $2a - b^2$ is a square smaller than b^2.

Proof. The case (a) is immediate, for the only possibility is to write $m = b^2 + 2b^2$. Let us proceed to the other case, considering the system (4) with $t = 2$. Observe that $2a - b^2 = 2(x_1^2 + x_2^2) - (x_1 + x_2)^2 = (x_1 - x_2)^2$. Thus if m admits a 2-*squared partition*, then $2a - b^2 = (x_1 - x_2)^2$. Since $x_1, x_2 \in \mathbb{N}$, we have that $|x_1 - x_2| < x_1 + x_2 = b$. Conversely, consider $2a - b^2 = d^2 < b^2$ and take $x_1 = (b+d)/2$ and $x_2 = (b - d)/2$. Since $b \equiv d \pmod 2$ and $b > d$, we have that x_1 and x_2 are positive integers.

Next we present some combinatorial lemmas that will be helpful for our study of numbers m admitting t-squared partitions.

Lemma 2.5. *Let $c_1, c_2, \ldots, c_s \in \mathbb{N}$, with $s \geq 2$, and assume $c_1 \geq \cdots \geq c_s$. Then*

$$c_1^2 + c_2^2 + \cdots c_s^2 \leq \left((\sum_{i=1}^{s} c_i) - 1 \right)^2 + 1.$$

Proof. (Induction on s). Let $s = 2$, then

$$(c_1 + c_2 - 1)^2 + 1 = (c_1 + c_2)^2 - 2(c_1 + c_2) + 2 \geq c_1^2 + c_2^2,$$

since $c_1, c_2 \in \mathbb{N}$. Now, let $b = c_1 + c_2 + \cdots + c_s$. By the induction hypothesis, we have

$$\begin{aligned} c_1^2 + \cdots + c_{s-1}^2 + c_s^2 &\leq ((b - c_s) - 1)^2 + 1 + c_s^2 \leq \\ &\leq (b-1)^2 + 1 - 2c_s((b-1) - c_s) \leq (b-1)^2 + 1, \end{aligned}$$

since $b > c_s$.

Lemma 2.6. *Let $c_1, c_2, \ldots, c_s \in \mathbb{N}$, with $s \geq 2$, and assume that they are not all equal. Then*

$$2 \sum_{1=i<j}^{s} c_i c_j + (s-1) \leq (s-1) \sum_{i=1}^{s} c_i^2. \tag{5}$$

Proof. (Induction on s). The case $s = 2$ follows from $(c_1 - c_2)^2 \geq 1$. Let us assume that there is only one c_j different from the others, say $c_1 = \cdots = c_{s-1} \neq c_s$. In this case the LHS of (5) is equal to

$$(s-1)\{(s-2)c_1^2 + 2c_1 c_s + 1\}$$

and the RHS of (5) is equal to $(s-1)\{(s-1)c_1^2 + c_s^2\}$. Now it is simple to see that the inequality in (5) holds since $(c_1 - c_s)^2 \geq 1$.

Let us assume $c_1 \geq \cdots \geq c_s$ and write $c_j = c_s + \delta_j$, for $j = 1, \ldots, s-1$. Hence we have

$$2 \sum_{1=i<j}^{s} c_i c_j = s(s-1)c_s^2 + 2(s-1)c_s(\sum_{j=1}^{s-1} \delta_j) + 2 \sum_{1=i<j}^{s-1} \delta_i \delta_j, \tag{6}$$

and

$$(s-1) \sum_{i=1}^{s} c_i^2 = s(s-1)c_s^2 + 2(s-1)c_s(\sum_{j=1}^{s-1} \delta_j) + (s-1) \sum_{i=1}^{s-1} \delta_i^2. \tag{7}$$

Since the δ_j's are not all equal (for there are at least two distinct c_j's), the result follows from the induction hypothesis, since

$$2\sum_{1=i<j}^{s-1}\delta_i\delta_j+(s-2)\le(s-2)\sum_{i=1}^{s-1}\delta_i^2<(s-1)\sum_{i=1}^{s-1}\delta_i^2,$$

(see (6) and (7) above). completing the proof.

Theorem 2.7. *Let $m\in\mathbb{N}$. Then m admits a s-squared partition only if m can be written as $m=b^2+2a$, with $a,b\in\mathbb{N}$ and*

$$(i)\quad \left\lceil\sqrt{\frac{m}{3}}\;\right\rceil\le\ b\ \le\lfloor\sqrt{m+1}\rfloor-1.$$

$$(ii)\quad (\left\lceil\frac{b}{s}\right\rceil)^2+(s-1)(\left\lfloor\frac{b}{s}\right\rfloor)^2\ \le\ a\ \le(b-1)^2+1.$$

Proof. Let $m=b^2+2a$, and $c_1,\dots,c_s\in\mathbb{N}$ be a solution for (4). From the inequalities stated in Lemma 2.1 we have

$$b^2+2b\le\ m\ \le 3b^2,$$

which gives (i), since $b^2+2b=(b+1)^2-1$. For the item (ii), the inequality on the RHS follows directly from Lemma 2.5. Now observe that

$$(\lceil\frac{b}{s}\rceil)^2+(s-1)(\lfloor\frac{b}{s}\rfloor)^2=\begin{cases}\dfrac{b^2}{s}, & \text{if } b\equiv 0 \pmod s\\[2ex] \dfrac{((b-r)+1)^2+(s-1)}{s}, & \text{if } b\equiv r\not\equiv 0 \pmod s).\end{cases}$$

In any case we have, (taking $r=1$)

$$(\lceil\frac{b}{s}\rceil)^2+(s-1)(\lfloor\frac{b}{s}\rfloor)^2\le\frac{b^2+(s-1)}{s}.$$

By Lemma 2.6, we have

$$b^2+(s-1)\le s\sum_{i=1}^{s}c_i^2=\ sa,$$

concluding the proof.

Our goal is to prove that any $m \in \mathbb{N}$, $m \equiv 0$ or $3 \pmod 4$, admits a t-squared partition, provided m is not one of the 12 exceptional values. For this purpose we need the following Theorem proved in Pall [Theorem 4, [9]] and a Lemma.

Theorem 2.8. *Let $a, b \in \mathbb{N}$, and assume that $a \equiv b \pmod 2$ and $7a \geq b^2 \geq 3a - 5$. Then the system (4), with $t = 7$, has a solution $c_1, \ldots, c_7 \in \mathbb{N} \cup \{0\}$.*

Lemma 2.9. *Let $m \in \mathbb{N}$, $m \geq 5$ and let*

$$c = \left\lceil \sqrt{\frac{3m-10}{5}} \right\rceil \quad and \quad d = \left\lfloor \sqrt{\frac{7m}{9}} \right\rfloor. \tag{8}$$

If $m \geq 290$ then $d \geq c + 1$.

Proof. Observe that

$$H(m) = \sqrt{\frac{7m}{9}} - \sqrt{\frac{3m-10}{5}} \; > (\sqrt{\frac{7}{9}} - \sqrt{\frac{3}{5}})\sqrt{m} > \frac{\sqrt{m}}{10},$$

hence $H(m)$ is an increasing function. Since $H(350) > 2$, consequently we have $d \geq c + 1$, for $m \geq 350$. For smaller values of m in the interval $[290\,,\,349]$, a computer search verified that $d \geq c + 1$, in all of these cases.

Theorem 2.10. *Let $m \in \mathbb{N}$ such that $m \equiv 0\, or 3 \pmod 4$ then m always admits a t-squared partition, unless*

$$m \in \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}.$$

Proof. First let us assume $m \geq 290$ and $m \equiv 0$ or $3 \pmod 4$. According to Lemma 2.9, the interval $[c, d]$ contains at least two consecutive natural numbers, so we can choose $b \in [c, d]$ such that $m \equiv b \pmod 2$. Now, it follows from (8) that

$$\frac{3}{5}m - 2 \leq \; b^2 \; \leq \frac{7}{9}m, \tag{9}$$

Let $a = (m - b^2)/2$, and recall that $m \equiv 0$ or $3 \pmod 4$ and $m \equiv b \pmod 2$. If $m \equiv 0 \pmod 4$, then we also have $b^2 \equiv 0 \pmod 4$, and if $m \equiv 3 \pmod 4$, then b is odd, and $b^2 \equiv 1 \pmod 4$. In any case we have $a \equiv b \pmod 2$.

It follows from (9) that a and b satisfy the following inequalities

$$b^2 < \frac{7}{9}m \implies 2b^2 < 7(m - b^2) \implies b^2 < 7a,$$

and

$$\frac{3}{5}m - 2 < b^2 \implies 3m - 10 < 5b^2 \implies 3a - 5 < b^2.$$

Hence, for this choice of a and b there exist a solution $c_1, \ldots, c_7 \in \mathbb{N} \cup \{0\}$ for the system (4) with $t = 7$, according to Theorem 2.8. With no loss in generality, let us assume $c_1 \geq \cdots \geq c_7 \geq 0$, and since $b \neq 0$, there must be an s such that $c_1 \geq \cdots \geq c_s \geq 1$ and $c_{s+1} = 0$. Therefore, m admits a s-squared partition, as desired.

For all other values of m in the interval $[3\,,\,290]$, a computer search was performed using the software MAPLE©, with the bounds for the parameters a and b given in Theorem 2.7, and assuming also that $m \equiv 0$ or $3 \pmod 4$ and $m \equiv b \equiv a \pmod 2$. For all values of $m \in [3\,,\,290]$, satisfying the conditions above we have found s-squared partitions, unless m is in the set $\{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}$.

Definition 2.11. *Let $m \in \mathbb{N}$ and define $f(m)$, the frequency of m, as the number of times m can be represented by a t-squared partition.*

Corollary 2.12. *Let $m \in \mathbb{N}$, such that $m \equiv 0$ or $3 \pmod 4$. Then $f(m)$ is equal to the number of non-negative solutions $(c_1, c_2, \ldots, c_b)$, assuming $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, of systems of the type*

$$\begin{cases} b = x_1 + \cdots + x_b \\ a = x_1^2 + \cdots + x_b^2, \end{cases} \tag{10}$$

for any pair a, b such that $a \equiv b \pmod 2$ and $m = b^2 + 2a$. Moreover, $f(m) \geq 1$ unless $m \in \{4, 7, 11, 16, 20, 23, 31, 40, 44, 55, 68, 95\}$.

Proof. Given a non-negative solution $(c_1, c_2, \ldots, c_b)$ with $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, we may assume that for some $t \geq 1$ we have $c_t \neq 0$ and $c_{t+1} = \cdots = c_b = 0$. This shows that m admits a t-squared partition and for any distinct non-negative solution $(c_1, c_2, \ldots, c_b)$ of (10), assuming $c_1 \geq c_2 \geq \cdots \geq c_b \geq 0$, we have a distinct t-squared partition of m. The final statement is a direct consequence of Theorem 2.10.

Example 2.13. *It is not a simple task to determine the frequency of a number, for it involves calculating the number of positive solutions of the system (10). But for small values of m it can be easily done, for example, a simple computation shows that $f(107) = 2$ and $f(144) = 4$. Below we have a list of the distinct t-squared partitions of 107 and 144.*

$$\begin{aligned} 107 &= (5+2)^2 \quad + \quad 2 \times (5^2 + 2^2) \\ 107 &= (2+2+1+1+1+1+1)^2 \\ &\quad + \; 2 \times (2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2) \end{aligned}$$

$$144 = (6+2)^2 \quad + \quad 2 \times (6^2 + 2^2)$$
$$144 = (3+2+2+2+1)^2 \quad + \quad 2 \times (3^2 + 2^2 + 2^2 + 2^2 + 1^2)$$
$$144 = (3+3+1+1+1+1)^2 \quad + \quad 2 \times (3^2 + 3^2 + 1^2 + 1^2 + 1^2 + 1^2)$$
$$144 = (4+1+1+1+1+1+1)^2 \\ \quad + \; 2 \times (4^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)$$

For the interested reader we recommend the papers of Kloosterman [6] and Pall [9] where conditions for the existence of integer solutions and formulas for the number of integer solutions for the system (10) are presented.

2.1. The Special Set $\mathcal{U}_n(m, \beta, \delta)$

Let $m \in \mathbb{N}$. For a fixed pair a, b such that $a \equiv b \pmod 2$ and $m = b^2 + 2a$ define $\mathcal{A}(m)$ as the set of all non-negative solutions $\vec{x}$ of (10) such that

$$\vec{x} = (c_1, c_2, \ldots, c_b), \;\text{ and }\; c_1 \geq c_2 \geq \cdots \geq c_b \geq 0.$$

According to Corollary 2.12, we have $|\mathcal{A}(m)| = f(m)$. For $\vec{x} \in \mathcal{A}(m)$, denote by $w(\vec{x})$ the number of nonzero coordinates of $\vec{x}$, by $b(\vec{x})$ the sum of the coordinates of $\vec{x}$ and by $c(\vec{x})$ the biggest coordinate of $\vec{x}$. Hence if $\vec{x} = (c_1, c_2, \ldots, c_b) \in \mathcal{A}(m)$ and $w = w(\vec{x})$ then

$$\begin{aligned} &c_1 \geq c_2 \geq \cdots \geq c_w \geq 1, \;\text{ and }\; c_{w+1} = \cdots = c_b = 0, \\ &b(\vec{x}) = \sum_{i=1}^{b} c_i = c_1 + \cdots + c_w \;\text{ and }\; c(\vec{x}) = c_1. \end{aligned} \tag{11}$$

For fixed $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$, take any m, such that $m \equiv 0$ or $3 \pmod 4$ and consider the set $\mathcal{A}(m)$. For any $\vec{x} \in \mathcal{A}(m)$, $\vec{x}$ written as in (11), and assuming (and abusing notation) $w = w(\vec{x})$, $b = b(\vec{x})$ and $c = c(\vec{x})$, we define the set $\mathcal{U}_n(m, \beta, \delta)$ as

$$\{\vec{x} \in \mathcal{A}(m) \mid b + c + \delta \leq n, \;\; c_w \geq \beta, \text{ and } \; c_j \geq c_{j+1} + \delta, \text{ for } 1 \leq j \leq w-1\}. \tag{12}$$

Lemma 2.14. *Let $n \in \mathbb{N}$, and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. If $m \geq n^2$ then*

$$\mathcal{U}_n(m, \beta, \delta) = \emptyset.$$

Proof. Suppose $m \geq n^2$ and let $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$. Writing $\vec{x} = (c_1, c_2, \ldots, c_w)$, we have

$$m = (c_1 + c_2 + \cdots + c_w)^2 + 2(c_1^2 + c_2^2 + \cdots + c_w^2). \tag{13}$$

Since $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$, we must have $(c_1 + c_2 + \cdots + c_w) + c_1 + \delta \leq n$, and then

$$(c_1 + c_2 + \cdots + c_w)^2 + 2(c_1 + \delta)(c_1 + c_2 + \cdots + c_w) + (c_1 + \delta)^2 \leq n^2.$$

Since

$$(c_1 + c_2 + \cdots + c_w)c_1 \geq (c_1^2 + c_2^2 + \cdots + c_w^2)$$

we must have (see (13)) $m < n^2$, a contradiction. Therefore the set $\mathcal{U}_n(m, \beta, \delta)$ must be empty.

3. Sets of Two-Line Matrices

Let $M \in \mathbb{M}(n, \beta, \delta)$ (see (1) and (2))

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s-1} & c_s \\ d_1 & d_2 & \cdots & d_{s-1} & d_s \end{pmatrix}, \tag{14}$$

and define

$$\ell(M) = (c_1 + d_1) + \cdots + (c_s + d_s) = n. \tag{15}$$

From this point onwards, we will always consider that all matrices in the set $\mathbb{M}(n, \beta, \delta)$ are written as in (14), so we will refer to the entries of a matrix $M \in \mathbb{M}$ in terms of c_j's and d_j's.

Definition 3.1. *Let $n \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. Define*

$$\mathbb{M}_0(n, \beta, \delta) = \{M \in \mathbb{M}(n, \beta, \delta) \mid d_1 = 0\}. \tag{16}$$

Lemma 3.2. *Let $M \in \mathbb{M}_0(n, \beta, \delta)$, then*

(i) $c_{s-1} \geq \beta + \delta$, *and* $c_j \geq c_{j+1} + \delta$, *for* $1 \leq j \leq s - 2$;

(ii) $\ell(M) = 2c_1 + c_2 + \cdots + c_{s-1} - (s-1)\delta$;

Proof. The first statement follows from (2), since $d_j \in \mathbb{N} \cup \{0\}$ for $j = 2, \ldots, s$ (we are assuming $d_1 = 0$). Again from (2), we have $d_{j+1} = c_j - c_{j+1} - \delta$, for $j = 1, \ldots s - 1$, thus (see (15))

$$\ell(M) = \sum_{j=1}^{s-1} c_j + \beta + \sum_{j=2}^{s} d_j = \sum_{j=1}^{s-1} c_j + c_1 - (s-1)\delta.$$

Remark 3.3. *The matrix* $\begin{pmatrix} \beta \\ n - \beta \end{pmatrix}$ *is the only one-column matrix in the set* $\mathbb{M}(n, \beta, \delta)$, *and since $n > \beta$, this matrix does not belong to $\mathbb{M}_0(k, \beta, \delta)$, for any k.*

Furthermore, observe that if $i \neq j$ then $\mathbb{M}_0(i, \beta, \delta) \cap \mathbb{M}_0(j, \beta, \delta) = \emptyset$, otherwise we would have a matrix M such that $\ell(M) = i$ and $\ell(M) = j$, which is impossible.

Lemma 3.4. *Let $n \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. With the notation of Definition 3.1, we have*

$$|\,\mathbb{M}(n, \beta, \delta)\,| \;=\; \sum_{j=1}^{n} |\,\mathbb{M}_0(j, \beta, \delta)\,| + 1.$$

Proof. Let $\mathbb{M}^*(n, \beta, \delta)$ be the subset of $\mathbb{M}(n, \beta, \delta)$ of all matrices with at least two columns. In order to complete this proof we present the following 1-1 correspondence between $\mathbb{M}^*(n, \beta, \delta)$ and the disjoint union $\bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta)$:

$$\begin{array}{ccc} M & \longleftrightarrow & M_0 \\ \begin{pmatrix} c_1 & c_2 & \cdots & c_{s-1} & c_s \\ d_1 & d_2 & \cdots & d_{s-1} & d_s \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} c_1 & c_2 & \cdots & c_{s-1} & c_s \\ 0 & d_2 & \cdots & d_{s-1} & d_s \end{pmatrix}. \end{array}$$

Since $\ell(M) = n$ then $\ell(M_0) \leq n$, hence $M_0 \in \bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta)$. On the other hand given any $M_0 \in \bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta)$, we can find $d_1 \in \mathbb{N} \cup \{0\}$, such that $\ell(M_0) + d_1 = n$, and determine the matrix $M \in \mathbb{M}^*(n, \beta, \delta)$. Now the result follows from the fact that $\mathbb{M}^*(n, \beta, \delta) = \mathbb{M}(n, \beta, \delta) - 1$, according to Remark 3.3.

The next theorem establishes an 1-1 correspondence between subsets of $\mathcal{A}(m)$ and subsets of $\mathbb{M}(n, \beta, \delta)$.

Theorem 3.5. *Let $n \in \mathbb{N}$ and $\beta, \delta \in \mathbb{N} \cup \{0\}$, with $n > \beta$. There exits an 1-1 correspondence between vectors $\vec{x}$ in $\bigcup_{m=1}^{n^2-1} \mathcal{U}_n(m, \beta, \delta)$ and two-line matrices M in $\bigcup_{j=1}^{n} \mathbb{M}_0(j, \beta, \delta)$.*

Proof. Let $m \in [1,\, n^2 - 1]$ (see Lemma 2.14). For any $\vec{x} \in \mathcal{U}_n(m, \beta, \delta)$, (see (11) and (12)) we associate the $2 \times (w + 1)$ matrix

$$M(\vec{x}) = \begin{pmatrix} (c_1 + \delta) & (c_2 + \delta) & \cdots & (c_w + \delta) & \beta \\ 0 & d_2 & \cdots & d_w & d_{w+1} \end{pmatrix},$$

where $d_{w+1} = c_w - \beta$, and $d_j = c_j - c_{j-1} - \delta$, for $2 \leq j \leq w$. It follows from the definition of the set $\mathcal{U}_n(m, \beta, \delta)$ that the $d_j \in \mathbb{N} \cup \{0\}$, for $j = 2, \ldots, w + 1$, and (see (12))

$$\begin{aligned} \ell(M(\vec{x})) \;\; &= \textstyle\sum_{i=1}^{w} (c_i + \delta) + \beta + \sum_{j=2}^{w+1} d_j \\ &= 2c_1 + c_2 + \cdots + c_w + \delta = b(\vec{x}) + c(\vec{x}) + \delta \leq n. \end{aligned}$$

Hence $M(\vec{x}) \in \mathbb{M}_0(\ell_x, \beta, \delta)$, with $\ell_x = \ell(M(\vec{x})) \leq n$ (see (16)).

Now take a matrix $M \in \mathbb{M}_0(r, \beta, \delta)$, with $r \leq n$,

$$M = \begin{pmatrix} c_1 & c_2 & \cdots & c_s & \beta \\ 0 & d_2 & \cdots & d_s & d_{s+1} \end{pmatrix},$$

and define $\vec{x}_M = (c_1^*, \ldots, c_s^*) = ((c_1 - \delta), \ldots, (c_s - \delta))$. According to Lemma 3.2 we have, for any $j = 1, \ldots, s-1$,

$$c_j \geq c_{j+1} + \delta \implies c_j^* \geq c_{j+1}^* + \delta \quad \text{and} \quad c_s \geq \beta + \delta \implies c_s^* \geq \beta,$$

and

$$\begin{aligned} \ell(M) &= 2c_1 + c_2 + \cdots + c_s - s\delta \\ &= (c_1 - \delta) + (c_1 - \delta) + (c_2 - \delta) + \cdots + (c_s - \delta) + \delta \\ &= b(\vec{x}) + c_1^* + \delta \ = r \ \leq n. \end{aligned} \tag{17}$$

Now define

$$m_M = (c_1^* + \cdots + c_s^*)^2 + 2((c_1^*)^2 + \cdots + (c_s^*)^2). \tag{18}$$

We want to prove that $\vec{x}_M \in \mathbb{U}(m_M, \beta, \delta)$, and the only thing left to be proved is that $m_M \in [1, n^2 - 1]$. Observe that (see (17))

$$\begin{aligned} n^2 \geq \ell(M)^2 = & \ (b(\vec{x}_M) + c_1^* + \delta)^2, \\ = & \ (c_1^* + \cdots + c_s^*)^2 + 2(c_1^* + \delta)(c_1^* + \cdots + c_s^*) + (c_1^* + \delta)^2 \\ > & \ (c_1^* + \cdots + c_s^*)^2 + 2((c_1^*)^2 + \cdots + (c_s^*)^2) = m_M, \end{aligned}$$

since $c_1^* \geq c_2^* \geq \cdots \geq c_s^*$, completing the proof.

Corollary 3.6. *Under the same hypothesis of Theorem 3.5 we have*

$$\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m, \beta, \delta)| = \sum_{j=1}^{n} |\mathbb{M}_0(j, \beta, \delta)|.$$

Proof. It is a straight forward consequence of Theorem 3.5, since these sets are all disjoint.

4. Main Theorem

Now we are ready to state and prove our main Theorem presenting new formulas for the number of partitions of n into distinct parts, and the partitions of n arising from the two classic Rogers-Ramanujan Identities.

Theorem 4.1. *Let n be a natural number. Then*

(a) The number of partitions of n into distinct parts is equal to $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m, 1, 1)|$ *+ 1.*

(b) *The number of partitions of* n *where the difference between two parts is at least two is equal to* $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,2,1)| + 1$.

(c) *The number of partitions of* n *where the difference between two parts is at least two and each part is greater than one is equal to* $\sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,2,2)| + 1$.

Proof. By the definition of $\mathbb{M}(n,\beta,\delta)$, we have that the number of partitions of n into distinct parts is equal to $|\mathbb{M}(n,1,1)|$, the number of partitions of n where the difference between two parts is at least two is equal to $|\mathbb{M}(n,2,1)|$, and the number of partitions of n where the difference between two parts is at least two and each part is greater than one is equal to $|\mathbb{M}(n,2,2)|$. Now the conclusion follows from Lemma 3.4 and Corollary 3.6, since

$$|\mathbb{M}(n,\beta,\delta)| = \sum_{j=1}^{n} |\mathbb{M}_0(j,\beta,\delta)| + 1 = \sum_{m=1}^{n^2-1} |\mathcal{U}_n(m,\beta,\delta)| + 1.$$

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