J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.3, No.1 (2014), pp. 75-80

A note on certain transformations involving Lambert series

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Received June 8, 2014

Abstract: In this paper, making use of known summation formula [1] for bilateral basic hypergeometric series [5], an attempt has been made to established certain interesting results involving Lambert series [1] and continued fractions in q-series [5].
Key words and phrases: Bilateral basic hypergeometric series, q-series, Lambert series, Continued fractions.
2000 A.M.S. subject classification: 33D15

1. Introduction, Notations and Definitions

For real or complex q, $|q| < 1$, the q -shifted factorial is defined by

$$(a,q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{n-1}) & \text{if } n \in N. \end{cases} \tag{1.1}$$

$$[a;q]_\infty = \prod_{r=0}^{\infty}(1-aq^r) \tag{1.2}$$

and

$$[a_1,a_2,\ldots,a_r;q]_n = \prod_{k=0}^{r}[a_k;q]_n \tag{1.3}$$

The Roger-Ramanujan's continued fraction is given by

$$R(q) = \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots} \tag{1.4}$$

Using Rogers-Ramanujan identity, viz.

$$G(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{[q;q]_n} = \frac{1}{[q;q^4;q^5]_\infty} \tag{1.5}$$

and

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[q;q]_n}=\frac{1}{[q^2;q^3;q^5]_\infty} \tag{1.6}$$

Roger proved the following:

$$R(q)=q^{1/5}\frac{H(q)}{G(q)} \tag{1.7}$$

Comparing (1.5) and (1.6),we get the following identity:

$$\frac{H(q)}{G(q)}=\frac{[q;q^4;q^5]_\infty}{[q^2;q^3;q^5]_\infty}=\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots} \tag{1.8}$$

[2; Corollary (6.2.6) p. 153]

We shall make use of the following identity.

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2}-\frac{bq^n}{(1-bq^n)^2}\right]=\frac{a[ab,q/ab,b/a,aq/b;q]_\infty[q;q]_\infty^4}{[a,b,q/a,q/b;q]_\infty^2} \tag{1.9}$$

[1;(4.5)p.197]

The following results are also needed in our analysis,

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(1-cq^n)}=\frac{[q;q]_\infty^2}{[c;q/c;q]_\infty} \tag{1.10}$$

[2;(12.2.9)p.264]

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{n(q/a)^n}{(1-aq^n)}=\frac{[q;q]_\infty^4}{[a,q/a;q]_\infty^2} \tag{1.11}$$

[4;eq.(1.4)]

$$\frac{[q,q^5;q^6]_\infty}{[q^3;q^6]_\infty^2}=\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+\cdots} \tag{1.12}$$

[2,(6.2.37)p.154)]

$$\frac{[q,q^7;q^8]_\infty}{[q^3,q^5;q^8]_\infty}=\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+\cdots} \tag{1.13}$$

[2,(6.2.38)p.154)]

and

$$\frac{[q;q^2]_\infty}{[q^2;q^4]_\infty^2} = \frac{1}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\frac{q^2+q^4}{1+\cdots} \tag{1.14}$$

[2;(6.2.22)p.150)]

2. Main Results

(i) Using (1.10) and (1.9) we have the following transformation formula:

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right] = \frac{a[ab,q/ab,b/a,aq/b;q]_\infty}{[q;q]_\infty^4}\times$$

$$\left\{\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(1-cq^n)}\right\}^2\left\{\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{m(m-1)/2}}{(1-cq^m)}\right\}^2 \tag{2.1}$$

(ii) Using (1.7) in (2.1) we have the following transformation formula:

$$\left\{\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right]\right\}\left\{\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(1-abq^n)^2}\right\}\times$$

$$\left\{\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{m(m-1)/2}}{(1-bq^m/a)}\right\}$$

$$= a\left\{\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(1-aq^n)}\right\}^2\left\{\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{m(m-1)/2}}{(1-bq^m)}\right\}^2 \tag{2.2}$$

(iii) Using (1.11) and (1.9) we have the following transformation formula:

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right]$$

$$= \frac{a[ab,q/ab,b/a,aq/b;q]_\infty}{[q;q]_\infty^4}\left\{\sum_{n=-\infty}^{\infty}\frac{n(q/a)^n}{(1-aq^n)}\right\}\left\{\sum_{m=-\infty}^{\infty}\frac{m(q/b)^m}{(1-bq^m)}\right\} \tag{2.3}$$

(iv) Using (1.10) and (2.3) we have the following transformation formula:

$$\left\{\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right]\right\}\left\{\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(1-abq^n)^2}\right\}\times$$

$$\left\{\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{m(m-1)/2}}{(1-bq^m/a)}\right\}$$
$$=a\left\{\sum_{n=-\infty}^{\infty}\frac{n(q/a)^n}{(1-aq^n)}\right\}\left\{\sum_{m=-\infty}^{\infty}\frac{m(q/b)^m}{(1-bq^m)}\right\} \tag{2.4}$$

(v) Taking q^k for q and $a=q^i$ in (1.9),we have:

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{kn+i}}{(1-q^{kn+i})^2}-\frac{bq^{kn}}{(1-bq^{kn})^2}\right]=\frac{q^i[bq^i,q^{k-i}/b,bq^{-i},q^{i+k}/b;q^k]_\infty[q^k;q^k]_\infty^4}{[q^i,b,q^{k-i},q^k/b;q^k]_\infty^2} \tag{2.5}$$

(vi) Taking $i=2$ for q and $k=5$ in (2.5),we have:

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{bq^{5n}}{(1-bq^{5n})^2}\right]=\frac{q^2[bq^2,q^3/b,bq^{-2},q^7/b;q^5]_\infty[q^5;q^5]_\infty^4}{[q^2,b,q^3,q^5/b;q^5]_\infty^2} \tag{2.6}$$

(vii) Again taking $i=1$ for q and $k=5$ in (2.5),we have:

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{bq^{5n}}{(1-bq^{5n})^2}\right]=\frac{q[bq,q^4/b,bq^{-1},q^6/b;q^5]_\infty[q^5;q^5]_\infty^4}{[q,b,q^4,q^5/b;q^5]_\infty^2} \tag{2.7}$$

(viii) Now dividing (2.6) by (2.7) and using (1.8) we have:

$$\frac{\sum_{n=-\infty}^{\infty}\left[\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{bq^{5n}}{(1-bq^{5n})^2}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{q^{5n+1}}{(1-q^{5n+1})^2}-\frac{bq^{5n}}{(1-bq^{5n})^2}\right]}=\frac{[bq^2,bq^3,q^2/b,q^3/b;q^5]_\infty[q,q^4;q^5]_\infty^2}{[bq,bq^4,q^4/b,q/b;q^5]_\infty[q^2,q^3;q^5]_\infty^2}$$
$$=\frac{[bq^2,bq^3,q^2/b,q^3/b;q^5]_\infty}{[bq,bq^4,q^4/b,q/b;q^5]_\infty}\left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+\ldots}\right\}^2 \tag{2.8}$$

(ix) Taking $i=3$ for q and $k=6$ in (2.5),we have:

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+3}}{(1-q^{6n+3})^2}-\frac{bq^{6n}}{(1-bq^{6n})^2}\right]=\frac{q^3[bq^3,q^3/b,bq^{-3},q^9/b;q^6]_\infty[q^6;q^6]_\infty^4}{[q^3,b,q^3,q^6/b;q^6]_\infty^2} \tag{2.9}$$

(x) Again taking $i=1$ and $k=6$ in (2.5) we have:

$$\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{bq^{6n}}{(1-bq^{6n})^2}\right]=\frac{q[bq,q^5/b,bq^{-1},q^7/b;q^6]_\infty[q^6;q^6]_\infty^4}{[q,b,q^5,q^6/b;q^6]_\infty^2} \tag{2.10}$$

Now dividing (2.9) by (2.10) and using (1.12) we have:

$$\frac{\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+3}}{(1-q^{6n+3})^2}-\frac{bq^{6n}}{(1-bq^{6n})^2}\right]}{\sum_{n=-\infty}^{\infty}\left[\frac{q^{6n+1}}{(1-q^{6n+1})^2}-\frac{bq^{6n}}{(1-bq^{6n})^2}\right]}=\frac{q^3[bq^3,q^3/b,;q^6]_\infty^2}{[bq,bq^5,q/b,q^5/b;q^6]_\infty}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+\cdots}\right\}^2 \tag{2.11}$$

Acknowledgment:

The authors are thankful to Dr.S.N.Singh, Ex. Reader and Head, Department of Mathematics, T.D.P.G. College, Jaunpur (U.P.), India, for his noble guidance during the preparation of this paper.

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