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INTEGRAL REPRESENTATIONS OF G-FUNCTIONS THROUGH AN INTERESTING IDEA

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Abstract: Some fifteen G-functions are given two integral representations and a series form each through an interesting idea of constructing statistical distributions of products and ratios of statistically independently distributed real positive scalar random variables which are equivalent to computing the Mellin convolutions of products and ratios involving two functions. These integral representations and the explicit series forms may not be available in the literature. For special values of the parameters some integral representations and series forms are available in the literature for some of the G-functions considered in this paper. But the representations given in this paper are for general parameters.

Keywords and Phrases: G-functions, integral representations, statistical distributions, distributions of products and ratios, Mellin convolutions, series forms.

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1. Introduction

In this paper, real scalar mathematical as well as random variables, will be denoted by small letters $x, y, ..., x_1, x_2, ...$ Let $x_1 > 0, x_2 > 0$ be real scalar positive random variables, statistically independently distributed with density functions $f_1(x_1)$ and $f_2(x_2)$ respectively so that the joint density is $f_1(x_1)f_2(x_2)$. A density function $f(x)$ is a real-valued scalar function of x such that $f(x) \geq 0$ for all x and $\int_x f(x)\mathrm{d}x = 1$. Let $u = x_1x_2$ the product. Then the density of u, denoted by $g(u)$, is available by considering the one to one transformation $u = x_1x_2, v = x_1$

or $u = x_1x_2, v = x_2$ so that $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u \wedge \mathrm{d}v$ where the wedge product of differentials is defined as $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$ so that $\mathrm{d}x \wedge \mathrm{d}x = 0, \mathrm{d}y \wedge \mathrm{d}y = 0$ where x and y are real scalar variables. For Jacobians of transformations, involving matrix transformations, wedge product etc see Mathai (1997). Then the density $g(u)$ has the following integral representations:

$$g(u) = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v})\mathrm{d}v \tag{1.1}$$

$$= \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v)\mathrm{d}v. \tag{1.2}$$

Let us evaluate (1.1) and (1.2) for the following basic densities:

$$f_1(x) = \frac{a^\gamma}{\Gamma(\gamma)} x^{\gamma-1}\mathrm{e}^{-ax}, a > 0, \gamma > 0, 0 \leq x < \infty \tag{1.3}$$

$$f_2(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 \leq x \leq 1, \alpha > 0, \beta > 0 \tag{1.4}$$

$$f_3(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0 \leq x < \infty, \alpha > 0, \beta > 0. \tag{1.5}$$

In statistical densities the parameters are usually real but the various results in this paper will hold for complex parameters also, the conditions are to be changed correspondingly. When $x_1 > 0$ and $x_2 > 0$ are independently distributed then the $(s-1)$th moment of u is the following, where $E(\cdot)$ denotes the expected value of $(\cdot)$:

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{s-1}) \text{ due to independence} \tag{1.6}$$

where, for example,

$$E(x_1^{s-1}) = \int_0^\infty x_1^{s-1} f_1(x)\mathrm{d}x = M_{f_1}(s) = \text{ Mellin transform of } f_1 \tag{1.7}$$

with Mellin parameter s and

$$E(x_2^{s-1}) = \int_0^\infty x_2^{s-1} f_2(x)\mathrm{d}x = M_{f_2}(s) \tag{1.8}$$

whenever the Mellin transforms or the $(s-1)$th moments exist. If we take the Mellin transform of $g(u)$, with Mellin parameter s, then we have

$$M_g(s) = M_{f_1}(s)M_{f_2}(s). \tag{1.9}$$

This is the Mellin convolution of a product property involving two functions.

The series representations in Sections 2 and 3 are obtained by using the following results which will be stated here as lemmas.

Lemma 1.1. *For* $m = \pm\lambda, \lambda = 1, 2, ...$

$$\lim_{\alpha\to -m}(\alpha+m)\Gamma(\alpha+m) = \lim_{\alpha\to -m}\frac{(\alpha+m)(\alpha+m-1)...\alpha\Gamma(\alpha+m)}{(\alpha+m-1)...\alpha}$$
$$= \lim_{\alpha\to -m}\frac{\Gamma(\alpha+m+1)}{(\alpha+m-1)...\alpha} = \frac{(-1)^m}{m!}. \tag{1.10}$$

Lemma 1.2. *For* $m = 1, 2, ..., \alpha \neq -\lambda, \lambda = 0, 1, 2, ...$

$$\Gamma(\alpha - m) = \frac{(-1)^m\Gamma(\alpha)}{(-\alpha+1)_m} \tag{1.11}$$

where the Pochhammer symbol is

$$(a)_m = a(a+1)...(a+m-1), a \neq 0, (a)_0 = 1. \tag{1.12}$$

2. Integral Representations of G-functions Through Mellin Convolutions of Products

Case 2.1: (1.3) versus (1.3). Let $u = x_1x_2$ where x_1 and x_2 be statistically independently distributed real scalar random variables with x_1 having the density in (1.3) with the parameters $(a_1 > 0, \gamma_1 > 0)$ and x_2 having the density in (1.3) with the parameters $(a_2 > 0, \gamma_2 > 0)$. Then from (1.1) and (1.2) the density of u, denoted by $g(u)$, is the following:

$$g(u) = c\int_v \frac{1}{v}v^{\gamma_1-1}\mathrm{e}^{-a_1v}(\frac{u}{v})^{\gamma_2-1}\mathrm{e}^{-a_2\frac{u}{v}}\mathrm{d}v, c = \frac{(a_1^{\gamma_1}a_2^{\gamma_2})}{\Gamma(\gamma_1)\Gamma(\gamma_2)}$$
$$= cu^{\gamma_2-1}\int_v v^{\gamma_1-\gamma_2-1}\mathrm{e}^{-a_1v-a_2\frac{u}{v}}\mathrm{d}v \tag{2.1}$$
$$= c\int_v (\frac{u}{v})^{\gamma_1-1}\mathrm{e}^{-a_1\frac{u}{v}}v^{\gamma_2-1}\mathrm{e}^{-a_2v}\mathrm{d}v$$
$$= cu^{\gamma_1-1}\int_v v^{\gamma_2-\gamma_1-1}\mathrm{e}^{-a_1\frac{u}{v}-a_2v}\mathrm{d}v. \tag{2.2}$$

But from (1.6)

$$\begin{aligned}E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{s-1})\\ &= a_1^{-(s-1)}\frac{\Gamma(\gamma_1+s-1)}{\Gamma(\gamma_1)}a_2^{-(s-1)}\frac{\Gamma(\gamma_2+s-1)}{\Gamma(\gamma_2)}\\ &= \frac{a_1a_2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\Gamma(\gamma_1+s-1)\Gamma(\gamma_2+s-1)(a_1a_2)^{-s}\end{aligned}$$

for $a_j>0, \Re(\Gamma_j+s-1)>0, j=1,2$, where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Then from the inverse Mellin transform, the density $g(u)$ is the following:

$$g(u) = \frac{a_1a_2}{\Gamma(\gamma_1)\Gamma(\gamma_2)}G_{0,2}^{2,0}\left[a_1a_2u\big|_{\gamma_1-1,\gamma_2-1}\right] \tag{2.3}$$

where

$$G_{0,2}^{2,0}\left[a_1a_2u\big|_{\gamma_1-1,\gamma_2-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\gamma_1+s-1)\Gamma(\gamma_2+s-1)u^{-s}\mathrm{d}s$$

with $a_j>0, \gamma_j>0, j=1,2$. A general G-function is defined and denoted as follows:

$$\begin{aligned}G(z) = G_{p,q}^{m,n}(z) &= G_{p,q}^{m,n}\left[z\big|_{b_1,\dots,b_q}^{a_1,\dots,a_p}\right]\\ &= \frac{1}{2\pi i}\int_L \frac{\{\prod_{j=1}^m\Gamma(b_j+s)\}\{\prod_{j=1}^n\Gamma(1-a_j-s)\}}{\{\prod_{j=m+1}^q\Gamma(1-b_j-s)\}\{\prod_{j=n+1}^p\Gamma(a_j+s)\}}z^{-s}\mathrm{d}s\end{aligned} \tag{2.4}$$

where $i=\sqrt{(-1)}$ the b_j's and a_j's are complex parameters and the poles of $\prod_{j=1}^m\Gamma(b_j+s)$ are separated from those of $\prod_{j=1}^n\Gamma(1-a_j-s)$ to the left and right of the contour L. Types of contour L are described and the conditions of existence of the G-function are given in detail in Mathai (1993) and hence the details are not given here.

For $\gamma_1-\gamma_2\neq\pm\lambda, \lambda=0,1,2,\ldots$ the poles of the integrand are simple and in this case the sum of the residues at the poles of $\Gamma(\gamma_1+s-1)$ is the following:

$$\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\Gamma(\gamma_2-\gamma_1-\nu)(a_1a_2u)^{\gamma_1-1+\nu} = \Gamma(\gamma_2-\gamma_1)(a_1a_2u)^{\gamma_1-1}{}_0F_1(\,;\gamma_1-\gamma_2+1;a_1a_2u).$$

We have a similar series for poles of $\Gamma(\gamma_2+s-1)$. Hence

$$\begin{aligned}G_{0,2}^{2,0}\left[a_1a_2u\big|_{\gamma_1-1,\gamma_2-1}\right] &= \Gamma(\gamma_2-\gamma_1)(a_1a_2u)^{\gamma_1-1}{}_0F_1(\,;\gamma_1-\gamma_2+1;a_1a_2u)\\ &+ \Gamma(\gamma_1-\gamma_2)(a_1a_2u)^{\gamma_2-1}{}_0F_1(\,;\gamma_2-\gamma_1+1;a_1a_2u).\end{aligned} \tag{2.5}$$

Hence we have the following result:

Theorem 2.1. *For* $\gamma_1-\gamma_2\neq\pm\lambda,\lambda=0,1,2,...,\gamma_j>0,a_j>0,j=1,2$ *the G-function*

$$\begin{aligned}G_{0,2}^{2,0}\left[a_1a_2u\big|_{\gamma_1-1,\gamma_2-1}\right]&=\Gamma(\gamma_2-\gamma_1)(a_1a_2u)^{\gamma_1-1}{}_0F_1(\ ;\gamma_1-\gamma_2+1;a_1a_2u)\\&+\Gamma(\gamma_1-\gamma_2)(a_1a_2u)^{\gamma_2-1}{}_0F_1(\ ;\gamma_2-\gamma_1+1;a_1a_2u)\\&=a_1^{\gamma_1-1}a_2^{\gamma_2-1}u^{\gamma_2-1}\int_v v^{\gamma_1-\gamma_2-1}\mathrm{e}^{-a_1v-a_2\frac{u}{v}}\mathrm{d}v\\&=a_1^{\gamma_1-1}a_2^{\gamma_2-1}u^{\gamma_1-1}\int_v v^{\gamma_2-\gamma_1-1}\mathrm{e}^{-a_1\frac{u}{v}-a_2v}\mathrm{d}v.\end{aligned}\tag{2.6}$$

Note that the integral representations in (2.1) and (2.2) are also connected to various problems in different areas. It is a basic Krätzel integral and Krätzel transforms are associated with it, see Krätzel (1979). The integrand there, normalized is the inverse Gaussian density. A slightly generalized form of (2.1) is the reaction-rate probability integral in nuclear reaction-rate theory, see Mathai and Haubold (1988). The form in (2.1) is also connected to the unconditional density in Bayesian analysis when the conditional and marginal densities belong to exponential densities. The integrals in (2.1) and (2.2) are also called generalized gamma integral, ultra gamma integral, Kobayashi integral, Bessel integral etc by various such names in different areas. But the series representation in (2.5) shows that it is a Bessel series in the simple poles case and hence the name Bessel integral is the most appropriate name to call the integrals in (2.1) and (2.2). When poles are of higher order, some poles are of order 1 and other poles are of order 2, one gets the logarithmic series for Bessel functions, involving logarithms, gamma and psi functions, the details for constructing such series when poles of the integrand are of general types may be seen from Mathai (1993). For some special parameters, the series form in (2.5) is available in Mathai (1993) but (2.5) here is for general parameters. Some properties of Mellin convolutions of products and ratios may be seen from Mathai (2018).

Case 2.2: (1.3) versus (1.4). Let $u=x_1x_2$ where $x_1>0$ and $x_2>0$ be independently distributed real scalar random variables with x_1 having the density in (1.3) with parameters $(a>0,\gamma>0)$ and x_2 having the density in (1.4) with parameters $(\alpha>0,\beta>0)$. Let $g(u)$ be the density of u. In all cases to be discussed in this section, we will use the same notations $x_1,x_2,u,g(u)$ and c in all situations in order to avoid multiplicity of notations. The results will be clear from the contexts. Then, following through the same procedure as above we have the following integral representations for $g(u)$:

$$\begin{aligned} g(u) &= c\int_v \frac{1}{v}v^{\gamma-1}\mathrm{e}^{-av}(\frac{u}{v})^{\alpha-1}(1-\frac{u}{v})^{\beta-1}\mathrm{d}v, c=\frac{a^{\gamma}}{\Gamma(\gamma)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= cu^{\alpha-1}\int_v v^{\gamma-\alpha-1}\mathrm{e}^{-av}(1-\frac{u}{v})^{\beta-1}\mathrm{d}v \qquad (2.7) \\ &= c\int_v \frac{1}{v}(\frac{u}{v})^{\gamma-1}\mathrm{e}^{-a\frac{u}{v}}v^{\alpha-1}(1-v)^{\beta-1}\mathrm{d}v \\ &= cu^{\gamma-1}\int_v v^{\alpha-\gamma-1}\mathrm{e}^{-a\frac{u}{v}}(1-v)^{\beta-1}\mathrm{d}v. \qquad (2.8) \end{aligned}$$

The integral in (2.7), when one density is the type-1 beta density and the other is an arbitrary density, can be shown to be connected to Erdélyi-Kober fractional integral of order α of the second kind, see Mathai (2014).

$$\begin{aligned} E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{s-1}) \text{ due to independence} \\ &= a^{-(s-1)}\frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)}\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+s-1)}, \end{aligned}$$

for $\Re(\alpha+s-1)>0, \Re(\gamma+s-1)>0$. Then

$$M_g(s) = M_{f_1}(s)M_{f_2}(s).$$

Therefore

$$g(u) = \frac{a\Gamma(\alpha+\beta)}{\Gamma(\gamma)\Gamma(\alpha)}G_{1,2}^{2,0}\left[au\Big|_{\gamma-1,\alpha-1}^{\alpha+\beta-1}\right]. \qquad (2.9)$$

where

$$G_{1,2}^{2,0}\left[au\Big|_{\gamma-1,\alpha-1}^{\alpha+\beta-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\gamma+s-1)\Gamma(\alpha+s-1)}{\Gamma(\alpha+\beta+s-1)}(au)^{-s}\mathrm{d}s.$$

For $\gamma-\alpha \neq \pm\lambda, \lambda = 0,1,2,...$ the poles of the integrand are simple and in this case the sum of the residues at the poles of $\Gamma(\gamma+s-1)$ and at the poles of $\Gamma(\alpha+s-1)$

are the following:

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(au)^{\gamma-1+\nu}\frac{\Gamma(\alpha-\gamma-\nu)}{\Gamma(\alpha+\beta-\gamma-\nu)}$$
$$=(au)^{\gamma-1}\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)}{}_1F_1(\gamma-\alpha-\beta+1;\gamma-\alpha+1;-au),0<u<\infty$$
$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(au)^{\alpha-1+\nu}\frac{\Gamma(\gamma-\alpha-\nu)}{\Gamma(\beta-\nu)}$$
$$=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}(au)^{\alpha-1}{}_1F_1(1-\beta;1+\alpha-\gamma;-au),0<u<\infty.$$

$$G_{1,2}^{2,0}\left[au\Big|_{\gamma-1,\alpha-1}^{\alpha+\beta-1}\right]=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)}(au)^{\gamma-1}{}_1F_1(\gamma-\alpha-\beta+1;\gamma-\alpha+1;-au)$$
$$+\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}(au)^{\alpha-1}{}_1F_1(1-\beta:1+\alpha-\gamma;-au),0<u<\infty. \tag{2.10}$$

Hence we have the following result:

Theorem 2.2. *For* $\gamma-\alpha\neq\pm\lambda,\lambda=0,1,2,...,a>0,\alpha>0,\beta>0,\gamma>0$ *the G-function*

$$G_{1,2}^{2,0}\left[au\Big|_{\gamma-1,\alpha-1}^{\alpha+\beta-1}\right]=\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)}(au)^{\gamma-1}{}_1F_1(\gamma-\alpha-\beta+1;\gamma-\alpha+1;-au)$$
$$+\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}(au)^{\alpha-1}{}_1F_1(1-\beta:1+\alpha-\gamma;-au),0<u<\infty$$
$$=\frac{a^{\gamma-1}}{\Gamma(\beta)}u^{\alpha-1}\int_v v^{\gamma-\alpha-1}\mathrm{e}^{-av}(1-\frac{u}{v})^{\beta-1}\mathrm{d}v$$
$$=\frac{a^{\gamma-1}}{\Gamma(\beta)}u^{\gamma-1}\int_v v^{\alpha-\gamma-1}\mathrm{e}^{-a\frac{u}{v}}(1-v)^{\beta-1}\mathrm{d}v. \tag{2.11}$$

Case 2.3. (1.3) versus (1.5). Again let $u=x_1x_2$ where $x_1>0$ and $x_2>0$ be independently distributed real scalar random variables with x_1 having the density in (1.3) with the parameters $(a>0,\gamma>0)$ and x_2 having the density in (1.5) with the parameters $(\alpha>0,\beta>0)$. Let $g(u)$ again denote the density of u. Then $g(u)$

has the following integral representations:

$$\begin{aligned} g(u) &= c\int_v \frac{1}{v}v^{\gamma-1}\mathrm{e}^{-av}(\frac{u}{v})^{\alpha-1}(1+\frac{u}{v})^{-(\alpha+\beta)}\mathrm{d}v, c=\frac{a^{\gamma}}{\Gamma(\gamma)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= cu^{\alpha-1}\int_v v^{\gamma-\alpha-1}\mathrm{e}^{-av}(1+\frac{u}{v})^{-(\alpha+\beta)}\mathrm{d}v \qquad (2.12) \\ &= c\int_v \frac{1}{v}(\frac{u}{v})^{\gamma-1}\mathrm{e}^{-a\frac{u}{v}}v^{\alpha-1}(1+v)^{-(\alpha+\beta)}\mathrm{d}v \\ &= cu^{\gamma-1}\int_v v^{\alpha-\gamma-1}\mathrm{e}^{-a\frac{u}{v}}(1+v)^{-(\alpha+\beta)}\mathrm{d}v. \qquad (2.13) \end{aligned}$$

Now,

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{s-1}) = a^{-(s-1)}\frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)}\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\beta-s+1)}{\Gamma(\beta)}$$

for $\Re(\gamma+s-1)>0, \Re(\alpha+s-1)>0, \Re(\beta-s+1)>0$. Therefore

$$g(u) = \frac{a}{\Gamma(\alpha)\Gamma(\gamma)\Gamma(\beta)}G_{1,2}^{2,1}\left[au\Big|_{\gamma-1,\alpha-1}^{-\beta}\right] \qquad (2.14)$$

where

$$G_{1,2}^{2,1}\left[au\Big|_{\gamma-1,\alpha-1}^{-\beta}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\gamma+s-1)\Gamma(\alpha+s-1)\Gamma(\beta-s+1)(au)^{-s}\mathrm{d}s.$$

For $\gamma-\alpha \neq \pm\lambda, \lambda = 0,1,2,...$ the poles of $\Gamma(\gamma+s-1)\Gamma(\alpha+s-1)$ are simple. Then the sum of the residues at the poles of $\Gamma(\gamma+s-1)$ and $\Gamma(\alpha+s-1)$ are the following:

$$\begin{aligned} &\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\alpha-\gamma-\nu)\Gamma(\beta+\gamma+\nu)(au)^{\gamma-1+\nu} \\ &\quad = \Gamma(\alpha-\gamma)\Gamma(\beta+\gamma)(au)^{\gamma-1}{}_1F_1(\gamma+\beta;\gamma-\alpha+1;au) \\ &\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(au)^{\alpha-1+\nu}\Gamma(\gamma-\alpha-\nu)\Gamma(\alpha+\beta+\nu) \\ &\quad = \Gamma(\gamma-\alpha)\Gamma(\alpha+\beta)(au)^{\alpha-1}{}_1F_1(\alpha+\beta;\alpha-\gamma+1;au). \end{aligned}$$

The sum of these two ${}_1F_1$'s is the value of the G-function for $0<u<\infty$. The continuation part gives a divergent series. That is,

$$\begin{aligned} G_{1,2}^{2,1}\left[au\Big|_{\gamma-1,\alpha-1}^{-\beta}\right] &= \Gamma(\alpha-\gamma)\Gamma(\beta+\gamma)(au)^{\gamma-1}{}_1F_1(\gamma+\beta;\gamma-\alpha+1;au) \\ &+\Gamma(\gamma-\alpha)\Gamma(\alpha+\beta)(au)^{\alpha-1}{}_1F_1(\alpha+\beta;\alpha-\gamma+1;au). \qquad (2.15) \end{aligned}$$

When the poles of the integrand are not simple, then some of the poles will be of order 1 and the remaining are of order two. Then we have the logarithmic series, containing logarithms, gamma and psi functions. We have the following result for the case of simple poles:

Theorem 2.3. *For $\gamma-\alpha\neq\pm\lambda,\lambda=0,1,2,...,a>0,\alpha>0,\beta>0,\gamma>0$ the G-function*

$$\begin{aligned}
G_{1,2}^{2,1}\left[au\big|_{\gamma-1,\alpha-1}^{-\beta}\right] &= \Gamma(\alpha-\gamma)\Gamma(\beta+\gamma)(au)^{\gamma-1}{}_1F_1(\gamma+\beta;\gamma-\alpha+1;au)\\
&+\Gamma(\gamma-\alpha)\Gamma(\alpha+\beta)(au)^{\alpha-1}{}_1F_1(\alpha+\beta;\alpha-\gamma+1;au)\\
&= a^{\gamma-1}\Gamma(\alpha+\beta)u^{\alpha-1}\int_v v^{\gamma-\alpha-1}\mathrm{e}^{-av}(1+\frac{u}{v})^{-(\alpha+\beta)}\mathrm{d}v\\
&= a^{\gamma-1}\Gamma(\alpha+\beta)u^{\gamma-1}\int_v v^{\alpha-\gamma-1}\mathrm{e}^{-\frac{u}{v}}(1+v)^{-(\alpha+\beta)}\mathrm{d}v. \qquad (2.16)
\end{aligned}$$

Case 2.4: (1.4) versus (1.4). Again let $u=x_1x_2$ where $x_1>0$ and $x_2>0$ be real scalar random variables, independently distributed with x_1 having the density in (1.4) with the parameters $(\alpha_1>0,\beta_1>0)$ and x_2 having the density in (1.4) with the parameters $(\alpha_2>0,\beta_2>0)$ respectively. Again let $g(u)$ denote the density of u. Then we have the following integral representations:

$$\begin{aligned}
g(u) &= c\int_v \frac{1}{v}v^{\alpha_1-1}(1-v)^{\beta_1-1}(\frac{u}{v})^{\alpha_2-1}(1-\frac{u}{v})^{\beta_2-1}\mathrm{d}v, c=\{\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}\}\\
&= cu^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1-v)^{\beta_1-1}(1-\frac{u}{v})^{\beta_2-1}\mathrm{d}v \qquad (2.17)\\
&= c\int_v \frac{1}{v}(\frac{u}{v})^{\alpha_1-1}(1-\frac{u}{v})^{\beta_1-1}v^{\alpha_2-1}(1-v)^{\beta_2-1}\mathrm{d}v\\
&= cu^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1-v)^{\beta_2-1}(1-\frac{u}{v})^{\beta_1-1}\mathrm{d}v. \qquad (2.18)
\end{aligned}$$

$$\begin{aligned}
E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{s-1})\\
&= \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)}\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_1+s-1)}\frac{\Gamma(\alpha_2+s-1)}{\Gamma(\alpha_2)}\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2+\beta_2+s-1)}
\end{aligned}$$

for $\Re(\alpha_j+s-1)>0, j=1,2$. Then

$$g(u)=\frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}G_{2,2}^{2,0}\left[u\big|_{\alpha_1-1,\alpha_2-1}^{\alpha_1+\beta_1-1,\alpha_2+\beta_2-1}\right] \qquad (2.19)$$

where

$$G_{2,2}^{2,0}\left[u\Big|_{\alpha_1-1,\alpha_2-1}^{\alpha_1+\beta_1-1,\alpha_2+\beta_2-1}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha_1+s-1)\Gamma(\alpha_2+s-1)}{\Gamma(\alpha_1+\beta_1+s-1)\Gamma(\alpha_2+\beta_2+s-1)}u^{-s}\mathrm{d}s.$$

For $\alpha_1-\alpha_2\neq\pm\lambda,\lambda=0,1,2,...$ the poles of the integrand are simple and in this case, going through the same procedure as before, we have

$$\begin{aligned}&G_{2,2}^{2,0}\left[u\Big|_{\alpha_1-1,\alpha_2-1}^{\alpha_1+\beta_1-1,\alpha_2+\beta_2-1}\right]\\&=\frac{\Gamma(\alpha_2-\alpha_1)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2-\alpha_1)}u^{\alpha_1-1}{}_2F_1(1-\beta_1,\alpha_1-\alpha_2-\beta_2+1;\alpha_1-\alpha_2+1;u)\\&+\frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(\alpha_1+\beta_1-\alpha_2)\Gamma(\beta_2)}u^{\alpha_2-1}{}_2F_1(1-\beta_2,\alpha_2-\alpha_1-\beta_1+1;\alpha_2-\alpha_1+1;u),0<u<1.\end{aligned}\tag{2.20}$$

Hence we have the following theorem.

Theorem 2.4. *For $\alpha_1-\alpha_2\neq\pm\lambda,\lambda=0,1,2,...,\alpha_j>0,\beta_j>0,j=1,2$ the G-function*

$$\begin{aligned}&G_{2,2}^{2,0}\left[u\Big|_{\alpha_1-1,\alpha_2-1}^{\alpha_1+\beta_1-1,\alpha_2+\beta_2-1}\right]\\&=\frac{\Gamma(\alpha_2-\alpha_1)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2-\alpha_1)}u^{\alpha_1-1}{}_2F_1(1-\beta_1,\alpha_1-\alpha_2-\beta_2+1;\alpha_1-\alpha_2+1;u)\\&+\frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(\alpha_1+\beta_1-\alpha_2)\Gamma(\beta_2)}u^{\alpha_2-1}{}_2F_1(1-\beta_2,\alpha_2-\alpha_1-\beta_1+1;\alpha_2-\alpha_1+1;u),0<u<1\\&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1-v)^{\beta_1-1}(1-\frac{u}{v})^{\beta_2-1}\mathrm{d}v\\&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1-v)^{\beta_2-1}(1-\frac{u}{v})^{\beta_1-1}\mathrm{d}v.\end{aligned}\tag{2.21}$$

Case 2.5: (1.4) versus (1.5). Again, let $u=x_1x_2$ where $x_1>0$ and $x_2>0$ be real scalar random variables, independently distributed, with x_1 having the density in (1.4) with the parameters $(\alpha_1>0,\beta_1>0)$ and x_2 having the density in (1.5) with the parameters $(\alpha_2>0,\beta_2>0)$ respectively. Again, let $g(u)$ denote the density of u. Then we have the following integral representations:

$$
\begin{aligned}
g(u) &= c\int_v \frac{1}{v} v^{\alpha_1-1}(1-v)^{\beta_1-1}(\frac{u}{v})^{\alpha_2-1}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v, c=\{\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}\} \\
&= cu^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1-v)^{\beta_1-1}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v \qquad (2.22)\\
&= c\int_v \frac{1}{v}(\frac{u}{v})^{\alpha_1-1}(1-\frac{u}{v})^{\beta_1-1}v^{\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v \\
&= cu^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1-\frac{u}{v})^{\beta_1-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v. \qquad (2.23)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{s-1}) \\
&= \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)}\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_1+s-1)}\frac{\Gamma(\alpha_2+s-1)}{\Gamma(\alpha_2)}\frac{\Gamma(\beta_2-s+1)}{\Gamma(\beta_2)}
\end{aligned}
$$

for $\Re(\alpha_1+s-1)>0, \Re(\alpha_2+s-1)>0, \Re(\beta_2-s+1)>0$. Then

$$
g(u) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2)} G_{2,2}^{2,1}\left[\big|_{\alpha_1-1,\alpha_2-1}^{-\beta_2,\alpha_1+\beta_1-1}\right] \qquad (2.24)
$$

where

$$
G_{2,2}^{2,1}\left[u\big|_{\alpha_1-1,\alpha_2-1}^{-\beta_2,\alpha_1+\beta_1-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha_1+s-1)\Gamma(\alpha_2+s-1)\Gamma(\beta_2-s+1)}{\Gamma(\alpha_1+\beta_1+s-1)}u^{-s}\mathrm{d}s.
$$

For $\alpha_1-\alpha_2 \neq \pm\lambda, \lambda=0,1,2,...$ the poles of $\Gamma(\alpha_1+s-1)\Gamma(\alpha_2+s-1)$ are simple and in this case let us evaluate the G-function. Proceeding as before, we have

$$
\begin{aligned}
&G_{2,2}^{2,1}\left[u\big|_{\alpha_1-1,\alpha_2-1}^{-\beta_2,\alpha_1+\beta_1-1}\right] \\
&= \frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_2+\alpha_1)}{\Gamma(\beta_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\beta_2, 1-\beta_1; \alpha_1-\alpha_2+1; -u) \\
&+ \frac{\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_2+\alpha_2)}{\Gamma(\alpha_1+\beta_1-\alpha_2)}u^{\alpha_2-1}{}_2F_1(\alpha_2+\beta_2, \alpha_2-\alpha_1-\beta_1+1; \alpha_2-\alpha_1+1; -u),
\end{aligned}
$$

for $0<u<1$. The continuation part is available by evaluating the G-function as the sum of the residues at the poles of $\Gamma(\beta_2-s+1)$ and it is the following:

$$
\frac{\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\beta_2)}(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2, \alpha_2+\beta_2; \alpha_1+\beta_1+\beta_2; -\frac{1}{u}), u \geq 1.
$$

Hence

$$
\begin{aligned}
&G^{2,1}_{2,2}\left[u\Big|^{-\beta_2,\alpha_1+\beta_1-1}_{\alpha_1-1,\alpha_2-1}\right]\\
&=\frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_2+\alpha_1)}{\Gamma(\beta_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\beta_2,1-\beta_1;\alpha_1-\alpha_2+1;-u)\\
&+\frac{\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_2+\alpha_2)}{\Gamma(\alpha_1+\beta_1-\alpha_2)}u^{\alpha_2-1}{}_2F_1(\alpha_2+\beta_2,\alpha_2-\alpha_1-\beta_1+1;\alpha_2-\alpha_1+1;-u),\\
&\quad 0<u<1\\
&=\frac{\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\beta_2)}(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2,\alpha_2+\beta_2;\alpha_1+\beta_1+\beta_2;-\frac{1}{u}),u\geq 1.
\end{aligned}
\tag{2.25}
$$

Hence we have the following result:

Theorem 2.5. *For* $\alpha_1-\alpha_2\neq\pm\lambda,\lambda=0,1,2,...,\alpha_j>0,\beta_j>0,j=1,2$ *the G-function*

$$
\begin{aligned}
&G^{2,1}_{2,2}\left[u\Big|^{-\beta_2,\alpha_1+\beta_1-1}_{\alpha_1-1,\alpha_2-1}\right]\\
&=\frac{\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_2+\alpha_1)}{\Gamma(\beta_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\beta_2,1-\beta_1;\alpha_1-\alpha_2+1;-u)\\
&+\frac{\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_2+\alpha_2)}{\Gamma(\alpha_1+\beta_1-\alpha_2)}u^{\alpha_2-1}{}_2F_1(\alpha_2+\beta_2,\alpha_2-\alpha_1-\beta_1+1;\alpha_2-\alpha_1+1;-u),\\
&\quad 0<u<1\\
&=\frac{\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\beta_2)}(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2,\alpha_2+\beta_2;\alpha_1+\beta_1+\beta_2;-\frac{1}{u}),u\geq 1\\
&=\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1-v)^{\beta_1-1}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v\\
&=\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1-\frac{u}{v})^{\beta_1-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v.
\end{aligned}
\tag{2.26}
$$

Case 2.6. (1.5) versus (1.5). Again let $u=x_1x_2$ where let $x_1>0$ and $x_2>0$ be independently distributed with x_1 having the density in (1.5) with the parameters $(\alpha_1>0,\beta_1>0)$ and x_2 having the density in (1.5) with the parameters $(\alpha_2>0,\beta_2>0)$. Let $g(u)$ again denote the density of u. Then $g(u)$ has the following integral representations:

$$g(u)=c\int_v \frac{1}{v}v^{\alpha_1-1}(1+v)^{-(\alpha_1+\beta_1)}(\frac{u}{v})^{\alpha_2-1}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v, c=\{\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}\}$$

$$=cu^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1+v)^{-(\alpha_1+\beta_1)}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v \quad (2.27)$$

$$=c\int_v \frac{1}{v}(\frac{u}{v})^{\alpha_1-1}(1+\frac{u}{v})^{-(\alpha_1+\beta_1)}v^{\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v$$

$$=cu^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1+\frac{u}{v})^{-(\alpha_1+\beta_1)}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v. \quad (2.28)$$

Now,

$$E(u^{s-1})=E(x_1^{s-1})E(x_2^{s-1})$$
$$=\frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)}\frac{\Gamma(\beta_1-s+1)}{\Gamma(\beta_1)}\frac{\Gamma(\alpha_2+s-1)}{\Gamma(\alpha_2)}\frac{\Gamma(\beta_2-s+1)}{\Gamma(\beta_2)}$$

for $\Re(\alpha_j+s-1)>0, \Re(\beta_j-s+1)>0, j=1,2$. Then

$$g(u)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}G_{2,2}^{2,2}\left[u\big|_{\alpha_1-1,\alpha_2-1}^{-\beta_1,-\beta_2}\right] \quad (2.29)$$

where

$$G_{2,2}^{2,2}\left[u\big|_{\alpha_1-1,\alpha_2-1}^{-\beta_1,-\beta_2}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{\prod_{j=1}^{2}\Gamma(\alpha_j+s-1)\Gamma(\beta_j-s+1)\}u^{-s}\mathrm{d}s.$$

For $\alpha_1-\alpha_2\neq\pm\lambda, \lambda=0,1,2,...$ the poles of $\Gamma(\alpha_1+s-1)\Gamma(\alpha_2+s-1)$ are simple and then evaluating the sum of the residues at the poles of $\Gamma(\alpha_1+s-1)$ and $\Gamma(\alpha_2+s-1)$ and then simplifying in terms of hypergeometric functions, we have the following:

$$\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_1+\alpha_1)\Gamma(\beta_2+\alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1+\alpha_1,\beta_2+\alpha_1;\alpha_1-\alpha_2+1;u)$$
$$+\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1+\alpha_2)\Gamma(\beta_2+\alpha_2)u^{\alpha_2-1}{}_2F_1(\beta_1+\alpha_2,\beta_2+\alpha_2;\alpha_2-\alpha_1+1;u),$$

for $0<u<1$. Similarly for $\beta_1-\beta_2\neq\pm\lambda, \lambda=0,1,2,...$ the poles of the continuation part are simple. In this simple poles case the continuation part is the following:

$$\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_1)\Gamma(\beta_2-\beta_1)(\frac{1}{u})^{\beta_1+1}{}_2F_1(\alpha_1+\beta_1,\alpha_2+\beta_1;\beta_1-\beta_2+1;\frac{1}{u})$$
$$+\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)\Gamma(\beta_1-\beta_2)(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2,\alpha_2+\beta_2;\beta_2-\beta_1+1;\frac{1}{u}), u\geq 1.$$

Hence we have the following:

$$G^{2,2}_{2,2}\left[u\Big|^{-\beta_1,-\beta_2}_{\alpha_1-1,\alpha_2-1}\right]$$
$$=\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_1+\alpha_1)\Gamma(\beta_2+\alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1+\alpha_1,\beta_2+\alpha_1;\alpha_1-\alpha_2+1;u)$$
$$+\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1+\alpha_2)\Gamma(\beta_2+\alpha_2)u^{\alpha_2-1}{}_2F_1(\beta_1+\alpha_2,\beta_2+\alpha_2;\alpha_2-\alpha_1+1;u),$$
$$0<u<1$$
$$=\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_1)\Gamma(\beta_2-\beta_1)(\frac{1}{u})^{\beta_1+1}{}_2F_1(\alpha_1+\beta_1,\alpha_2+\beta_1;\beta_1-\beta_2+1;\frac{1}{u})$$
$$+\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)\Gamma(\beta_1-\beta_2)(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2,\alpha_2+\beta_2;\beta_2-\beta_1+1;\frac{1}{u}),u\geq 1. \tag{2.30}$$

Hence we have the following result:

Theorem 2.6. *For* $\alpha_1-\alpha_2\neq\pm\lambda,\lambda=0,1,2,...,\beta_1-\beta_2\neq\pm\mu,\mu=0,1,2,...,\alpha_j>0,\beta_j>0,j=1,2$ *the G-function*

$$G^{2,2}_{2,2}\left[u\Big|^{-\beta_1,-\beta_2}_{\alpha_1-1,\alpha_2-1}\right]$$
$$=\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_1+\alpha_1)\Gamma(\beta_2+\alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1+\alpha_1,\beta_2+\alpha_1;\alpha_1-\alpha_2+1;u)$$
$$+\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1+\alpha_2)\Gamma(\beta_2+\alpha_2)u^{\alpha_2-1}{}_2F_1(\beta_1+\alpha_2,\beta_2+\alpha_2;\alpha_2-\alpha_1+1;u),$$
$$0<u<1$$
$$=\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_1)\Gamma(\beta_2-\beta_1)(\frac{1}{u})^{\beta_1+1}{}_2F_1(\alpha_1+\beta_1,\alpha_2+\beta_1;\beta_1-\beta_2+1;\frac{1}{u})$$
$$+\Gamma(\alpha_1+\beta_2)\Gamma(\alpha_2+\beta_2)\Gamma(\beta_1-\beta_2)(\frac{1}{u})^{\beta_2+1}{}_2F_1(\alpha_1+\beta_2,\alpha_2+\beta_2;\beta_2-\beta_1+1;\frac{1}{u}),u\geq 1$$
$$=\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)u^{\alpha_2-1}\int_v v^{\alpha_1-\alpha_2-1}(1+v)^{-(\alpha_1+\beta_1)}(1+\frac{u}{v})^{-(\alpha_2+\beta_2)}\mathrm{d}v$$
$$=\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)u^{\alpha_1-1}\int_v v^{\alpha_2-\alpha_1-1}(1+\frac{u}{v})^{-(\alpha_1+\beta_1)}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v. \tag{2.2.31}$$

By considering other densities, other than (1.3), (1.4), (1.5), and then looking at the Mellin convolutions of products we can derive other integral and series representations of other G-functions by using the same procedure as described in this section.

3. Integral Representations of G-functions Through Mellin Convolutions of Ratios for Two Functions

Let $u=\frac{x_1}{x_2}$ where $x_1>0$ and $x_2>0$ be real scalar random variables independently distributed with density functions $f_1(x_1)$ and $f_2(x_2)$ respectively. In this section also we will continue using the symbols u, x_1, x_2, c in order to avoid multiplicity of symbols but the meanings will be clear from the contexts. Let $x_2 = v$. Then $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = v\mathrm{d}u \wedge \mathrm{d}v$. Then the density of u, denoted by $g(u)$, is the following:

$$g(u) = \int_v v f_1(uv) f_2(v) \mathrm{d}v. \tag{3.1}$$

If $x_1 = v$ then $\mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{v}{u^2}\mathrm{d}u \wedge \mathrm{d}v$ and then $g(u)$ is the following:

$$g(u) = \int_v (\frac{v}{u^2}) f_1(v) f_2(\frac{v}{u}) \mathrm{d}v. \tag{3.2}$$

Hence we have two different integral representations for $g(u)$ as in (3.1) and (3.2). Note that

$$E(u^{s-1}) = E(x_1^{s-1}) E(x_2^{-s+1}) \text{ due to independence,}$$

whenever the expected values exist. Then in terms of Mellin transforms of g, f_1, f_2 we have

$$M_g(s) = M_{f_1}(s) M_{f_2}(2-s) \tag{3.3}$$

whenever the Mellin transforms exist. Our aim in this section is to examine (3.1) and (3.2) for the models in (1.3),(1.4),(1.5) and obtain several interesting integral representations for G-functions. Most of these integral representations do not seem to be available in the literature.

Case 3.1: (1.3) versus (1.3) Let $u = \frac{x_1}{x_2}$ where $x_1 > 0$ and $x_2 > 0$ be real scalar random variables independently distributed with x_1 having the density in (1.3) with parameters $(a_1 > 0, \gamma_1 > 0)$ and x_2 having the density in (1.3) with the parameters $(a_2 > 0, \gamma_2 > 0)$ respectively. Then

$$\begin{aligned}
g(u) &= c \int_v v(uv)^{\gamma_1 - 1} \mathrm{e}^{-a_1 uv} v^{\gamma_2 - 1} \mathrm{e}^{-a_2 v} \mathrm{d}v, c = \prod_{j=1}^{2} \frac{a_j^{\gamma_j}}{\Gamma(\gamma_j)} \\
&= c u^{\gamma_1 - 1} \int_v v^{\gamma_1 + \gamma_2 - 1} \mathrm{e}^{-a_1 uv - a_2 v} \mathrm{d}v \\
&= c\Gamma(\gamma_1 + \gamma_2) u^{\gamma_1 - 1} (a_2 + a_1 u)^{-(\gamma_1 + \gamma_2)}, 0 < u < \infty \qquad (3.4) \\
&= c \int_v (\frac{v}{u^2}) v^{\gamma_1 - 1} \mathrm{e}^{-a_1 v} (\frac{v}{u})^{\gamma_2 - 1} \mathrm{e}^{-a_2 \frac{v}{u}} \mathrm{d}v \\
&= c u^{-\gamma_2 - 1} \int_v v^{\gamma_1 + \gamma_2 - 1} \mathrm{e}^{-a_1 v - a_2 \frac{v}{u}} \mathrm{d}v \\
&= c\Gamma(\gamma_1 + \gamma_2) u^{-\gamma_2 - 1} (a_1 + \frac{a_2}{u})^{-(\gamma_1 + \gamma_2)}, 0 < u < \infty. \qquad (3.5)
\end{aligned}$$

Note that both (3.4) and (3.5) are of type-2 beta form for the same original conditions $a_j > 0, \gamma_j > 0, j = 1, 2$. By interchanging (a_1, γ_1) and (a_2, γ_2) we obtain the density of $\frac{x_2}{x_1}$ and the integral representations for its density.

$$\begin{aligned} E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{-s+1}) \text{ due to independence} \\ &= a_1^{-(s-1)}\frac{\Gamma(\gamma_1+s-1)}{\Gamma(\gamma_1)}a_2^{s-1}\frac{\Gamma(\gamma_2-s+1)}{\Gamma(\gamma_2)} \\ &= \frac{a_1}{a_2}(\frac{a_1}{a_2})^{-s}\frac{\Gamma(\gamma_1+s-1)\Gamma(\gamma_2-s+1)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \end{aligned}$$

for $\Re(\gamma_1 + s - 1) > 0, \Re(\gamma_2 - s + 1) > 0$. Hence the density $g(u)$ for the case of (1.3) versus (1.3) is the following:

$$\begin{aligned} g(u) &= (\frac{a_1}{a_2})\frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\gamma_1+s-1)\Gamma(\gamma_2-s+1)(\frac{a_1}{a_2}u)^{-s}\mathrm{d}s \\ &= (\frac{a_1}{a_2})\frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)}G_{1,1}^{1,1}\left[u\Big|_{\gamma_1-1}^{-\gamma_2}\right]. \end{aligned} \tag{3.6}$$

Evaluating the G-function as the sum of the residues at the poles of $\Gamma(\gamma_1 + s - 1)$ we have the following:

$$\begin{aligned} G_{1,1}^{1,1}\left[u\Big|_{\gamma_1-1}^{-\gamma_2}\right] &= \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}(\frac{a_1u}{a_2})^{\gamma_1-1+\nu}\Gamma(\gamma_2+\gamma_1+\nu) \\ &= (\frac{a_1u}{a_2})^{\gamma_1-1}\Gamma(\gamma_1+\gamma_2){}_1F_0(\gamma_1+\gamma_2;\ ;-\frac{a_1}{a_2}u) \\ &= (\frac{a_1}{a_2}u)^{\gamma_1-1}\Gamma(\gamma_1+\gamma_2)(1+\frac{a_1}{a_2}u)^{-(\gamma_1+\gamma_2)}, 0 < \frac{a_1}{a_2}u < 1. \end{aligned} \tag{i}$$

Since its continuation part also gives the same result the result will hold for $0 < u < \infty$. The continuation part is available by evaluating the G-function as the sum of the residues at the poles of $\Gamma(\gamma_2 - s + 1)$, which is the following:

$$\begin{aligned} G_{1,1}^{1,1}\left[u\Big|_{\gamma_1-1}^{-\gamma_2}\right] &= \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}(\frac{a_1}{a_2}u)^{-\gamma_2-1-\nu}\Gamma(\gamma_1+\gamma_2+\nu) \\ &= \Gamma(\gamma_1+\gamma_2)(\frac{a_1}{a_2}u)^{-\gamma_2-1}{}_1F_0(\gamma_1+\gamma_2;\ ;-\frac{a_2}{a_1u}) \\ &= (\frac{a_2}{a_1u})^{\gamma_2+1}(1+\frac{a_2}{a_1u})^{-(\gamma_1+\gamma_2)}, \frac{a_2}{a_1u} < 1. \end{aligned} \tag{ii}$$

Note that both the forms in (i) and (ii) above are one and the same and same as the ones available from the integral representations and also they are of type-2 beta form and hence they hold for $0 < u < \infty$. We have the following result:

Theorem 3.1. *For $a_j > 0, \gamma_j > 0, j = 1, 2$ the G-function*

$$\begin{aligned} G^{1,1}_{1,1}\left[u\big|^{-\gamma_2}_{\gamma_1-1}\right] &= (\frac{a_1}{a_2}u)^{\gamma_1-1}\Gamma(\gamma_1+\gamma_2)(1+\frac{a_1}{a_2}u)^{-(\gamma_1+\gamma_2)}, 0 < u < \infty \\ &= (\frac{a_2}{a_1 u})^{\gamma_2+1}(1+\frac{a_2}{a_1 u})^{-(\gamma_1+\gamma_2)}, 0 < u < \infty \\ &= a_1^{\gamma_1-1}a_2^{\gamma_2+1}u^{\gamma_1-1}\int_v v^{\gamma_1+\gamma_2-1}\mathrm{e}^{-a_1uv-a_2v}\mathrm{d}v, 0 < u < \infty \\ &= a_1^{\gamma_1-1}a_2^{\gamma_2+1}u^{-\gamma_2-1}\int_v v^{\gamma_1+\gamma_2-1}\mathrm{e}^{-a_1v-a_2\frac{v}{u}}\mathrm{d}v, 0 < u < \infty. \end{aligned} \tag{3.7}$$

Case 3.2: (1.3) versus (1.4). Let $u = \frac{x_1}{x_2}$ where $x_1 > 0$ and $x_2 > 0$ be independently distributed real scalar random variables, with x_1 having the density in (1.3) with the parameters $(a > 0, \gamma > 0)$ and x_2 having the density in (1.4) with the parameters $(\alpha > 0, \beta > 0)$ respectively. Let the density of u be again denoted as $g(u)$. Then $g(u)$ is the following:

$$\begin{aligned} g(u) &= c\int_v v(uv)^{\gamma-1}\mathrm{e}^{-auv}v^{\alpha-1}(1-v)^{\beta-1}\mathrm{d}v, c = \frac{a^\gamma}{\Gamma(\gamma)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\ &= cu^{\gamma-1}\int_v v^{\alpha+\gamma-1}(1-v)^{\beta-1}\mathrm{e}^{-auv}\mathrm{d}v \qquad (3.8) \\ &= c\int_v (\frac{v}{u^2})v^{\gamma-1}\mathrm{e}^{-av}(\frac{v}{u})^{\alpha-1}(1-\frac{v}{u})^{\beta-1}\mathrm{d}v \\ &= cu^{-\alpha-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-av}(1-\frac{v}{u})^{\beta-1}\mathrm{d}v. \qquad (3.9) \end{aligned}$$

$$\begin{aligned} E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{-s+1}) \\ &= a^{-(s-1)}\frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)}\frac{\Gamma(\alpha-s+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-s+1)} \end{aligned}$$

for $\Re(\gamma+s-1) > 0, \Re(\alpha-s+1) > 0$. Therefore

$$g(u) = \frac{a\Gamma(\alpha+\beta)}{\Gamma(\gamma)\Gamma(\alpha)}G^{1,1}_{1,2}\left[au\big|^{-\alpha}_{\gamma-1,-\alpha-\beta}\right] \tag{3.10}$$

where

$$G_{1,2}^{1,1}\left[au\Big|_{\gamma-1,-\alpha-\beta}^{-\alpha}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\gamma+s-1)\frac{\Gamma(\alpha-s+1)}{\Gamma(\alpha+\beta-s+1)}(au)^{-s}\mathrm{d}s.$$

Evaluating the G-function as the sum of residues at the poles of $\Gamma(\gamma+s-1)$ we have the following:

$$\begin{aligned}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(au)^{\gamma-1+\nu}&\frac{\Gamma(\alpha+\gamma+\nu)}{\Gamma(\alpha+\beta+\gamma+\nu)}\\ &=\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha+\beta+\gamma)}{}_1F_1(\alpha+\gamma;\alpha+\beta+\gamma;-au)\end{aligned}\tag{3.11}$$

for $0<u<\infty$. Hence

$$G_{1,2}^{1,1}\left[au\Big|_{\gamma-1,-\alpha-\beta}^{-\alpha}\right]=\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha+\beta+\gamma)}{}_1F_1(\alpha+\gamma;\alpha+\beta+\gamma;-au)\tag{3.12}$$

for $0<u<\infty$. The continuation is a divergent series. Hence we have the following result:

Theorem 3.2. *For $a>0,\gamma>0,\alpha>0,\beta>0$ the G-function*

$$\begin{aligned}G_{1,2}^{1,1}\left[au\Big|_{\gamma-1,-\alpha-\beta}^{-\alpha}\right]&=\frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha+\beta+\gamma)}{}_1F_1(\alpha+\gamma;\alpha+\beta+\gamma;-au),0<u<\infty\\ &=\frac{a^{\gamma-1}}{\Gamma(\beta)}u^{\gamma-1}\int_v v^{\alpha+\gamma-1}(1-v)^{\beta-1}\mathrm{e}^{-auv}\mathrm{d}v\\ &=\frac{a^{\gamma-1}}{\Gamma(\beta)}u^{-\alpha-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-av}(1-\frac{v}{u})^{\beta-1}\mathrm{d}v.\end{aligned}\tag{3.13}$$

Case 3.3: (1.4) versus (1.3). Let $u=\frac{x_1}{x_2}$ where $x_1>0$ and $x_2>0$ be independently distributed real scalar random variables with x_1 having the density in (1.4) with the parameters $(\alpha>0,\beta>0)$ and x_2 having the density in (1.3) with the parameters $(a>0,\gamma>0)$ respectively. Let $g(u)$ again denote the density of u. The properties of u in Case 3.3 are also available from the properties of $\frac{1}{u}$ in Case

3.2. We have $g(u)$ in Case 3.3 as the following:

$$
\begin{aligned}
g(u) &= c\int_v v(uv)^{\alpha-1}(1-uv)^{\beta-1}v^{\gamma-1}\mathrm{e}^{-av}\mathrm{d}v, c = \frac{a^\gamma}{\Gamma(\gamma)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= cu^{\alpha-1}\int_v v^{\alpha+\gamma-1}(1-uv)^{\beta-1}\mathrm{e}^{-av}\mathrm{d}v \qquad (3.14)\\
&= c\int_v(\frac{v}{u^2})v^{\alpha-1}(1-v)^{\beta-1}(\frac{v}{u})^{\gamma-1}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v \\
&= cu^{-\gamma-1}\int_v v^{\alpha+\gamma-1}(1-v)^{\beta-1}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v. \qquad (3.15)
\end{aligned}
$$

But

$$
E(u^{s-1}) = E(x_1^{s-1})E(x_2^{-s+1}) = \frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+s-1)}a^{s-1}\frac{\Gamma(\gamma-s+1)}{\Gamma(\gamma)}.
$$

for $\Re(\alpha+s-1)>0, \Re(\gamma-s+1)>0$. Therefore

$$
g(u) = \frac{\Gamma(\alpha+\beta)}{a\Gamma(\alpha)\Gamma(\gamma)}G_{2,1}^{1,1}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\gamma,\alpha+\beta-1}\right] \qquad (3.16)
$$

where

$$
G_{2,1}^{1,1}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\gamma,\alpha+\beta-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+s-1)\Gamma(\gamma-s+1)}{\Gamma(\alpha+\beta+s-1)}(\frac{u}{a})^{-s}\mathrm{d}s = G(\frac{u}{a}) \text{ say.}
$$

Sum of residues at the poles of $\Gamma(\alpha+s-1)$ gives a divergent series. Evaluating the G-function as the sum of residues at the poles of $\Gamma(\gamma-s+1)$ will give the following:

$$
\begin{aligned}
G(\frac{u}{a}) &= \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}(\frac{u}{a})^{-\gamma-1-\nu}\frac{\Gamma(\alpha+\gamma+\nu)}{\Gamma(\alpha+\beta+\gamma+\nu)} \\
&= \frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha+\beta+\gamma)}(\frac{u}{a})^{-\gamma-1}{}_1F_1(\alpha+\gamma;\alpha+\beta+\gamma;-\frac{a}{u}), 0<u<\infty. \qquad (3.17)
\end{aligned}
$$

Then we have the following result:

Theorem 3.3. *For $\alpha>0,\beta>0,\gamma>0,a>0$ the G-function*

$$
\begin{aligned}
G_{2,1}^{1,1}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\gamma,\alpha+\beta-1}\right] &= \frac{\Gamma(\alpha+\gamma)}{\Gamma(\alpha+\beta+\gamma)}(\frac{u}{a})^{-\gamma-1}{}_1F_1(\alpha+\beta;\alpha+\beta+\gamma;-\frac{a}{u}), 0<u<\infty \\
&= \frac{a^{\gamma+1}}{\Gamma(\beta)}u^{\alpha-1}\int_v v^{\alpha+\gamma-1}(1-uv)^{\beta-1}\mathrm{e}^{-av}\mathrm{d}v \\
&= \frac{a^{\gamma+1}}{\Gamma(\beta)}u^{-\gamma-1}\int_v v^{\alpha+\gamma-1}(1-v)^{\beta-1}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v. \qquad (3.18)
\end{aligned}
$$

Case 3.4: (1.3) versus (1.5). Let $u = \frac{x_1}{x_2}$ where $x_1 > 0$ and $x_2 > 0$ be independently distributed real scalar random variables with x_1 having the density in (1.3) with the parameters $(a > 0, \gamma > 0)$ and x_2 having the density in (1.5) with the parameters $(\alpha > 0, \beta > 0)$ respectively. As before, let $g(u)$ denote the density of u then $g(u)$ is the following:

$$\begin{aligned} g(u) &= c\int_v v(uv)^{\gamma-1}\mathrm{e}^{-auv}v^{\alpha-1}(1+v)^{-(\alpha+\beta)}\mathrm{d}v, c = \frac{a^{\gamma}\Gamma(\alpha+\beta)}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\beta)} \\ &= cu^{\gamma-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-auv}(1+v)^{-(\alpha+\beta)}\mathrm{d}v \qquad (3.19) \\ &= c\int_v (\frac{v}{u^2})v^{\gamma-1}\mathrm{e}^{-av}(\frac{v}{u})^{\alpha-1}(1+\frac{v}{u})^{-(\alpha+\beta)}\mathrm{d}v \\ &= cu^{-\alpha-1}\int_v v^{\alpha+\gamma-1}(1+\frac{v}{u})^{-(\alpha+\beta)}\mathrm{e}^{-av}\mathrm{d}v. \qquad (3.20) \end{aligned}$$

But

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{-s+1}) = a^{-(s-1)}\frac{\Gamma(\gamma+s-1)}{\Gamma(\gamma)}\frac{\Gamma(\alpha-s+1)}{\Gamma(\alpha)}\frac{\Gamma(\beta+s-1)}{\Gamma(\beta)}$$

for $\Re(\gamma+s-1) > 0, \Re(\alpha-s+1) > 0, \Re(\beta+s-1) > 0$. Then

$$g(u) = \frac{a}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\beta)}G_{1,2}^{2,1}\left[au\big|_{\gamma-1,\beta-1}^{-\alpha}\right] = G(au) \text{ say} \qquad (3.21)$$

where

$$G(au) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\gamma+s-1)\Gamma(\beta+s-1)\Gamma(\alpha-s+1)(au)^{-s}\mathrm{d}s.$$

For $\gamma-\beta \neq \pm\lambda, \lambda = 0, 1, 2, ...$ the poles of $\Gamma(\gamma+s-1)\Gamma(\beta+s-1)$ are simple and evaluating the G-function as the sum of the residues at the poles of $\Gamma(\gamma+s-1)$ and $\Gamma(\beta+s-1)$ we have the following two series:

$$\begin{aligned} &\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\beta-\gamma-\nu)\Gamma(\alpha+\gamma+\nu) \\ &\quad = \Gamma(\beta-\gamma)\Gamma(\alpha+\gamma)(au)^{\gamma-1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;au); \\ &\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(au)^{\beta-1+\nu}\Gamma(\gamma-\beta-\nu)\Gamma(\alpha+\beta+\nu) \\ &\quad = \Gamma(\gamma-\beta)\Gamma(\alpha+\beta)(au)^{\beta-1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;au). \end{aligned}$$

Therefore

$$G_{1,2}^{2,1}\left[au\big|_{\gamma-1,\beta-1}^{-\alpha}\right]=\Gamma(\beta-\gamma)\Gamma(\alpha+\gamma)(au)^{\gamma-1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;au)$$
$$+\Gamma(\gamma-\beta)\Gamma(\alpha+\beta)(au)^{\beta-1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;au),\quad(3.22)$$

for $0<u<\infty$. Note that the continuation gives a divergent series. We have the following result:

Theorem 3.4. *For* $\gamma-\beta\neq\pm\lambda,\lambda=0,1,2,...,a>0,\gamma>0,\alpha>0,\beta>0$ *the G-function*

$$\begin{aligned}G_{1,2}^{2,1}\left[au\big|_{\gamma-1,\beta-1}^{-\alpha}\right]&=\Gamma(\beta-\gamma)\Gamma(\alpha+\gamma)(au)^{\gamma-1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;au)\\&+\Gamma(\gamma-\beta)\Gamma(\alpha+\beta)(au)^{\beta-1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;au)\\&=a^{\gamma-1}\Gamma(\alpha+\beta)u^{\gamma-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-auv}(1+v)^{-(\alpha+\beta)}\mathrm{d}v\\&=a^{\gamma-1}\Gamma(\alpha+\beta)u^{-\alpha-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-av}(1+\frac{v}{u})^{-(\alpha+\beta)}\mathrm{d}v.\end{aligned}\quad(3.23)$$

Case 3.5: (1.5) versus (1.3). Let $u=\frac{x_1}{x_2}$ where $x_1>0$ and $x_2>0$ be real scalar random variables independently distributed with x_1 having the density in (1.5) with the parameters $(\alpha>0,\beta>0)$ and x_2 having the density in (1.3) with the parameters $(a>0,\gamma>0)$ respectively. Let $g(u)$ denote the density of u. Then

$$\begin{aligned}g(u)&=c\int_v v(uv)^{\alpha-1}(1+uv)^{-(\alpha+\beta)}v^{\gamma-1}\mathrm{e}^{-av}\mathrm{d}v, c=\frac{a^{\gamma}\Gamma(\alpha+\beta)}{\Gamma(\gamma)\Gamma(\alpha)\Gamma(\beta)}\\&=cu^{\alpha-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-av}(1+uv)^{-(\alpha+\beta)}\mathrm{d}v\quad(3.24)\\&=c\int_v(\frac{v}{u^2})v^{\alpha-1}(1+v)^{-(\alpha+\beta)}(\frac{v}{u})^{\gamma-1}\mathrm{e}^{-a\frac{u}{v}}\mathrm{d}v\\&=cu^{-\gamma-1}\int_v v^{\alpha+\gamma-1}(1+v)^{-(\alpha+\beta)}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v.\quad(3.25)\end{aligned}$$

$$E(u^{s-1})=E(x_1^{s-1})E(x_2^{-s+1})=\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\beta-s+1)}{\Gamma(\beta)}a^{s-1}\frac{\Gamma(\gamma-s+1)}{\Gamma(\gamma)}.$$

Then

$$g(u)=\frac{1}{a\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}G_{2,1}^{1,2}\left[\frac{u}{a}\big|_{\alpha-1}^{-\beta,-\gamma}\right]\quad(3.26)$$

where

$$G_{2,1}^{1,2}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\beta,-\gamma}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha+s-1)\Gamma(\beta-s+1)\Gamma(\gamma-s+1)(\frac{u}{a})^{-s}\mathrm{d}s.$$

The sum of residues at the poles of $\Gamma(\alpha+s-1)$ gives a divergent series. For $\beta-\gamma\neq\pm\lambda,\lambda=0,1,2,...$ the poles of $\Gamma(\beta-s+1)\Gamma(\gamma-s+1)$ are simple. In this case, evaluating the G-function as the sum of residues at the poles of $\Gamma(\beta-s+1)$ and $\Gamma(\gamma-s+1)$ we have the following:

$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(\frac{u}{a})^{-\beta-1-\nu}\Gamma(\alpha+\beta+\nu)\Gamma(\gamma-\beta-\nu)$$
$$=\Gamma(\alpha+\beta)\Gamma(\gamma-\beta)(\frac{u}{a})^{-\beta-1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;\frac{a}{u})$$
$$\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}(\frac{u}{a})^{-\gamma-1-\nu}\Gamma(\alpha+\gamma+\nu)\Gamma(\beta-\gamma-\nu)$$
$$=\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)(\frac{u}{a})^{-\gamma-1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;\frac{a}{u}).$$

Hence

$$\begin{aligned}G_{2,1}^{1,2}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\beta,-\gamma}\right]&=\Gamma(\alpha+\beta)\Gamma(\gamma-\beta)(\frac{a}{u})^{\beta+1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;\frac{a}{u})\\&+\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)(\frac{a}{u})^{\gamma+1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;\frac{a}{u}),0<u<\infty.\end{aligned}\tag{3.27}$$

Then we have the following result:
Theorem 3.5. *For $a>0,\gamma>0,\alpha>0,\beta>0,\beta-\gamma\neq\pm\lambda,\lambda=0,1,2,...$ the G-function*

$$\begin{aligned}G_{2,1}^{1,2}\left[\frac{u}{a}\Big|_{\alpha-1}^{-\beta,-\gamma}\right]&=\Gamma(\alpha+\beta)\Gamma(\gamma-\beta)(\frac{a}{u})^{\beta+1}{}_1F_1(\alpha+\beta;\beta-\gamma+1;\frac{a}{u})\\&+\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)(\frac{a}{u})^{\gamma+1}{}_1F_1(\alpha+\gamma;\gamma-\beta+1;\frac{a}{u}),0<u<\infty\\&=a^{\gamma+1}\Gamma(\alpha+\beta)u^{\alpha-1}\int_v v^{\alpha+\gamma-1}\mathrm{e}^{-av}(1+uv)^{-(\alpha+\beta)}\mathrm{d}v\\&=a^{\gamma+1}\Gamma(\alpha+\beta)u^{-\gamma-1}\int_v v^{\alpha+\gamma-1}(1+v)^{-(\alpha+\beta)}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v.\end{aligned}\tag{3.28}$$

Note that the results in the Case 3.5 are also available by taking $\frac{1}{u}$ of case 3.4.

Case 3.6: (1.4) versus (1.4). Let $u = \frac{x_1}{x_2}$ where $x_1 > 0$ and $x_2 > 0$ be real scalar random variables independently distributed with x_1 having the density in (1.4) with the parameters $(\alpha_1 > 0, \beta_1 > 0)$ and x_2 having the density in (1.4) with the parameters $(\alpha_2 > 0, \beta_2 > 0)$. Again, let $g(u)$ denote the density of u. Then

$$\begin{aligned} g(u) &= c\int_v v(uv)^{\alpha_1-1}(1-uv)^{\beta_1-1}v^{\alpha_2-1}(1-v)^{\beta_2-1}\mathrm{d}v, c = \prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)} \\ &= cu^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1-uv)^{\beta_1-1}(1-v)^{\beta_2-1}\mathrm{d}v \qquad (3.29) \\ &= c\int_v (\frac{v}{u^2})v^{\alpha_1-1}(1-v)^{\beta_1-1}(\frac{v}{u})^{\alpha_2-1}(1-\frac{v}{u})^{\beta_2-1}\mathrm{d}v \\ &= cu^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1-v)^{\beta_1-1}(1-\frac{v}{u})^{\beta_2-1}\mathrm{d}v. \qquad (3.30) \end{aligned}$$

$$\begin{aligned} E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{-s+1}) \\ &= \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)}\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_1+s-1)}\frac{\Gamma(\alpha_2-s+1)}{\Gamma(\alpha_2)}\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_2+\beta_2-s+1)} \end{aligned}$$

for $\Re(\alpha_1+s-1) > 0, \Re(\alpha_2-s+1) > 0$. Then

$$g(u) = \frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}G_{2,2}^{1,1}\left[u\Big|_{\alpha_1-1,-\alpha_2-\beta_2}^{-\alpha_2,\alpha_1+\beta_1-1}\right] \qquad (3.31)$$

where

$$G_{2,2}^{1,1}\left[u\Big|_{\alpha_1-1,-\alpha_2-\beta_2}^{-\alpha_2,\alpha_1+\beta_1-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha_1+s-1)\Gamma(\alpha_2-s+1)}{\Gamma(\alpha_1+\beta_1+s-1)\Gamma(\alpha_2+\beta_2-s+1)}u^{-s}\mathrm{d}s.$$

The sum of residues at the poles of $\Gamma(\alpha_1+s-1)$ is the following:

$$\begin{aligned} &\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\frac{\Gamma(\alpha_1+\alpha_2+\nu)}{\Gamma(\beta_1-\nu)\Gamma(\alpha_2+\beta_2+\alpha_1+\nu)} \\ &\quad = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2+\alpha_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2, 1-\beta_1; \alpha_1+\alpha_2+\beta_2; u) \end{aligned}$$

for $0 < u < 1$. The sum of residues at the poles of $\Gamma(\alpha_2-s+1)$ will give the continuation part which can be written as the following:

$$\begin{aligned} &\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}u^{-\alpha_2-1-\nu}\frac{\Gamma(\alpha_1+\alpha_2+\nu)}{\Gamma(\alpha_1+\beta_1+\alpha_2+\nu)\Gamma(\beta_2-\nu)} \\ &\quad = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_2)\Gamma(\alpha_1+\beta_1+\alpha_2)}u^{-\alpha_2-1}{}_2F_1(\alpha_1+\alpha_2, 1-\beta_2; \alpha_1+\alpha_2+\beta_1; \frac{1}{u}), u \geq 1. \end{aligned}$$

Therefore

$$\begin{aligned}
&G^{1,1}_{2,2}\left[u\big|^{-\alpha_2,\alpha_1+\beta_1-1}_{\alpha_1-1,-\alpha_2-\beta_2}\right]\\
&=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2+\alpha_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1+\alpha_2+\beta_2;u),0<u<1\\
&=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_2)\Gamma(\alpha_1+\beta_1+\alpha_2)}(\frac{1}{u})^{\alpha_2+1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_2;\alpha_1+\beta_1+\alpha_2;\frac{1}{u}),u\geq 1.
\end{aligned} \tag{3.32}$$

The expression for $u \geq 1$ will reduce to the case for $u \leq 1$. Hence we have the following result:

Theorem 3.6. *For $\alpha_j > 0, \beta_j > 0, j = 1, 2$ the G-function*

$$\begin{aligned}
&G^{1,1}_{2,2}\left[u\big|^{-\alpha_2,\alpha_1+\beta_1-1}_{\alpha_1-1,-\alpha_2-\beta_2}\right]\\
&=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2+\alpha_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1+\alpha_2+\beta_2;u),0<u<1\\
&=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\beta_2)\Gamma(\alpha_1+\beta_1+\alpha_2)}(\frac{1}{u})^{\alpha_2+1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_2;\alpha_1+\beta_1+\alpha_2;\frac{1}{u}),u\geq 1\\
&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1-uv)^{\beta_1-1}(1-v)^{\beta_2-1}\mathrm{d}v\\
&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1-v)^{\beta_1-1}(1-\frac{v}{u})^{\beta_2-1}\mathrm{d}v.
\end{aligned} \tag{3.33}$$

Note that we can also interchange the parameters (α_1, β_1) with (α_2, β_2).

Case 3.7: (1.4) versus (1.5). Let $u = \frac{x_1}{x_2}$ where $x_1 > 0$ and $x_2 > 0$ be independently distributed real scalar random variables with x_1 having the density in (1.4) with the parameters $(\alpha_1 > 0, \beta_1 > 0)$ and x_2 having the density in (1.5) with the parameters $(\alpha_2 > 0, \beta_2 > 0)$. Again, let $g(u)$ denote the density of u. Then

$$g(u) = c\int_v v(uv)^{\alpha_1-1}(1-uv)^{\beta_1-1}v^{\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v, c=\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}$$

$$= cu^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1-uv)^{\beta_1-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v \tag{3.34}$$

$$= c\int_v(\frac{v}{u^2})v^{\alpha_1-1}(1-v)^{\beta_1-1}(\frac{v}{u})^{\alpha_2-1}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v$$

$$= cu^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1-v)^{\beta_1-1}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v. \tag{3.35}$$

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{-s+1})$$
$$= \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_1+s-1)} \frac{\Gamma(\alpha_2-s+1)}{\Gamma(\alpha_2)} \frac{\Gamma(\beta_2+s-1)}{\Gamma(\beta_2)}$$

for $\Re(\alpha_1+s-1)>0, \Re(\alpha_2-s+1)>0, \Re(\beta_2+s-1)>0$. Then

$$g(u) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2)} G_{2,2}^{2,1}\left[u\Big|_{\alpha_1-1,\beta_2-1}^{-\alpha_2,\alpha_1+\beta_1-1}\right] \tag{3.36}$$

where

$$G_{2,2}^{2,1}\left[u\Big|_{\alpha_1-1,\beta_2-1}^{-\alpha_2,\alpha_1+\beta_1-1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha_1+s-1)\Gamma(\beta_2+s-1)\Gamma(\alpha_2-s+1)}{\Gamma(\alpha_1+\beta_1+s-1)} u^{-s}\mathrm{d}s.$$

For $\alpha_1-\beta_2 \neq \pm\lambda, \lambda=0,1,2,...$ the poles of $\Gamma(\alpha_1+s-1)\Gamma(\beta_2+s-1)$ are simple. The sum of the residues at the poles of $\Gamma(\alpha_1+s-1)$ and $\Gamma(\beta_2+s-1)$ are the following:

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{\Gamma(\beta_2-\alpha_1-\nu)\Gamma(\alpha_2+\alpha_1+\nu)}{\Gamma(\beta_1-\nu)}$$
$$= \frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2-\alpha_1)}{\Gamma(\beta_1)} u^{\alpha_1-1} {}_2F_1(\alpha_1+\alpha_2, 1-\beta_1; \alpha_1-\beta_2+1; -u), 0<u<1$$
$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{\beta_2-1+\nu} \frac{\Gamma(\alpha_1-\beta_2-\nu)\Gamma(\alpha_2+\beta_2+\nu)}{\Gamma(\alpha_1+\beta_1-\beta_2-\nu)}$$
$$= \frac{\Gamma(\alpha_1-\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1-\beta_2)} u^{\beta_2-1} {}_2F_1(\alpha_2+\beta_2, \beta_2-\alpha_1-\beta_1+1; \beta_2-\alpha_1+1; -u),$$

for $0<u<1$. The continuation is available by evaluating as the sum of residues at the poles of $\Gamma(\alpha_2-s+1)$ and it is the following:

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{\Gamma(\alpha_1+\alpha_2+\nu)\Gamma(\beta_2+\alpha_2+\nu)}{\Gamma(\alpha_1+\beta_1+\alpha_2+\nu)} u^{-\alpha_2-1-\nu}$$
$$= \frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\alpha_2)} u^{-\alpha_2-1} {}_2F_1(\alpha_1+\alpha_2, \alpha_2+\beta_2; \alpha_1+\beta_1+\alpha_2; -\frac{1}{u}), u \geq 1.$$

Therefore

$$
\begin{aligned}
&G_{2,2}^{2,1}\left[u\Big|_{\alpha_1-1,\beta_2-1}^{-\alpha_2,\alpha_1+\beta_1-1}\right]\\
&=\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2-\alpha_1)}{\Gamma(\beta_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1-\beta_2+1;-u)\\
&+\frac{\Gamma(\alpha_1-\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1-\beta_2)}u^{\beta_2-1}{}_2F_1(\alpha_2+\beta_2,\beta_2-\alpha_1-\beta_1+1;\beta_2-\alpha_1+1;-u),\\
&\quad 0<u<1\\
&=\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\alpha_2)}u^{-\alpha_2-1}{}_2F_1(\alpha_1+\alpha_2,\alpha_2+\beta_2;\alpha_1+\beta_1+\alpha_2;-\frac{1}{u}),u\ge 1.
\end{aligned}
\tag{3.37}
$$

We have the following result:

Theorem 3.7. *For $\alpha_j>0,\beta_j>0,j=1,2,\alpha_1-\beta_2\neq\pm\lambda,\lambda=0,1,2,...$ the G-function*

$$
\begin{aligned}
&G_{2,2}^{2,1}\left[u\Big|_{\alpha_1-1,\beta_2-1}^{-\alpha_2,\alpha_1+\beta_1-1}\right]\\
&=\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2-\alpha_1)}{\Gamma(\beta_1)}u^{\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1-\beta_2+1;-u)\\
&+\frac{\Gamma(\alpha_1-\beta_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1-\beta_2)}u^{\beta_2-1}{}_2F_1(\alpha_2+\beta_2,\beta_2-\alpha_1-\beta_1+1;\beta_2-\alpha_1+1;-u),\\
&\quad 0<u<1\\
&=\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\beta_1+\alpha_2)}u^{-\alpha_2-1}{}_2F_1(\alpha_1+\alpha_2,\alpha_2+\beta_2;\alpha_1+\beta_1+\alpha_2;-\frac{1}{u}),u\ge 1\\
&=\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1-uv)^{\beta_1-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v\\
&=\frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1-v)^{\beta_1-1}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v
\end{aligned}
\tag{3.38}
$$

Case 3.8: (1.5) versus (1.4). Let $u=\frac{x_1}{x_2}$ where $x_1>0$ and $x_2>0$ be independently distributed real scalar random variables with x_1 having the density in (1.5) with the parameters $(\alpha_2>0,\beta_2>0)$ and x_2 having the density in (1.4) with the parameters $(\alpha_1>0,\beta_1>0)$ respectively. Let $g(u)$ again denote the density of u.

Then

$$
\begin{aligned}
g(u) &= c\int_v v(uv)^{\alpha_2-1}(1+uv)^{-(\alpha_2+\beta_2)}v^{\alpha_1-1}(1-v)^{\beta_1-1}\mathrm{d}v, c=\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)} \\
&= cu^{\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1+uv)^{-(\alpha_2+\beta_2)}(1-v)^{\beta_1-1}\mathrm{d}v \qquad (3.39) \\
&= c\int_v (\frac{v}{u^2})v^{\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}(\frac{v}{u})^{\alpha_1-1}(1-\frac{v}{u})^{\beta_1-1}\mathrm{d}v \\
&= cu^{-\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}(1-\frac{v}{u})^{\beta_1-1}\mathrm{d}v. \qquad (3.40)
\end{aligned}
$$

$$
\begin{aligned}
E(u^{s-1}) &= E(x_1^{s-1})E(x_2^{-s+1}) \\
&= \frac{\Gamma(\alpha_2+s-1)}{\Gamma(\alpha_2)}\frac{\Gamma(\beta_2-s+1)}{\Gamma(\beta_2)}\frac{\Gamma(\alpha_1-s+1)}{\Gamma(\alpha_1)}\frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1+\beta_1-s+1)}
\end{aligned}
$$

for $\Re(\alpha_2+s-1)>0, \Re(\alpha_1-s+1)>0, \Re(\beta_2-s+1)>0$. Then

$$
g(u) = \frac{\Gamma(\alpha_1+\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2)}G_{2,2}^{1,2}\left[u\Big|_{\alpha_2-1,-\alpha_1-\beta_1}^{-\beta_2,-\alpha_1}\right] \qquad (3.41)
$$

where

$$
G_{2,2}^{1,2}\left[u\Big|_{\alpha_2-1,-\alpha_1-\beta_1}^{-\beta_2,-\alpha_1}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha_2+s-1)\frac{\Gamma(\beta_2-s+1)\Gamma(\alpha_1-s+1)}{\Gamma(\alpha_1+\beta_1-s+1)}u^{-s}\mathrm{d}s.
$$

Evaluating as the sum of residues at the poles of $\Gamma(\alpha_2+s-1)$ we have the following:

$$
\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}u^{\alpha_2-1+\nu}\frac{\Gamma(\beta_2+\alpha_2+\nu)\Gamma(\alpha_1+\alpha_2+\nu)}{\Gamma(\alpha_1+\beta_1+\alpha_2+\nu)}.
$$

Therefore

$$
G_{2,2}^{1,2}\left[u\Big|_{\alpha_2-1,-\alpha_1-\beta_1}^{-\beta_2,-\alpha_1}\right] = \frac{\Gamma(\alpha_2+\beta_2)\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\beta_1+\alpha_2)}u^{\alpha_2-1}{}_2F_1(\alpha_2+\beta_2,\alpha_1+\alpha_2;\alpha_1+\beta_1+\alpha_2;-u)
$$

for $0<u<1$. The continuation part is available from the poles of $\Gamma(\beta_2-s+1)\Gamma(\alpha_1-s+1)$. For $\beta_2-\alpha_1\neq\pm\lambda, \lambda=0,1,2,...$ the poles are simple and in this case the sum of residues at the poles of $\Gamma(\beta_2 - s + 1)$ and $\Gamma(\alpha_1 - s + 1)$ are the following:

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{-\beta_2-1-\nu} \frac{\Gamma(\alpha_1 - \beta_2 - \nu)\Gamma(\alpha_2 + \beta_2 + \nu)}{\Gamma(\alpha_1 + \beta_1 - \beta_2 - \nu)}$$
$$= \frac{\Gamma(\alpha_1 - \beta_2)\Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_1 + \beta_1 - \beta_2)} u^{-\beta_2-1} {}_2F_1(\alpha_2 + \beta_2, \beta_2 - \alpha_1 - \beta_1 + 1; \beta_2 - \alpha_1 + 1; -\frac{1}{u}), u \geq 1$$
$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{-\alpha_1-1-\nu} \frac{\Gamma(\alpha_1 + \alpha_2 + \nu)\Gamma(\beta_2 - \alpha_1 - \nu)}{\Gamma(\beta_1 - \nu)}$$
$$= \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\beta_2 - \alpha_1)}{\Gamma(\beta_1)} u^{-\alpha_1-1} {}_2F_1(\alpha_1 + \alpha_2, 1 - \beta_1; \alpha_1 - \beta_2 + 1; -\frac{1}{u}), u \geq 1.$$

Hence

$$G_{2,2}^{1,2}\left[u\Big|_{\alpha_2-1,-\alpha_1-\beta_1}^{-\beta_2,-\alpha_1}\right]$$
$$= \frac{\Gamma(\alpha_2 + \beta_2)\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1 + \beta_1 + \alpha_2)} u^{\alpha_2-1} {}_2F_1(\alpha_2 + \beta_2, \alpha_1 + \alpha_2; \alpha_1 + \beta_1 + \alpha_2; -u), 0 < u < 1$$
$$= \frac{\Gamma(\alpha_2 + \beta_2)\Gamma(\alpha_1 - \beta_2)}{\Gamma(\alpha_1 + \beta_1 - \beta_2)} u^{-\beta_2-1} {}_2F_1(\alpha_2 + \beta_2, \beta_2 - \alpha_1 - \beta_1 + 1; \beta_2 - \alpha_1 + 1; -\frac{1}{u})$$
$$+ \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\beta_2 - \alpha_1)}{\Gamma(\beta_1)} u^{-\alpha_1-1} {}_2F_1(\alpha_1 + \alpha_2, 1 - \beta_1; \alpha_1 - \beta_2 + 1; -\frac{1}{u}), u \geq 1. \tag{3.42}$$

Therefore we have the following result:

Theorem 3.8. *For $\alpha_j > 0, \beta_j > 0, j = 1, 2, \beta_2 - \alpha_1 \neq \pm\lambda, \lambda = 0, 1, 2, ...$ the G-function*

$$G_{2,2}^{1,2}\left[u\Big|_{\alpha_2-1,-\alpha_1-\beta_1}^{-\beta_2,-\alpha_1}\right]$$
$$= \frac{\Gamma(\alpha_2 + \beta_2)\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1 + \beta_1 + \alpha_2)} u^{\alpha_2-1} {}_2F_1(\alpha_2 + \beta_2, \alpha_1 + \alpha_2; \alpha_1 + \beta_1 + \alpha_2; -u), 0 < u < 1$$

$$= \frac{\Gamma(\alpha_2+\beta_2)\Gamma(\alpha_1-\beta_2)}{\Gamma(\alpha_1+\beta_1-\beta_2)}u^{-\beta_2-1}{}_2F_1(\alpha_2+\beta_2,\beta_2-\alpha_1-\beta_1+1;\beta_2-\alpha_1+1;-\frac{1}{u})$$
$$+\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2-\alpha_1)}{\Gamma(\beta_1)}u^{-\alpha_1-1}{}_2F_1(\alpha_1+\alpha_2,1-\beta_1;\alpha_1-\beta_2+1;-\frac{1}{u}),u\geq 1$$
$$= \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1+uv)^{-(\alpha_2+\beta_2)}(1-v)^{\beta_1-1}\mathrm{d}v$$
$$= \frac{\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1)}u^{-\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}(1-\frac{v}{u})^{\beta_1-1}\mathrm{d}v. \quad (3.43)$$

Case 3.9: (1.5) versus (1.5). Let $u=\frac{x_1}{x_2}$ where $x_1>0$ and $x_2>0$ be independently distributed real scalar random variables with x_1 having the density in (1.5) with the parameters $(\alpha_1>0,\beta_1>0)$ and x_2 having the density in (1.5) with the parameters $(\alpha_2>0,\beta_2>0)$ respectively. Again, let $g(u)$ denote the density of u. Then

$$g(u) = c\int_c v(uv)^{\alpha_1-1}(1+uv)^{-(\alpha_1+\beta_1)}v^{\alpha_2-1}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v, c=\prod_{j=1}^{2}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j)}$$
$$= cu^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1+uv)^{-(\alpha_1+\beta_1)}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v \quad (3.44)$$
$$= c\int_v (\frac{v}{u^2})v^{\alpha_1-1}(1+v)^{-(\alpha_1+\beta_1)}(\frac{v}{u})^{\alpha_2-1}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v$$
$$= cu^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1+v)^{-(\alpha_1+\beta_1)}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v. \quad (3.45)$$

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{-s+1}) = \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)}\frac{\Gamma(\beta_1-s+1)}{\Gamma(\beta_1)}\frac{\Gamma(\alpha_2-s+1)}{\Gamma(\alpha_2)}\frac{\Gamma(\beta_2+s-1)}{\Gamma(\beta_2)}$$

for $\Re(\alpha_1+s-1)>0,\Re(\beta_2+s-1)>0,\Re(\beta_1-s+1)>0,\Re(\alpha_2-s+1)>0$. Then

$$g(u)=\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\alpha_2)\Gamma(\beta_2)}G_{2,2}^{2,2}\left[u\big|_{\alpha_1-1,\beta_2-1}^{-\beta_1,-\alpha_2}\right] \quad (3.46)$$

where

$$G_{2,2}^{2,2}\left[u\big|_{\alpha_1-1,\beta_2-1}^{-\beta_1,-\alpha_2}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha_1+s-1)\Gamma(\beta_2+s-1)\Gamma(\beta_1-s+1)\Gamma(\alpha_2-s+1)u^{-s}\mathrm{d}s.$$

For $\alpha_1 - \beta_2 \neq \pm\lambda, \lambda = 0, 1, 2, ...$ the poles of $\Gamma(\alpha_1 + s - 1)\Gamma(\beta_2 + s - 1)$ are simple. Then evaluating the G-function as the sum of residues at these poles we have the following:

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{\alpha_1-1+\nu}\Gamma(\beta_2 - \alpha_1 - \nu)\Gamma(\beta_1 + \alpha_1 + \nu)\Gamma(\alpha_2 + \alpha_1 + \nu)$$
$$= \Gamma(\beta_2 - \alpha_1)\Gamma(\beta_1 + \alpha_1)\Gamma(\alpha_2 + \alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1 + \alpha_1, \alpha_2 + \alpha_1; \alpha_1 - \beta_2 + 1; u),$$
$$0 < u < 1$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{\beta_2-1+\nu}\Gamma(\alpha_1 - \beta_2 - \nu)\Gamma(\beta_1 + \beta_2 + \nu)\Gamma(\alpha_2 + \beta_2 + \nu)$$
$$= \Gamma(\alpha_1 - \beta_2)\Gamma(\beta_1 + \beta_2)\Gamma(\alpha_2 + \beta_2)u^{\beta_2-1}{}_2F_1(\beta_1 + \beta_2, \alpha_2 + \beta_2; \beta_2 - \alpha_1 + 1; u),$$

for $0 < u < 1$. For $\beta_1 - \alpha_2 \neq \pm\lambda, \lambda = 0, 1, 2, ...$ the poles of $\Gamma(\beta_1 - s + 1)\Gamma(\alpha_2 - s + 1)$ are simple. Then evaluating the G-function as the sum of the residues at these poles we have the following:

$$\sum_{\nu=0}^{\infty} u^{-\beta_1-1-\nu} \frac{(-1)^\nu}{\nu!} \Gamma(\alpha_1 + \beta_1 + \nu)\Gamma(\beta_2 + \beta_1 + \nu)\Gamma(\alpha_2 - \beta_1 - \nu)$$
$$= \Gamma(\alpha_1 + \beta_1)\Gamma(\beta_2 + \beta_1)\Gamma(\alpha_2 - \beta_1)u^{-\beta_1-1}{}_2F_1(\alpha_1 + \beta_1, \beta_2 + \beta_1; \beta_1 - \alpha_2 + 1; \frac{1}{u}), u \geq 1$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} u^{-\alpha_2-1-\nu}\Gamma(\alpha_1 + \alpha_2 + \nu)\Gamma(\beta_2 + \alpha_2 + \nu)\Gamma(\beta_1 - \alpha_2 - \nu)$$
$$= \Gamma(\alpha_1 + \alpha_2)\Gamma(\beta_2 + \alpha_2)\Gamma(\beta_1 - \alpha_2)u^{-\alpha_2-1}{}_2F_1(\alpha_1 + \alpha_2, \beta_2 + \alpha_2; \alpha_2 - \beta_1 + 1; \frac{1}{u}), u \geq 1.$$

Therefore

$$\begin{aligned}
& G^{2,2}_{2,2}\left[u\Big|^{-\beta_1,-\alpha_2}_{\alpha_1-1,\beta_2-1}\right] \\
&= \Gamma(\beta_2 - \alpha_1)\Gamma(\beta_1 + \alpha_1)\Gamma(\alpha_2 + \alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1 + \alpha_1, \alpha_2 + \alpha_1; \alpha_1 - \beta_2 + 1; u) \\
&+ \Gamma(\alpha_1 - \beta_2)\Gamma(\beta_1 + \beta_2)\Gamma(\alpha_2 + \beta_2)u^{\beta_2-1}{}_2F_1(\beta_1 + \beta_2, \alpha_2 + \beta_2; \beta_2 - \alpha_1 + 1; u), \\
&\quad 0 < u < 1 \\
&= \Gamma(\alpha_1 + \beta_1)\Gamma(\beta_2 + \beta_1)\Gamma(\alpha_2 - \beta_1)u^{-\beta_1-1}{}_2F_1(\alpha_1 + \beta_1, \beta_2 + \beta_1; \beta_1 - \alpha_2 + 1; \frac{1}{u}) \\
&+ \Gamma(\alpha_1 + \alpha_2)\Gamma(\beta_2 + \alpha_2)\Gamma(\beta_1 - \alpha_2)u^{-\alpha_2-1}{}_2F_1(\alpha_1 + \alpha_2, \beta_2 + \alpha_2; \alpha_2 - \beta_1 + 1; \frac{1}{u}),
\end{aligned} \tag{3.47}$$

for $u \geq 1$. Hence we have the following result:

Theorem 3.9. *For* $\alpha_1-\beta_2 \neq \pm\lambda, \lambda = 0,1,2,...., \alpha_2-\beta_1 \neq \pm\mu, \mu = 0,1,2,..., \alpha_j > 0, \beta_j > 0, j = 1,2$ *the G-function*

$$\begin{aligned}
&G_{2,2}^{2,2}\left[u\Big|_{\alpha_1-1,\beta_2-1}^{-\beta_1,-\alpha_2}\right] \\
&= \Gamma(\beta_2-\alpha_1)\Gamma(\beta_1+\alpha_1)\Gamma(\alpha_2+\alpha_1)u^{\alpha_1-1}{}_2F_1(\beta_1+\alpha_1,\alpha_2+\alpha_1;\alpha_1-\beta_2+1;u) \\
&+ \Gamma(\alpha_1-\beta_2)\Gamma(\beta_1+\beta_2)\Gamma(\alpha_2+\beta_2)u^{\beta_2-1}{}_2F_1(\beta_1+\beta_2,\alpha_2+\beta_2;\beta_2-\alpha_1+1;u), \\
&\quad 0 < u < 1 \\
&= \Gamma(\alpha_1+\beta_1)\Gamma(\beta_2+\beta_1)\Gamma(\alpha_2-\beta_1)u^{-\beta_1-1}{}_2F_1(\alpha_1+\beta_1,\beta_2+\beta_1;\beta_1-\alpha_2+1;\frac{1}{u}) \\
&+ \Gamma(\alpha_1+\alpha_2)\Gamma(\beta_2+\alpha_2)\Gamma(\beta_1-\alpha_2)u^{-\alpha_2-1}{}_2F_1(\alpha_1+\alpha_2\beta_2+\alpha_2;\alpha_2-\beta_1+1;\frac{1}{u}), \\
&\quad u \geq 1 \\
&= \Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)u^{\alpha_1-1}\int_v v^{\alpha_1+\alpha_2-1}(1+uv)^{-(\alpha_1+\beta_1)}(1+v)^{-(\alpha_2+\beta_2)}\mathrm{d}v \\
&= \Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)u^{-\alpha_2-1}\int_v v^{\alpha_1+\alpha_2-1}(1+v)^{-(\alpha_1+\beta_1)}(1+\frac{v}{u})^{-(\alpha_2+\beta_2)}\mathrm{d}v.
\end{aligned} \tag{3.48}$$

Note that in this theorem we can interchange the parameters (α_1,β_1) and (α_2,β_2). We can also obtain several other results on G-functions by considering other densities other than (1.3),(1.4), (1.5). Note that we need not consider densities but take integrable functions with Mellin transforms existing for positive variables. Then also we can arrive at various integral representations and series forms for various types of G-functions. We can also obtain double integral representations by taking three positive variables instead of two variables and by considering Mellin convolutions of products and ratios. In this case, ratios can be defined in many different ways. These situations will not be discussed here.

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