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A NOVEL APPROACH FOR SOLVING FRACTIONAL BAGLEY-TORVIK EQUATIONS

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Abstract: In this paper, we obtained the solution of Bagley-Torvik equations which belongs to a class of fractional ordinary differential equation by the use of Elzaki transformation method. Here the fractional derivatives are defined in Caputo sense. Graphically comparison of the obtained solution by proposed method with the existing methods is also discussed.

Keywords and Phrases: Fractional Bagley-Torvik equation, Elzaki transformation, Caputo fractional derivative.

2010 Mathematics Subject Classification: 26A33, 34A08, 44A15.

1. Introduction

Over the last decades, researchers have been investigated many fractional differential equations, which is used in numerous fields. Due to so much useful, much research has been done in this field. Several phenomenons in natural science and engineering are written in the form of fractional differential equations. Fractional differential equations have been used within many Mathematical models, for more detail; see [10, 11, 22, 28]. Various important researches on fractional calculus has been deliberated in the past years and a lot of books have been written by various authors, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 17, 19, 24, 26]. In the literature of fractional differentiations and integrations there are several integral transforms like Laplace, Fourier, Mellin, Sumudu to name but a few. A new integral transform namely Elzaki transform [13] which is a modified form of classical Laplace and Sumudu transform and has some good features. Elzaki transform has been efficiently used to solve the integral equations and differential equations in fractional calculus.

The main purpose of this paper is to demonstrate how applicable the Elzaki transform is in solving fractional ordinary differential equation i.e. Bagley-Torvik equations.

Bagley-Torvik equation (BTE) firstly appeared in their seminal work [27] where they proposed to model viscoelastic behavior of geological strata, metals and glasses by using fractional differential equations, showing that this approach is effective in describing structures containing elastic and viscoelastic components. It plays important role in many engineering and applied science problems. In particular, the equation with 1/2 - order derivative or 3/2 - order derivative can model the frequency-dependent damping materials quite satisfactorily. It can also describe motion of real physical systems, the modeling of the motion of a rigid plate immersed in a Newtonian fluid and a gas in a fluid, respectively [24, 27].

The generic form of Bagley-Torvik equation [24] is

$$m\frac{d^2y(x)}{dx^2}+c\frac{d^\alpha y(x)}{dx^\alpha}+ky(x)=f(x), \quad where \ \ \alpha=\frac{1}{2}or\frac{3}{2} \tag{1.1}$$

Subject to initial conditions y(0) $=\delta_0$, $\frac{dy(x)}{dx}|_{x=0}=\delta_1$,

Here m, c, k, f(x) and y(x) denote the mass, damping, stiffness coefficients, external force and displacement function respectively, $\frac{d^\alpha y(x)}{dx^\alpha}$ is the fractional derivative of order $\alpha \in$ (0,2). Here δ_0 and δ_1 are real constants.

2. Basic features of fractional calculus

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers. In this section, we give some basic definitions of fractional calculus [16, 17, 24], as follows

Definition 2.1. *A real function f(t), $t>0$ is said to be in the space C_μ if $\mu \in R$, there exists a real number $p>\mu$ and the function $f_1(t) \in C[0,\infty)$ such that $f(t) = t^p f_1(t)$. Moreover, if $f^{(n)} \in C_\mu$ then f(t) is said to be in the space ${C_\mu}^n$, $n \in N$.*

Definition 2.2. *Fractional integral of Riemann-Liouville type of order $\alpha \geq 0$, for a function f(t) is*

$$I^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, & \alpha>0\\ f(t), & \alpha=0\end{cases}$$

Where $\Gamma(\cdot)$ *denotes the Gamma function.*

Definition 2.3. *The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *for a function* $f(t)$ *is defined as*

$$D^{\alpha} f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t),\ n \in N,\ n - 1 < \alpha \leq n.$$

Definition 2.4. *The Caputo fractional derivative of order* $\alpha > 0$ *is defined as*

$$D^{\alpha} f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n, n \in N \\ \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & 0 \leq n-1 < \alpha < n, \end{cases}$$

Where n *is an integer,* $t > 0$ *and* $f(t) \in C_1{}^n$.

3. Basic properties of Elzaki transformation

Definition 3.1. *The Elzaki transform of* $f(t)$ *is defined as [13]*

$$E[f(t)] = E[f(t), v] = T(v) = v \int\limits_0^t f(t) e^{-\frac{t}{v}} dt, k_1 < v < k_2, k_1 k_2 > 0, 0 \leq t < \infty, \tag{3.1}$$

where $f(t)$ *is taken from the set* A, *which is defined as*
$A = \left[f(t);\ \exists\ M,\ k_j > 0,\ j = 1,\ 2,\ |f(t)| < M\ e^{\frac{|t|}{k_j}},\ \text{if } t \in (-1)^j \times [0, \infty) \right]$
here, constant M *must be finite number,* k_1 *and* k_2 *may be finite or infinite.*

In order to obtain the Elzaki transform of the Caputo fractional derivative, we apply the Laplace transform formula for the Caputo fractional derivative (see [24]).

$$L[D^{\alpha} f(t), s] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), n-1 < \alpha \leq n. \tag{3.2}$$

Theorem 3.1. *([14]) Let* $A = \left[f(t);\ \exists\ M,\ K_j > 0,\ j = 1,\ 2,\ |f(t)| < M\ e^{\frac{|t|}{k_j}},\ \text{if } t \in (-1)^j \times [0, \infty) \right]$ *and* $f(t) \in A$. *If* $F(s)$ *is the Laplace transform of* $f(t)$, *then the Elzaki transform* $T(v)$ *of* $f(t)$ *is given by*

$$T(\vartheta) = \vartheta F\Big(\frac{1}{\vartheta}\Big). \tag{3.3}$$

Theorem 3.2. *Suppose T(ϑ) is the Elzaki transform of the function f(t). Then*

$$E[D^{\alpha}f(t),\vartheta]=\frac{T(\vartheta)}{\vartheta^{\alpha}}-\sum_{k=0}^{n-1}\vartheta^{k-\alpha+2}f^{(k)}(0), n-1<\alpha\leq n. \tag{3.4}$$

Proof. From Theorem 3.1, we have

$$\mathrm{E}[\mathrm{D}^{\alpha}\mathrm{f(t)},\ \vartheta]=\vartheta\mathrm{L}\Big[\mathrm{D}^{\alpha}\mathrm{f(t)},\ \tfrac{1}{\vartheta}\Big].$$

Now, applying equation (3.2), we get

$$\mathrm{E}[\mathrm{D}^{\alpha}\mathrm{f(t)},\ \vartheta]=\vartheta\Bigg(\Big(\tfrac{1}{\vartheta}\Big)^{\alpha}\ \mathrm{F}\Big(\tfrac{1}{\vartheta}\Big)\ \text{-}\sum_{k=0}^{n-1}\Big(\tfrac{1}{\vartheta}\Big)^{\alpha-k-1}\ \mathrm{f}^{(k)}(0)\Bigg)$$

$$=\frac{\vartheta F\Big(\frac{1}{\vartheta}\Big)}{\vartheta^{\alpha}}\ \text{-}\sum_{k=0}^{n-1}\vartheta^{k-\alpha+2}\ \mathrm{f}^{(k)}(0).$$

For more results, concern to Elzaki transform, see [13, 14, 15].

4. Implementation of Elzaki Transform Method on Bagley-Torvik equations

In this section, ETM is implemented to Bagley-Torvik equations of fractional order in the following examples.

Example 4.1. Let us take following Bagley-Torvik equation given by Pedas and Tamme [18]

$$D^2y(t)+D^{\frac{3}{2}}y(t)+y(t)=\frac{15}{4}\sqrt{t}+\frac{15}{8}\sqrt{\Pi}t+t^2\sqrt{(t)}, hereD^n=\frac{d^n}{dt^n}, andt\geq 0, \tag{4.1}$$

With the initial condition being y(0) = 0 = $\mathrm{y}^{(1)}(0)$. Here $\mathrm{y}^1(0)=\frac{dy(x)}{dx}|_{x=0}$.

In order to solve (4.1), taking Elzaki transform of both sides of eq. (4.1), we get

$\mathrm{E}[\mathrm{D}^2\mathrm{y(t)}]+\mathrm{E}[\mathrm{D}^{\frac{3}{2}}\mathrm{y(t)}]+\mathrm{E}[\mathrm{y(t)}]=\mathrm{E}[\frac{15}{4}\sqrt{t}]+\mathrm{E}[\frac{15}{8}\sqrt{\Pi}\mathrm{t}]+\mathrm{E}[\mathrm{t}^2\sqrt{(t)}]$,

On using (3.4), also see [13, 15], yields

$\frac{T(\vartheta)}{v^2}$ - $\sum_{k=0}^{1}\vartheta^{k-\alpha+2}\ \mathrm{y}^{(k)}(0)$ + $\frac{T(\vartheta)}{v^{\frac{3}{2}}}$ - $\sum_{k=0}^{1}\vartheta^{k+\frac{1}{2}}\ \mathrm{y}^{(k)}(0)$ + $\mathrm{T}(\vartheta)=\frac{15}{8}\ \Gamma\Big(\frac{3}{2}\Big)\ \vartheta^{\frac{5}{2}}$
$+\ \frac{15}{2}\ \sqrt{\pi}\ \vartheta^3+\Gamma\Big(\frac{7}{2}\Big)\ \vartheta^{\frac{9}{2}}$

using given initial condition, we get

$$\frac{T(\vartheta)}{\vartheta^2}+\frac{T(\vartheta)}{\vartheta^{\frac{3}{2}}}+T(\vartheta)=\frac{15}{8}\sqrt{\Pi}\vartheta^{\frac{5}{2}}+\frac{15}{8}\sqrt{\Pi}\vartheta^3+\frac{15}{8}\sqrt{\Pi}\vartheta^{\frac{9}{2}} \tag{4.2}$$

$$\left(\frac{1+\vartheta^{\frac{1}{2}}+\vartheta^2}{\vartheta^2}\right) T(\vartheta) = \frac{15}{8}\sqrt{\Pi}\vartheta^{\frac{5}{2}}(1 + \vartheta^{\frac{1}{2}} + \vartheta^2)$$

$$\Rightarrow T(\vartheta) = \frac{15}{8}\sqrt{\Pi}\vartheta^{\frac{9}{2}} \tag{4.3}$$

applying inverse Elzaki transform to (4.3) and on using results presented by Elzaki in [13, 15], we have

$$\mathrm{y(t)} = \mathrm{t}^{\frac{5}{2}}.$$

This is the analytical solution of eq. (4.1).

Example 4.2. Let us now solve the following Bagley-Torvik equation given in Parisa and Yadollah [23] and Mohammadi and Mohyud-Din [20] as

$$D^{\frac{3}{2}}y(t) + y(t) = \frac{2t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + t^2 - t, t \geq 0, \tag{4.4}$$

with the initial condition being y(0) = 0, $\mathrm{y}^{(1)}(0)$ = -1.
Applying Elzaki transform on both sides of eq. (4.4) with the initial condition, get

$$\frac{T(\vartheta)}{\vartheta^{\frac{3}{2}}} + \vartheta^{\frac{3}{2}} + \mathrm{T}(\vartheta) = 2\vartheta^{\frac{5}{2}} + 2\vartheta^4 - \vartheta^3$$

$$\Rightarrow \left(\frac{1}{\vartheta^{\frac{3}{2}}} + 1\right)\mathrm{T}(\vartheta) = 2\vartheta^{(\frac{5}{2})} + 2\vartheta^4 - \vartheta^3 - \vartheta^{\frac{3}{2}}$$

$$\Rightarrow T(\vartheta) = 2\vartheta^4 - \vartheta^3, \tag{4.5}$$

applying inverse Elzaki transform to (4.5) and on using the results presented by Elzaki in [13, 15], we get

$$\mathrm{y(t)} = \mathrm{t}^2 - \mathrm{t}.$$

This is the analytical solution of eq. (4.4).

Example 4.3. Further, consider the following Bagley-Torvik equation in Ford and Connolly [21]

$$D^2y(t) + D^{\frac{1}{2}}y(t) + y(t) = t^2 + 2 + \frac{2.6666666667}{\Gamma(\frac{1}{2})}t^{1.5}, \tag{4.6}$$

With the initial condition being y(0) = 0, $\mathrm{y}^{(1)}(0)$ = 0, t $\geq$ 0.
First, apply Elzaki transform on both sides of eq. (4.6) and initial condition to get

$$\frac{T(\vartheta)}{\vartheta^2} + \frac{T(\vartheta)}{\vartheta^{\frac{1}{2}}} + \mathrm{T}(\vartheta) = 2\vartheta^4 + 2\vartheta^2 + \frac{2.6666666667}{\sqrt{\pi}}\Gamma(\tfrac{5}{2})\vartheta^{\frac{7}{2}}$$

$$\left(\frac{(1+\vartheta^{(\frac{3}{2})}+\vartheta^2}{\vartheta^2}\right)\mathrm{T}(\vartheta) = 2\vartheta^4 + 2\vartheta^2 + 2\vartheta^{\frac{7}{2}}$$

$$\Rightarrow T(\vartheta) = 2\vartheta^4 \tag{4.7}$$

Applying inverse Elzaki transform to (4.7), we get

$$\mathrm{y(t)} = \mathrm{t}^2$$

This is the analytical solution of eq. (4.6).

Example 4.4. Finally, let us take the Bagley-Torvik equation in Ford and Connolly [21]

$$D^2y(t) + D^{\frac{1}{2}}y(t) + y(t) = t^3 + 6t + \frac{3.2}{\Gamma\left(\frac{1}{2}\right)}t^{2.5} \tag{4.8}$$

With the initial condition being $\mathrm{y}(0) = 0$, $\mathrm{y}^{(1)}(0) = 0$, $\mathrm{t} \geq 0$.
Taking Elzaki transform on both sides of eq. (4.8) and initial condition, the subsequent equation is obtained as

$$\left(\frac{1+\vartheta^2+\vartheta^{\frac{3}{2}}}{\vartheta^2}\right)\mathrm{T}(\vartheta) = 6\vartheta^5 + 6\vartheta^3 + 6\vartheta^{\frac{9}{2}}$$

$$\Rightarrow T(\vartheta) = 6\vartheta^5 \tag{4.9}$$

Applying inverse Elzaki transform to (4.9), we have

$$\mathrm{y(t)} = \mathrm{t}^3$$

This is the analytical solution of eq. (4.8).

Comparison of the present method with the existing methods in Fig. (a) for Example 4.1, Fig. (b) for Example 4.2, Fig. (c) for Example 4.3 and Fig. (d) for Example 4.4.

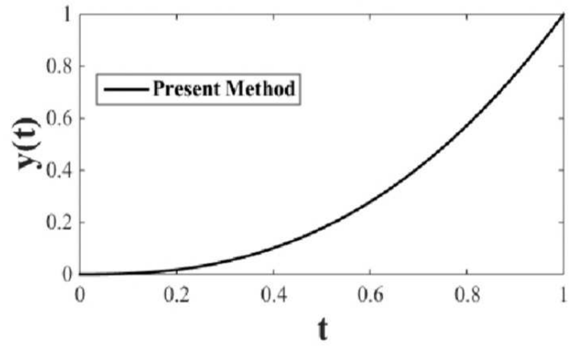

Fig. (a)

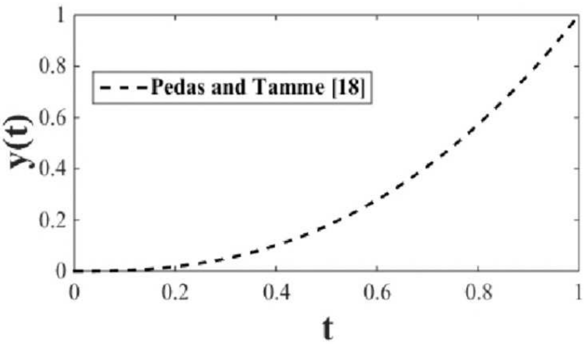

Fig. (a)

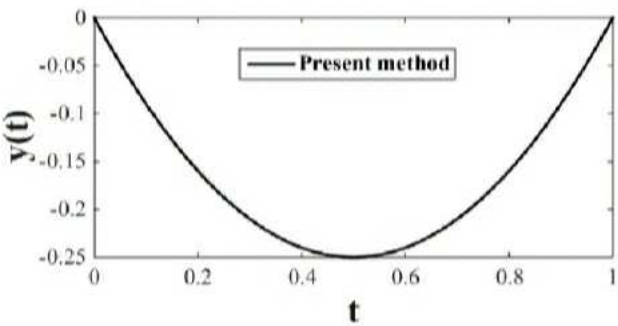

Fig. (b)

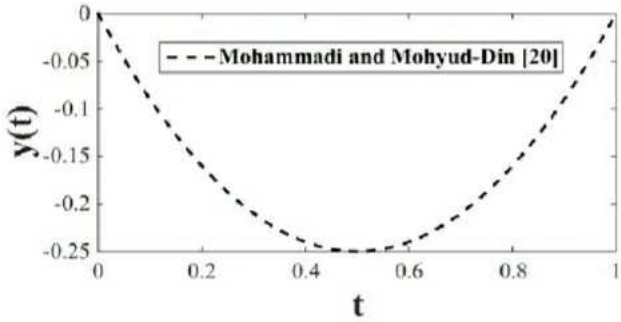

Fig. (b)

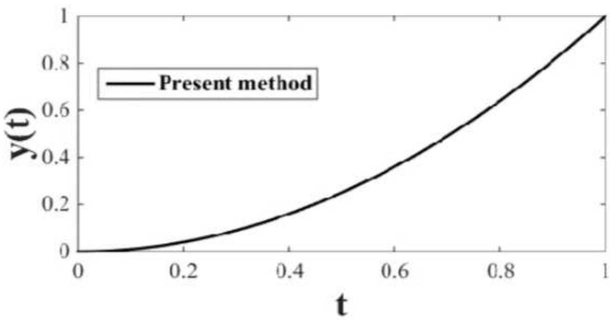

Fig. (c)

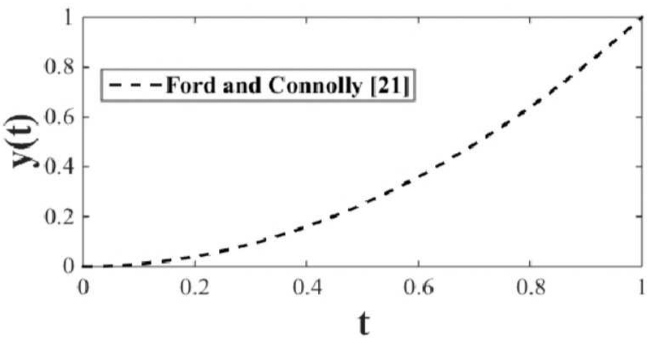

Fig. (c)

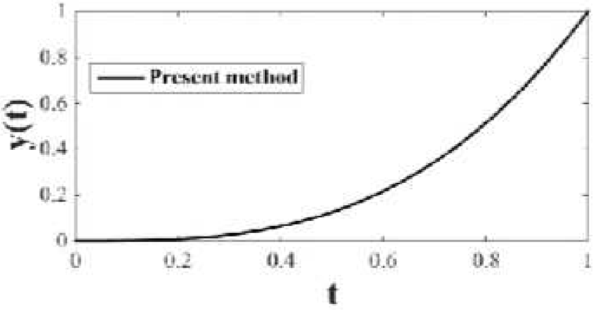

Fig. (d)

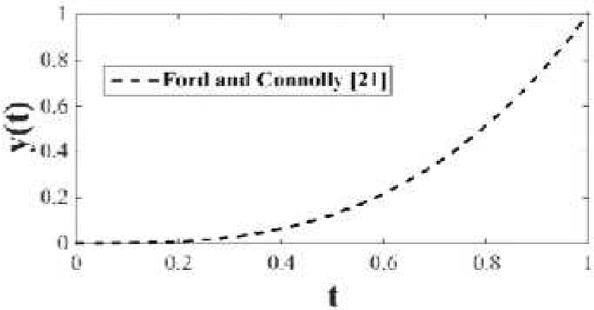

Fig. (d)

5. Conclusion

In this paper, Elzaki Transform Method is successfully applied to solve Bagley-Torvik equations. The proposed method is an easy, highly efficient and robust method for finding the solution of fractional ordinary differential equations.

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