South East Asian J. of Mathematics and Mathematical Sciences
Vol. 15, No. 3 (2019), pp. 241-248

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

THE GENERALIZED (s, t)- PELL MATRIX SEQUENCE

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(**Received:** *Sep. 22, 2019* **Accepted:** *Oct. 14, 2019* **Published:** *Apr. 30, 2020)*

Abstract: In this paper we defined generalized (s, t)-Pell matrix sequence which is generalized by (s, t)-Pell Matrix sequence and (s, t)-Pell-Lucas Matrix sequence. We also described some properties for generalized (s, t)-Pell matrix sequence and established relationship among (s, t)-Pell Matrix and (s, t)-Pell-Lucas Matrix sequence.

Keywords and Phrases: (s, t)-Fibonacci, (s, t)-Lucas, (s, t)-Pell, (s, t)-Pell Lucas, (s, t)-Pell matrix.

2010 Mathematics Subject Classification: 11B37, 11B39, 15A15.

1. Introduction, Notations and Definitions

Many scholars done fabulous work on Fibonacci, Lucas, Pell, Jacobsthal sequence etc by many sided of conditions [1-4]. The wonderful sequence Fibonacci is given by the equation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

From past years many scholars established the generalizations of Fibonacci, Lucas, Pell sequence etc by using parameters s and t then sequence called (s, t)-Fibonacci, (s, t)-Lucas, (s, t)-Pell sequence etc and we also describe the matrix sequence called as (s, t)- type matrix sequence like (s, t)-Fibonacci matrix sequence, (s, t)-Lucas Matrix Sequence, (s, t)-Pell matrix sequence etc.

In 2012 Gulec and Taskara in [5] defined (s, t)-Pell Sequence $\{p_n(s, t)\}_{n\in N}$ and (s, t)-Pell Lucas sequence $\{q_n(s, t)\}_{n\in N}$ and their matrix sequence (s, t)-Pell matrix sequence $\{P_n(s,t)\}_{n\in N}$ and (s,t)-Pell Lucas matrix sequence $\{Q_n(s,t)\}_{n\in N}$ respectively. For any real number s,t and $n \geq 2$, let $s^2+t>0, s>0$ and $t \neq 0$, then the (s,t)-Pell sequence $\{p_n(s,t)\}_{n\in N}$ and (s,t)-Pell-Lucas sequence $\{q_n(s,t)\}_{n\in N}$ are defined by

$$p_n(s,t) = 2sp_{n-1}(s,t) + tp_{n-2}(s,t), \tag{1}$$

$$q_n(s,t) = 2sq_{n-1}(s,t) + tq_{n-2}(s,t), \tag{2}$$

For some special values of s and t in (1), it is observable that the following results hold:

- If $s = \frac{1}{2}, t = 1$, the classic Fibonacci sequence is obtained.
- If $s = t = 1$, the classic Pell sequence is obtained.

Also some special values of s and t in (2), it is clear that the following results holds

- If $s = \frac{1}{2}, t = 1$, the classic Lucas sequence is obtained.
- If $s = t = 1$, the classic Pell-Lucas sequence is obtained.

In the following definition (s,t)-Pell matrix sequence $\{P_n(s,t)_{n\in N}\}$ and $(s,t)-$ Pell-Lucas matrix sequence $\{Q_n(s,t)\}_{n\in N}$ are defined respectively.

Proposition 1. Let us consider $s>0, t\neq 0$ and $s^2+t>0$ and $n\geq 2$ we have
1. $P_n(s,t) = 2sP_{n-1}(s,t) + tP_{n-2}(s,t)$,
with initial conditions $P_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $P_1(s,t) = \begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix}$,

2. $Q_n(s,t) = 2sQ_{n-1}(s,t)+tQ_{n-2}(s,t)$, with initial conditions $Q_0(s,t) = \begin{pmatrix} 2s & 2 \\ 2t & -2s \end{pmatrix}$,
$Q_1(s,t) = \begin{pmatrix} 4s^2+2t & 2s \\ 2st & 2t \end{pmatrix}$,
3. $P_n(s,t) = \begin{pmatrix} p_{n+1} & p_n \\ tp_n & tp_{n-1} \end{pmatrix}$ and $Q_n(s,t) = \begin{pmatrix} q_{n+1} & q_n \\ tq_n & tq_{n-1} \end{pmatrix}$,
4. $p_n = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$, $q_n = r_1^n + r_2^n$
where $r_1 = s + \sqrt{s^2+t}$ and $r_2 = s - \sqrt{s^2+t}$.
For $m,n \in Z^+$ $P_{n+m}(s,t) = P_n(s,t)P_m(s,t)$,
$P_m(s,t)Q_{n+1}(s,t) = Q_{n+1}(s,t)P_m(s,t)$.
In this paper we will use the notation P_n instead of $P_n(s,t)$ and Q_n instead of $Q_n(s,t)$.

2. Main Results

In this section we consider the following definition of generalized (s,t)-Pell number sequence and also defined the generalized (s,t)-Pell matrix sequence and relationships among them.

Definition 2.1. *For $n \geq 0$ any integer, let $a, b \in R$ and $s^2 + t > 0,\ s > 0,$ and $t \neq 0$ then the generalized (s,t)-Pell integer sequence $\{H_n(s,t)\}_{n\in N}$ defined by the following equation,*

$$H_{n+2}(s,t) = 2sH_{n+1}(s,t) + tH_n(s,t), \tag{2.1}$$

with initial conditions $H_0(s,t) = a,\ H_1(s,t) = bs.$

Definition 2.2. *For $n \geq 0$ any integer, let $a, b \in R$ and $s^2 + t > 0,\ s > 0,$ and $t \neq 0$ then the generalized (s,t)-Pell matrix sequence $\{T_n(s,t)\}_{n\in N}$ defined by the following equation,*

$$T_{n+2}(s,t) = 2sT_{n+1}(s,t) + tT_n(s,t), \tag{2.2}$$

with initial conditions $T_0(s,t) = \begin{pmatrix} bs & a \\ at & 2(b-a)s \end{pmatrix},\quad T_1(s,t) = \begin{pmatrix} 2bs^2 + at & bs \\ bst & at \end{pmatrix}.$

Now using this definitions we proved some theorems and established relations.

Theorem 2.3. *For any integer $n \geq 1$, we have*

$$T_n = \begin{pmatrix} H_{n+1} & 2H_n \\ 2tH_n & tH_{n-1} \end{pmatrix}.$$

Proof. Let us consider $n = 1$ in this theorem. Then we clearly have $H_0 = a, H_1 = bs$ and $H_2 = 2bs^2 + at$, then

$$T_1 = \begin{pmatrix} H_2 & 2H_1 \\ 2tH_1 & tH_0 \end{pmatrix} = \begin{pmatrix} 2bs^2 + at & 2bs \\ 2bst & at \end{pmatrix},$$

as a next step of that, for $n = 2$, we also get

$$T_2 = \begin{pmatrix} H_3 & 2H_2 \\ 2tH_2 & tH_1 \end{pmatrix} = \begin{pmatrix} 2bs^2 + 2sat + bst & 4bs^2 + 2at \\ 4bs^2t + 2at^2 & bst \end{pmatrix}$$

By following this procedure and considering induction method, let us assume that the theorem is proved for $n = i \in Z^+$, now we have to show that the case also holds for $n = i + 1$, therefore, we get

$$\begin{aligned} T_{i+1} =& 2sT_i + tT_{i-1}(s,t), \\ =& 2s\begin{pmatrix} H_{i+1} & 2H_i \\ 2tH_i & tH_{i-1} \end{pmatrix} + \begin{pmatrix} H_i & 2H_{i-1} \\ 2tH_{i-1} & tH_{n-2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 2sH_{i+1} & 4sH_i \\ 4stH_i & 2stH_{i-1} \end{pmatrix} + \begin{pmatrix} tH_i & 2tH_{i-1} \\ 2t^2H_{i-1} & t^2H_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} 2sH_{i+1} + tH_i & 4sH_i + 2tH_{i-1} \\ 4stH_i + 2t^2H_{i-1} & 2stH_{i-1} + t^2H_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} H_{i+2} & 2H_{i+1} \\ 2tH_{i+1} & tH_i \end{pmatrix}$$

Hence the result.

Corollary 2.4. *In above theorem 2.3, if we choose suitable values on s, t, a and b then it is obtained some special matrix sequence for example, by putting $s = t = 1$ and $a = 0$, $b = 1$, we obtain the (s, t) Pell matrix*

$$H_n = \begin{pmatrix} p_{n+1} & 2p_n \\ 2p_n & p_{n-1} \end{pmatrix}$$

where p_n is n^{th} (s, t)-Pell number and by putting $s = t = 1$ and $a = 2$, $b = 1$, we get the (s, t)-Pell Lucas matrix:

$$H_n = \begin{pmatrix} q_{n+1} & 2q_n \\ 2q_n & q_{n-1} \end{pmatrix}$$

where q_n is n^{th} (s, t)-Pell-Lucas number.

Theorem 2.5. *For $n \geq 1$ any integer, we have*
1. $H_n = bsp_n + atp_{n-1}$
2. $H_{n+1} + tH_{n-1} = bsq_n + atq_{n-1}$

Proof. To prove this theorem, we use the definition 2.1 with its initial conditions.
1. If we consider the initial conditions $H_1 = bs, H_2 = 2bs^2 + at$, then we observe that $H_1 = bs = (bs)p_1 + (at)p_0$ and $H_2 = 2s(bs) + (at) = bsp_2 + atp_1$.
By using the (s, t)-Pell sequence and following above procedure, we get $bsp_3 + atp_2$ which gives H_3. So by following above progresses, we obtain the general term $bsp_n + atp_{n-1}$ that implies H_n as required.
2. Now we replacing (s, t)-Pell's initial conditions p_0 and p_1 by (s, t)-Pell Lucas initial condition q_0 and q_1 in above, then we get $H_{n+1} + tH_{n-1} = bsq_n + atq_{n-1}$.

Theorem 2.6. *For $n \geq 1$ any integer, we have*
1. $T_n = bsP_n + atP_{n-1}$
2. $T_{n+1} + tT_{n-1} = bsQ_n + atQ_{n-1}$

Proof. 1. First we consider the initial values for

$$T_1 = \begin{pmatrix} 2bs^2 + at & bs \\ bst & at \end{pmatrix}$$

Then it is clear that

$$T_1 = bs\begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix} + at\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = bsP_1 + atP_0$$

Now we use above idea

$$T_2 = \begin{pmatrix} 4bs^3 + 2ast + bst & 2bs^2 + at \\ 2bs^2t + at^2 & bst \end{pmatrix}$$

then

$$T_2 = bs\begin{pmatrix} 4s^2 + t & 2s \\ 2st & t \end{pmatrix} + at\begin{pmatrix} 2s & 1 \\ t & 0 \end{pmatrix} = bsP_2 + atP_1$$

Therefore $T_1 = bsP_1 + atP_0$ and $T_2 = bsP_2 + atP_1$ by By using above process, we get the general terms $T_n = bs_n + at_{n-1}$.

2. Replacing the (s,t)-Pell initial conditions P_0 and P_1 by (s,t)-Pell- Lucas initial conditions Q_0 and Q_1, we get

$$T_{n+1} + tT_{n-1} = bsQ_n + atQ_{n-1}.$$

Theorem 2.7. *For $i, j \in N$, we have*

$$T_{i+j} = P_i T_j$$

Proof. To prove this equation, we have to follow induction method for $j = 0$,

$$\begin{aligned} P_1T_0 &= \begin{pmatrix} p_{i+1} & p_i \\ tp_i & tp_{i-1} \end{pmatrix}\begin{pmatrix} bs & a \\ at & 2(b-a)s \end{pmatrix}, \\ &= \begin{pmatrix} bsp_{i+1} + atp_i & ap_{i+1} + 2(b-a)sp_i \\ bstp_i + at^2p_{i-1} & atp_i + 2(b-a)stp_{i-1} \end{pmatrix} \\ &= T_i \end{aligned}$$

Let us assume that is true for all positive j, that is

$$T_{i+j} = P_i T_j$$

Now we prove it for $j+1$,

$$\begin{aligned} P_iT_{+1} &= P_i(2sT_j + tT_{j-1}) \\ &= 2sP_iT_j + tP_iT_{j-1} \\ &= 2sT_{i+j} + tT_{i+j-1} \\ &= T_{i+j+1} \end{aligned}$$

hence Proved.

Theorem 2.8. *Binet's Formula for generalized (s,t)-Pell numbers*

$$H_n = \frac{X\gamma_1^n - Y\gamma_2^n}{\gamma_1 - \gamma_2}$$

It is clear that the characteristic equation of (3) is $x^2 = 2sx + t$ *where* $\gamma_1 = s + \sqrt{s^2+t}$, $\gamma_2 = s - \sqrt{s^2+t}$ *are the roots.*
Then the Binet's Formula for n^{th} *the generalized* (s,t)*-Pell number is given by*

$$H_n = \frac{X\gamma_1^n - Y\gamma_2^n}{\gamma_1 - \gamma_2}$$

where $X = bs + \frac{at}{\gamma_1}$, $Y = bs + \frac{at}{\gamma_2}$

$$\begin{aligned} H_n =& (bs)p_n + (at)p_{n-1} \\ =& (bs)\frac{\gamma_1^n - \gamma_2^n}{\gamma_1 - \gamma_2} + (at)\frac{\gamma_1^{n-1} - \gamma_2^{n-1}}{\gamma_1 - \gamma_2} \\ =& \left[bs + \frac{at}{\gamma_1}\right]\gamma_1^n - \left[bs + \frac{at}{\gamma_1}\right]\gamma_2^n \\ =& \frac{\left[bs + \frac{at}{\gamma_1}\right]\gamma_1^n - \left[bs + \frac{at}{\gamma_1}\right]\gamma_2^n}{\gamma_1 - \gamma_2} \\ H_n =& \frac{X\gamma_1^n - Y\gamma_2^n}{\gamma_1 - \gamma_2}. \end{aligned}$$

Theorem 2.9. *For* $a, b \in R$, $n \in N$, $s > 0$, $t \neq 0$ *and* $s^2 + t > 0$, *we have*

$$T_1Q_n = T_{n+2} + tT_n$$

Proof.

$$\begin{aligned} T_1Q_n =& \begin{pmatrix} 2bs^2 + at & bs \\ bst & at \end{pmatrix} \begin{pmatrix} q_{n+1} & q_n \\ tq_n & tq_{n-1} \end{pmatrix}, \\ =& \begin{pmatrix} 2bs^2q_{n+1} + atq_{n+1} + bstq_n & 2bs^2q_n + atq_n + bstq_{n-1} \\ bstq_{n+1} + at^2q_{n-1} & bstq_n + at^2q_{n-1} \end{pmatrix}, \\ =& \begin{pmatrix} H_{n+3} & 2H_{n+2} \\ 2tH_{n+2} & tH_{n-1} \end{pmatrix} + t\begin{pmatrix} H_{n+1} & 2H_n \\ 2tH_n & tH_{n-1} \end{pmatrix}, \\ =& T_1Q_n = T_{n+2} + tT_n. \end{aligned}$$

Hence Proved

3. Acknowledgement

We would like to thankful to Late Dr. B. Singh, Prof. and Ex. Head, School of Studies in Mathematics, Vikram University Ujjain, India for sharing his pearls of wisdom with us.

References

[1] P. Catarino, A Note on $h(x)$-Fibonacci Quaternion Polynomials, Chaos, Solitons & Fractals, 77, (2011) 1-5.

[2] H. Civic, and R. Turkmen, On the (s, t)-Fibonacci and Matrix Sequences, Ars Combinatoria, 87, (2008) 161-173.

[3] H. Civciv, and R. Turkmen, Notes On the (s, t)-Lucas and Lucas Matrix Sequences, Ars Combinatoria, 89, (2008) 271-285.

[4] S. Halici and S. Oz, On Some Gaussian Pell and Pell-Lucas Numbers, Ordu Universitesi Bilim ve Teknoloji Dergisi, 6, (2016) 8-18.

[5] H. H. Hulec, and N. Taskara, On the (s, t) Pell and Pell-Lucas Sequences and Their Matrix Representations, Applied Mathematical Letters, 25, (2012) 1554-1559.

[6] A. Ipek, K. Ari, and R. Turkmen, The Generalized-(s, t) Fibonacci and Fibonacci Matrix Sequences, Transylvanian Journal of Mathematics and Mechanics, 7, (2015) 137-148.

[7] S. Srisawat, and W. Sriprad, On the (s, t) Pell and Pell-Lucas Numbers by Matrix Methods, Annales Mathematicae et Informaticae, 46, (2016) 195-204.