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NOTE ON CERTAIN OPERATORS OF JACOBI FORMS OF HALF INTEGRAL WEIGHT

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Abstract: In this note we characterise two operators I_m and K_m. on the space of Jacobi forms of half-integral weight.

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1. Introduction

In this note, we characterise certain operators I_m and K_m on the space of Jacobi forms of weight $k+1/2$($k>1$ is an integer), index m and level 4. The operator I_m has been introduced in [2] and proved that it maps Jacobi forms of weight $k+1/2$, index m, level 4 into the space of Jacobi forms of weight $k+1/2$, index 1, level $4m$ and character χ_m - a real character mod m or $4m$ according as $m \equiv 1(mod\ 4)$ or $m \equiv 2, 3(mod\ 4)$. It is also known that, the operator I_m preserves the space of cusp forms. It has a connection with the Eichler-Zagier maps: $\phi|\mathbb{Z}_m := \phi|I_m\mathbb{Z}_1$ where ϕ is a Jacobi form of weight $k+1/2$, index m, level 4 and $\mathbb{Z}_m$ is the Eichler-Zagier map as in [2]. We first prove that the index changing operator I_m preserves the

space of Eisenstein series.
Then, we consider an operator K_m which maps the space of Jacobi forms of weight $k+1/2$, index m, level 4 into the space of Jacobi forms of weight k, index m, level $4m$ and it also acts on the Fourier expansion $\phi = \sum C(D,r)e(n\tau + rz)$ and gives

$$\phi|K_m = \sum_{\substack{0\geq D, r\in\mathbb{Z} \\ D\equiv r^2(4m) \\ (D,m)=1}} C(D,r)e(n\tau + rz).$$

We then prove that its kernel is equal to the space of oldforms and it is injective on the space of newforms under the assumption that the Eichler- Zagier map Z_m is injective on the space of Jacobi forms of weight $k+1/2$, index m and level 4.

2. Notations

Throughout this paper, the letters k, m, N stand for natural numbers and $2|k$. ($k > 1, m \equiv 1(mod\, 4)$ is a square-free odd integer). Let τ be an element of $\mathbb{H}$, the complex upper half plane. Let $\mathbb{Z}$ be the ring of integers.
For a complex number z, we write $\sqrt{z}$ for the square root with argument in $(-\pi/2, \pi/2]$ and we set $z^{a/2} = (\sqrt{z})^a$ for any $a \in \mathbb{Z}$
For integers c, d, 4 divides c and d odd, let $\left(\frac{c}{d}\right)$ denote the generalized quadratic residue symbol. Let $d(c)$ denote $d(mod\, c)$, $c, d \in \mathbb{Z}$

Definition 2.1. *Modular forms of weight k, level N, character χ. For details we refer to [3]*

Let $f(z)$ be an analytic function on the upper half-plane $\mathbb{H}$ and at all rational points, and let $k > 1$ be an integer. Suppose that $f(z)$ satisfies the relation

$$f(\gamma z) = \chi(d)(cz+d)^k f(z) \textit{for all}\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

Then, $f(z)$ is called a modular form of weight k, level N and character χ with $\chi(-1) = (-1)^k$.
Let $M_k(N,\chi)$ denote the space of modular forms of weight k, level N and character χ. Let $S_k(N,\chi)$ denote the space of cusp forms in $M_k(N,\chi)$. For cusp forms f, g in the space $S_k(N,\chi)$, we denote their Petersson scalar product by $< f, g >$.

Definition 2.2. ***Poincarě series in** $S_k(N,\chi)$**:***

Let $k > 2$. For $n \in N$, define the n^{th} Poincarě series in $S_k(N,\chi)$ as follows:

$$P_{k,N,\chi;n}(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1 \\ N|c}} \bar{\chi}(d)(c\tau+d)^{-k} e^{2\pi i n\left(\frac{a\tau+b}{c\tau+d}\right)}$$

where in the above summation $(c,d) \in \mathbb{Z}^2$ *with* $(c,d) = 1$ *and* $N|c$ *which is equivalent that* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \mathbb{H}$ *where* $(a,b) \in \mathbb{Z}^2$ *with* $ad - bc = 1$*. We characterise the Poincarě series as* $< f, P_{k,N,\chi;n} >= \frac{\Gamma(k-1)}{i_N(4\pi n)^{k-1}} a_f(n)$*, for all* $f \in S_k(N,\chi)$ *with Fourier expansion*

$$f(\tau) = \sum_{n\geq 1} a_f(n) q^n.$$

Definition 2.3. ***Jacobi forms*** *[1]*
Let $J_{k+1/2,m}(\Gamma_0(4N))$ *denote the space of Jacobi forms of weight* $k+1/2$*, index* m*, for* $\Gamma_0(4N)$ *and* $J^{cusp}_{k+1/2,m}(\Gamma_0(4N)$ *denote the space of cusp forms in* $J_{k+1/2,m}(\Gamma_0(4N)$*. If* $\phi, \psi \in J^{cusp}_{k+1/2,m}(\Gamma_0(4N)$*, we denote* $< \phi, \psi >$ *the Petersson scalar product of* ϕ *and* ψ*.*
Let $J^{Eis}_{k+1/2,m}(\Gamma_0(4N)$ *be the space of Jacobi Eisenstein series in* $J_{k+1/2,m}(\Gamma_0(4N)$*. and it is the orthogonal compliment of* $J^{cusp}_{k+1/2,m}(\Gamma_0(4N)$*, with respect to Petersson scalar product.*

Definition 2.4. ***Poincarě series in*** $S_{k+1/2}(4N,\chi)$*:*
We define the n^{th} *Poincarě series in* $S_{k+1/2}(4N,\chi)$ *as follows:*

$$P_{k+1/2,4N,\chi;n}(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1, 4N|c}} \bar{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{k+1/2} (c\tau + d)^{-k-\frac{1}{2}} e\left(n \frac{a_0\tau + b_0}{c\tau + d}\right)$$

where the summation above varies for each coprime pair (c,d) *with* $4N|c$*, we make a fixed choice of* $(a_0, b_0) \in \mathbb{Z}^2$ *with* $a_0 d - b_0 c = 1$*. We characterize the Poincare series as follows:*

$$< f, P_{k+1/2,4N,\chi;n} >= \frac{\Gamma(k-1/2)}{i_{4N}(4\pi n)^{k-1/2}} a_f(n),$$

for any cusp form $f \in S_{k+1/2}(4N,\chi)$ *with Fourier expansion*

$$f(\tau) = \sum_{n\geq 1} a_f(n) q^n.$$

Definition 2.5. I_m ***Operator*** *[2]*
If $\phi \in J_{k+1/2,m}(4N,\chi)$*, define* I_m *by*

$$\phi|I_m(\tau,z) = \sum_{\lambda(m)} e(\lambda^2\tau + 2\lambda\tau)\phi(m\tau, z + \lambda\tau).$$

I_m maps $J^{cusp}_{k+1/2,m}(4N,\chi)$ into $J^{cusp}_{k+1/2,1}(4mN,\chi\chi_m)$. The Fourier development of $\phi|I_m$ is of the form

$$\phi|I_m(\tau,z) = \sum_{\substack{0<D,r\in\mathbb{Z}\\ D\equiv r^2(mod 4)}} \left(\sum_{\substack{s(mod\ 2m)\\ s\equiv r(2)}} c_\phi(D,s) \right) e\left(\frac{r^2-D}{4}\tau + rz\right).$$

Definition 2.6. K_m ***Operator***
If $\phi \in J_{k+1/2,m}(4,\chi)$, define

$$K_m = \phi - \sum_{g|m,g>1} \frac{1}{g} \sum_{\mu(g)} \phi| \left[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^*, (0,\mu/g), 1 \right].$$

A direct computation shows that K_m maps $J_{k+1/2,m}(4,\chi)$ into $J_{k+1/2,m}(4,\chi)$ and preserving cusp forms and Eisenstein series. For this we compute its Fourier coefficient on Jacobi form in the following.

Proof.

$$\sum_{g|m,g>1} \frac{1}{g} \sum_{\mu(g)} \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4mn}} c(n,r)e(n\tau+rz)|_{k,m} \left[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^*, (0,\mu/g), 1 \right]$$

$$\begin{aligned}
&= \sum_{g|m,g>1} \frac{1}{g} \sum_{\mu(g)} \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4mn}} c(n,r)e(n\tau + r(z+\mu/g)) \\
&= \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4mn}} c(n,r) \sum_{g|m,g>1} \frac{1}{g} \sum_{\mu(g)} e(r\mu/g)e(n\tau+rz) \\
&= \sum_{g|m} \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\le 4mn\\ g|r,g>1}} c(n,r)e(n\tau+rz) \\
&= \sum_{g|m} \sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2(4m)\\ g|D,g>1}} c(D,r)e(\frac{r^2-D}{4m}\tau+rz) \\
&= \sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2(4m)\\ (D,m)>1}} c(D,r)e(\frac{r^2-D}{4m}\tau+rz)
\end{aligned}$$

Thus,

$$\phi|K_m = \sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2(4m)\\ (D,m)=1}} c(D,r)e(\frac{r^2-D}{4m}\tau + rz)$$

3. Statement of Results

We first prove that I_m preserves the space of Eisenstein series.
The operator I_m maps Jacobi forms of weight $k+1/2$, index m, for $\Gamma_0(4N)$, character χ, into Jacobi forms of weight $k+1/2$, index 1 for $\Gamma_0(4mN)$, character $\chi\chi_m$ where χ_m is the quadratic character modulo m or $4m$ according as $m \equiv 1\,(mod\,4)$ or $m \equiv 2,3\,(mod\,4)$. It is also known that I_m preserves the space of cusp forms.
Let $\phi \in J^{Eis}_{k,m}(N)$.
Then, if $P_{D,r}$ is the $(D,r)^{th}$ Poincare series in $J^{cusp}_{k,m}(N)$, $P_{|D|} = P_{D,r}|Z_m$. This result has been proved in [2]. Using the action of Z_m on the Poincaré series and the definition of adjoint map we have a constant λ such that

$$P_D|Z^*_m = \lambda \sum_{\substack{r(mod\,2m),\\ D\equiv r^2(mod\,4m)}} P_{D,r}.$$

Now, for any ϕ in $J^{Eis}_{k+1/2,m}(4)$, we have

$$<\phi|\mathbb{Z}_m, P_D> = <\phi|I_m\mathbb{Z}_1, P_D>$$

$$<\phi|I_m, P_D|\mathbb{Z}^*_1> = <\phi|I_m, P_{D,r}>$$

On the other hand,

$$<\phi|\mathbb{Z}_m, P_D> = <\phi, P_D|\mathbb{Z}^*_m>$$

$$= \lambda <\phi, \sum_{\rho(2m)} P_{D,\rho}>$$

$$= \lambda \sum_{\rho(2m)} <\phi, P_{D,\rho}> = 0,$$

since $\phi \in J^{Eis}_{k,m}(N)$ and $P_{D,\rho} \in J^{cusp}_{k,m}(N)$ and $J^{Eis}_{k,m}(N)$ is the orthogonal compliment of $J^{cusp}_{k,m}(N)$ with respect to Petersson scalar product. Now,

$$0 = \sum_{\rho(2m)} <\phi, P_{D,\rho}> = <\phi, \sum_{\rho(2m)} P_{D,\rho}> = <\phi|\mathbb{Z}_m, P_{|D|}> = <\phi|I_m, P_{D,\rho}>$$

Hence, we have the required mapping property of I_m.
Define,

$$J^{new}_{k+1/2,m}(4) = \left\{\phi \in J_{k+1/2,m}(4) | \phi|\mathbb{Z}_m \in M^{new}_k(4m, \chi_m)\right\}$$

and

$$J^{old}_{k+1/2,m}(4) = \left\{\phi \in J_{k+1/2,m}(4) | \phi|\mathbb{Z}_m \in M^{old}_k(4m, \chi_m)\right\}$$

Then, we have

Theorem 3.1.

$$I_m : J^{Eis}_{k+1/2,m}(4N) \to J^{Eis}_{k+1/2,m}(4N),$$

$$J^{old}_{k+1/2,m}(4) = \bigoplus_{d^2|m} J_{k+1/2,m/d^2}(4)|B_{d^2}$$

and

$$kerK_m = J^{old}_{k+1/2,m}(4)$$

where

$$B_{d^2} : \phi_d(\tau, z) \mapsto \phi_d(\tau, dz).$$

Proof. $\phi \in J^{old}_{k+1/2,m}(4)$. Then, $\phi|\mathbb{Z}_m \in M^{old}_k(4m, \chi_m)$. Since $m = m_0 m_1^2$,

$$\phi|\mathbb{Z}_m = \sum_{d^2|m} f_d(d\tau), \quad f_d \in M_k(4m/d^2, \chi_{m_0}) \text{ and } d|m_1.$$

Thus,

$$\phi|\mathbb{Z}_m = \sum_{d^2|m} \phi_d|Z_{m/d^2}|B_{d^2} = \sum_{d^2|m} \phi_d|B_{d^2}|Z_m, \quad \phi_d \in J_{k+1/2,m/d^2}(4), \phi_d|Z_{m/d^2} = f_d.$$

Using, $\mathbb{Z}_m$ is injective on $J_{k+1/2,m}(4)$, we get

$$\phi = \sum_{d^2|m} \phi_d|B_{d^2}.$$

This characterises the space of old forms as stated above.

Note that $kerK_m$ contains $J^{old}_{k+1/2,m}(4)$.
Now, $\phi \in KerK_m$, then, $C_\phi(D, r) = 0$, $\forall (D, m) = 1$, therefore

$$a_{\phi|Z_m}(D) = 0, \forall (D, m) = 1.$$

$\phi|Z_m \in M_k(4m, \chi_m)$, using $\chi_m = \chi_{m_0}\chi^2_{m_1} = \chi_{m_0}$, $(m = m_1 m_1^2, m_0{-}square{-}free)$
Thus

$$\phi|Z_m \in M_k(4m, \chi_{m_0}), \quad a_{\phi_m}(|D|) = 0 \ \ \forall (D, m) = 1$$

$$\Longrightarrow \phi|\mathbb{Z}_m = \sum_{d^2|m} f_d(d\tau), \quad f_d \in M_k^{old}(4m/d^2, \chi_{m_0})$$

Hence, $\phi \in J^{old}_{k+1/2,m}(4m)$. This completes the proof of the theorem.

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