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EDGE VERTEX PRIME LABELING FOR SOME GRAPHS

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Abstract: In this paper, edge vertex prime labeling for helm graph and for even n, planter graph R_n are investigated. Further, a necessary condition for a graph to be an edge vertex prime graph is investigated.

Keywords and Phrases: Edge vertex prime labeling, helm graph, planter graph, prime labeling.

2010 Mathematics Subject Classification: 05C78.

1. Introduction

In this paper, all graphs are considered simple, undirected and finite. We follow Gross and Yeelen [4] for various graphs and graph theoretical notation and Burton [2] for number theoretical results.

An edge vertex prime labeling is a variation of prime labeling. *Prime labeling* is a bijection $f : V(G) \longrightarrow \{1, 2, ..., n\}$ such that for each edge $e = uv$, $\gcd(f(u), f(v)) = 1$ and corresponding graph is a *prime graph.* Roger Entringer introduced the notion of prime labeling and in 1982, Tout and et. al discussed it in their paper [10]. Lots of works done on prime labeling till today, which have given in dynamic survey on graph labeling by Galian [3].

Edge vertex prime labeling is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G) \cup E(G)|\}$ such that for any edge $e = uv$; $f(u), f(v)$ and $f(uv)$ are pairwise relatively prime and corresponding graph is an *edge vertex prime graph.* Edge vertex prime labeling was introduced by R. Jagadesh and J. Bhaskar Babujee [5] in 2017. They proved that paths, cycles and star-$K_{1,n}$ admit an edge vertex prime labeling. Y.

Parmar [7], [8] proved that wheel W_n, fan f_n, friendship F_n and complete bipartite graph $K_{2,n}$ are edge vertex prime graph for every n and the complete bipartite graph $K_{3,n}$ for $n = 3, 4, \ldots, 29$ is an edge vertex prime graph. N. Shrimali and Y. Parmar [9] proved that some trees like banana tree, bistar graph, centipede tree, coconut tree, double coconut tree, firecracker graph are edge vertex prime graph.

Definition 1.1. *Helm graph H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of the $n-$cycle.*

Definition 1.2. *Fan graph $f_n(n \geq 2)$ is obtained by joining all vertices of a path P_n to a further vertex, called center.*

Definition 1.3. *Planter graph $R_n(n \geq 3)$ is a graph obtained by joining a fan graph $(n \geq 2)$ and cycle graph C_n $(n \geq 3)$ with sharing a common vertex.*

In this paper, first we will show that helm graph and planter graphs for even n are edge vertex prime graphs. Then, we will give a necessary condition for being edge vertex prime graph. Further, we will prove that $P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$, $K_{1,n_1} \cup K_{2,n_2} \cup \ldots \cup K_{3,n_k}$ are not an edge vertex prime graphs.

2. Main Results

Theorem 2.1. *The Helm graph is an edge vertex prime graph.*

Proof. Let G be a helm graph with vertex set $V(G) = \{v, v_i, u_i/1 \leq i \leq n\}$, where v is the apex vertex, v_is are the vertices of cycle and u_is are the pendant vertices, $|V(G)| = 2n + 1$ and edge set
$E(G) = \{vv_i/1 \leq i \leq n\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{v_1v_n\}$, $|E(G)| = 3n$. Thus $|V(G) \cup E(G)| = 5n + 1$.
Define a bijection, $f : V(G) \cup E(G) \to \{1, 2, \ldots, 5n + 1\}$ as follows:
$f(v) = 1$

$$f(v_1) = \begin{cases} 3; & n \equiv 1(mod\ 6) \\ 5; & n \not\equiv 1(mod\ 6) \end{cases}$$

$$f(v_{2i-1}) = \begin{cases} 10i-7; & i+1 \equiv 0(mod\ 3) \\ 10i-5; & i \neq 1,\ i+1 \not\equiv 0(mod\ 3) \end{cases}$$

$$f(v_{2i}) = \begin{cases} 10i-1; & i \equiv 0(mod\ 3) \\ 10i-3; & i \not\equiv 0(mod\ 3) \end{cases}$$

$$f(u_1) = \begin{cases} 5; & n \equiv 1(mod\ 6) \\ 3; & n \not\equiv 1(mod\ 6) \end{cases}$$

$$f(u_{2i-1}) = \begin{cases} 10i-5; & i+1 \equiv 0(mod\ 3) \\ 10i-7; & i \neq 1,\ i+1 \not\equiv 0(mod\ 3) \end{cases}$$

$$f(u_{2i}) = \begin{cases} 10i-3; & i \equiv 0(mod\ 3) \\ 10i-1; & i \not\equiv 0(mod\ 3) \end{cases}$$

$$f(vv_2) = \begin{cases} 6; & n \equiv 1(mod\ 6) \\ 10; & n \not\equiv 1(mod\ 6) \end{cases}$$

$$f(vv_{2i-1}) = \begin{cases} 10i-8; & i = 1,2,3,\ldots,\lceil\frac{n-2}{2}\rceil \\ 10i-4; & i = \lceil\frac{n}{2}\rceil, n \equiv 1(mod\ 6) \\ 10i-8; & i = \lceil\frac{n}{2}\rceil, n \not\equiv 1(mod\ 6) \end{cases}$$

$f(vv_{2i}) = 10i, i = 2,\ldots,\lceil\frac{n-1}{2}\rceil$

$f(v_{2i-1}u_{2i-1}) = 10i-6, i = 1,2,3,\ldots,\lceil\frac{n}{2}\rceil$

$f(v_{2i}u_{2i}) = 10i-2, i = 1,2,\ldots,\lceil\frac{n-1}{2}\rceil$

$$f(v_1v_2) = \begin{cases} 10; & n \equiv 1(mod\ 6) \\ 6; & n \not\equiv 1(mod\ 6) \end{cases}$$

$f(v_{2i-1}v_{2i}) = 10i-4, i = 2,3,\ldots,\lceil\frac{n-1}{2}\rceil$

$f(v_{2i}v_{2i+1}) = 10i+1, i = 1,2,\ldots,\lceil\frac{n-2}{2}\rceil$

$$f(v_nv_1) = \begin{cases} 5n-3; & n \equiv 1(mod\ 6) \\ 5n+1; & n \not\equiv 1(mod\ 6) \end{cases}$$

In order to show that f is an edge vertex prime labeling, we need to show that for each edge $uv, f(u), f(uv)$ and $f(v)$ are pairwise relatively prime.
First, we prove that, for each edge $vv_i, f(v), f(vv_i), f(v_i)$ are pairwise relatively prime. Since, $f(v) = 1, \gcd(f(v), f(v_i)) = \gcd(f(v), f(vv_i)) = 1$. From the definition of f, we can see that, for both cases, $n \equiv 1(mod\ 6)$ and $n \not\equiv 1(mod\ 6)$, $\gcd(f(v_1), f(vv_1)) = 1$ and $\gcd(f(v_2), f(vv_2)) = 1$. Now, we prove that $\gcd(f(v_{2i-1}), f(vv_{2i-1})) = 1$ by considering the following two cases.

Case-1: $i+1 \equiv 0(mod\ 3)$: In this case, $f(v_{2i-1}) = 10i-7$ and $f(vv_{2i-1}) = 10i-8$ and so $\gcd(f(v_{2i-1}), f(vv_{2i-1})) = \gcd(10i-7, 10i-8) = 1$.

Case-2: $i+1 \not\equiv 0(mod\ 3), i \neq 1$: In this case, $f(v_{2i-1}) = 10i-5$ and $f(vv_{2i-1}) = 10i-8$ for $i \neq \lceil\frac{n}{2}\rceil$ and for $i = \lceil\frac{n}{2}\rceil$, $f(vv_{2i-1}) = 10i-4$ when $n \equiv 1(mod\ 6)$,whereas $f(vv_{2i-1}) = 10i-8$ when $n \not\equiv 1(mod\ 6)$. Thus, $f(vv_{2i-1}) = 10i-8$ or $10i-4$. Hence, $\gcd(f(v_{2i-1}), f(vv_{2i-1})) = 1$.
Now, for $i \equiv 0(mod\ 3)$, $\gcd(f(v_{2i}), f(vv_{2i})) = \gcd(10i-1, 10i) = 1$, as they are consecutive integers and for $i \not\equiv 0(mod\ 3)$, $\gcd(f(v_{2i}), f(vv_{2i})) = \gcd(10i-3, 10i) = 1$ as the difference between them is three and they are not multiple of three.
Thus, for each $vv_i, f(v), f(vv_i), f(v_i)$ are pairwise relatively prime.

Now, we prove that, for each edge $v_iv_{i+1}, f(v_i), f(v_iv_{i+1}), f(v_{i+1})$ are pairwise relatively prime. For $i \neq n$, labels of v_i and v_{i+1} are odd integers with even difference therefore $\gcd(f(v_i), f(v_{i+1})) = 1$. When $n \equiv 1(mod\ 6)$, $\gcd(f(v_1), f(v_1v_2)) = \gcd(3, 10) = 1$, $\gcd(f(v_2), f(v_1v_2)) = \gcd(7, 10) = 1$ and $\gcd(f(v_1), f(v_2)) = \gcd(3, 7) = 1$. When $n \not\equiv 1(mod\ 6), \gcd(f(v_1), f(v_1v_2)) = \gcd(5, 6) = 1$,
$\gcd(f(v_2), f(v_1v_2)) = \gcd(7, 6) = 1$ and $\gcd(f(v_1), f(v_2)) = \gcd(5, 7) = 1$.
For $i+1 \equiv 0(mod\ 3)$, $\gcd(f(v_{2i-1}), f(v_{2i-1}v_{2i})) = \gcd(10i-7, 10i-4) = 1$ as their difference is three and $10i-4$ can't be multiple of three and for $i+1 \not\equiv 0(mod\ 3)$, $\gcd(f(v_{2i-1}), f(v_{2i-1}v_{2i})) = \gcd(10i-5, 10i-4) = 1$ as they are consecutive integers. For $i \equiv 0(mod\ 3)$, $f(v_{2i}) = 10i-1$ and $f(v_{2i}v_{2i+1}) = 10i+1$ and so we are done. For $i \not\equiv 0(mod\ 3)$, $\gcd(f(v_{2i}), f(v_{2i}v_{2i+1})) = \gcd(10i-3, 10i+1) = 1$ because they are odd integers and their difference is four.
Finally, for each edge u_iv_i, $f(u_i), f(u_iv_i)$ and $f(v_i)$ are consecutive integers in which $f(u_i)$ and $f(v_i)$ are odd integers so we are done.

Hence, the Helm graph is an edge vertex prime graph.

Illustration 2.2. *Edge vertex prime labeling for Helm graph H_9.*

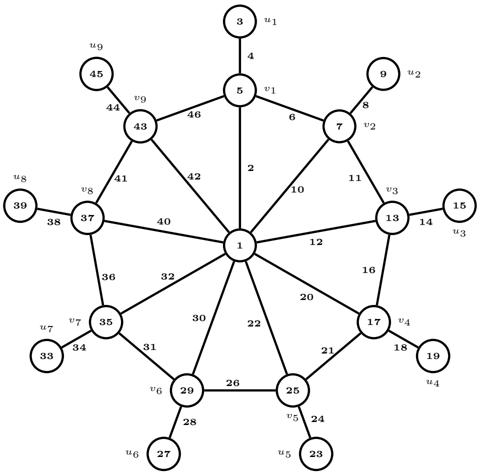

Figure 1: H_9

Illustration 2.3. *Edge vertex prime labeling for Helm graph H_7.*

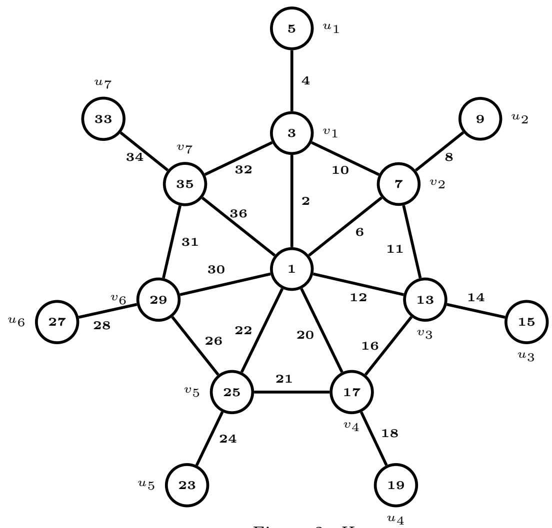

Figure 2: H_7

Further, in [1], if $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two connected graphs then the graph obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1 is denoted by $G_1 \hat{O} G_2$. The resultant graph $G = G_1 \hat{O} G_2$ contains $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges. In [6], Jagadesh and Babuji proved that, if $G(p, q)$ has edge vertex prime labeling with $p+q$ is odd then there exist a graph from the class $G \hat{O} C_n$ that admits edge vertex prime labeling. Planter graph R_n with odd n is a special case of this theorem. Here, we prove R_n is edge vertex prime for even n.

Theorem 2.4. *For even n, the planter graph R_n is an edge vertex prime graph.*
Proof. Let $G = R_n$ be a planter graph with vertex set
$V(G) = \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$ where v_is are vertices of fan graph and u_is are vertices of cycle and common vertex of fan and cycle is u_1, which is an apex vertex of fan. Edge set $E(G) = \{u_1v_i/1 \leq i \leq n\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_1u_n\}$. Thus $|V(G) \cup E(G)| = 5n - 1$.
Define bijection, $f : V(G) \cup E(G) \to \{1, 2, \ldots, 5n-1\}$ for even n, as follows:
$f(u_1) = 1$

$$f(v_i) = \begin{cases} 3i; & i \text{ is odd} \\ 3i-1; & i \text{ is even} \end{cases}$$

$$f(u_1v_i) = \begin{cases} 3i-1; & i \text{ is odd} \\ 3i; & i \text{ is even} \end{cases}$$

$f(v_iv_{i+1}) = 3i + 1;\ i = 1, 2, \ldots, n-1$
$f(u_i) = 3n + 2i - 1;\ i = 2, 3, \ldots, n$

$f(u_i u_{i+1}) = 3n + 2i;\ i = 1, 2, \ldots, n-1$
$f(u_1 u_n) = 3n + 1$
In order to show that, f is edge vertex prime labeling, we need to show that for each edge $uv, f(u), f(uv)$ and $f(v)$ are pairwise relatively prime.
Here, for each i, $\gcd(f(u_1), f(v_i)) = \gcd(f(u_1), f(u_1 v_i)) = 1$ as $f(u_1) = 1$ and $\gcd(f(v_i), f(u_1 v_i)) = 1$ as they are consecutive integers. For even i, $f(v_i), f(v_i v_{i+1})$ and $f(v_{i+1})$ are odd consecutive integers, hence condition is satisfied. And for odd i, $f(v_i)$, $f(v_i v_{i+1})$ and $f(v_{i+1})$ are consecutive integers starting with an odd integer and hence condition is satisfied. For $i \neq 1, n, f(u_i), f(u_i u_{i+1}), f(u_{i+1})$ are consecutive integers starting with and odd integer therefore condition is satisfied for an edge $u_i u_{i+1}$. And $\gcd(f(u_1), f(u_2)) = \gcd(f(u_1), f(u_1 u_2)) = \gcd(f(u_1), f(u_n)) = \gcd(f(u_1), f(u_1 u_n)) = 1$ as $f(u_1) = 1$, $\gcd(f(u_2), f(u_1 u_2)) = \gcd(3n+3, 3n+2) = 1$ as they are consecutive integers and we can easily verify that,
$\gcd(f(u_n), f(u_1 u_n)) = \gcd(5n-1, 3n+1) = 1$.

Hence, the planter graph for even n is an edge vertex prime graph.

Illustration 2.5. *Edge vertex prime labeling for Planter graph R_6.*

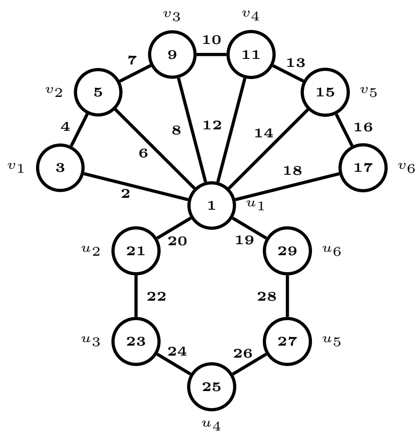

Figure 3: R_6

Lemma 2.6. *A necessary condition for a graph $G(n, m)$ without isolated vertices to be an edge vertex prime graph is $m \geq \lfloor \frac{m+n}{2} \rfloor$.*
Proof. Let G be a graph with n vertices and m edges. For edge vertex prime graph, the label on an edge and label on its end vertices must be pairwise relatively prime. To label each of these, we use integers $A = \{1, 2, \ldots, m+n\}$. Thus, to label each edge and its end vertices, call it triplet, we can use exactly one even integer from

the set A otherwise the triplet cannot be pairwise relatively prime. So when the number of edges is less than the number of even integers in the set A, then graph cannot be edge vertex prime graph. Hence, to become edge vertex prime graph, it is necessary that number of edges is greater or equal to the number of even integers in the set A. The number of even integers in the set A is $\lfloor \frac{m+n}{2} \rfloor$.
Thus, if graph G without isolated vertices, is edge vertex prime graph then $m \geq \lfloor \frac{m+n}{2} \rfloor$.

In consequence of the above lemma, we prove the following theorem.

Theorem 2.7. *Let $n_1, n_2, \ldots, n_k$ be positive integers then $P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$ is not an edge vertex prime graph.*

Proof. Let $G = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$ be a graph with order, $|V| = n = n_1 + n_2 + \ldots + n_k$ and size, $|E| = m = n_1 + n_2 + \ldots + n_k - k$.
So $|E(G) \cup V(G)| = m + n = 2(n_1 + n_2 + \ldots + n_k) - k$.
Here, $\frac{m+n}{2} = n_1 + n_2 + \ldots + n_k - \frac{k}{2}$ is always grater than $n_1 + n_2 + \ldots + n_k - k = m$.
By above lemma, G is not an edge vertex prime graph.

Theorem 2.8. *Let $n_1, n_2, \ldots, n_k$ be positive integers then $K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_k}$ is not an edge vertex prime graph.*

Proof. Let $G = K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_k}$ be a graph with order,
$|V| = n = n_1 + n_2 + \ldots + n_k + k$ and size, $|E| = m = n_1 + n_2 + \ldots + n_k$.
So $|E(G) \cup V(G)| = m + n = 2(n_1 + n_2 + \ldots + n_k) + k$.
Here, $\frac{m+n}{2} = n_1 + n_2 + \ldots + n_k + \frac{k}{2}$, which is always grater than $n_1 + n_2 + \ldots + n_k = m$.
Thus, $m < \lfloor \frac{m+n}{2} \rfloor$, so by above lemma, G is not an edge vertex prime graph.

One can see that, if $G_1(n, m)$ is an edge vertex prime graph with $m = \lfloor \frac{m+n}{2} \rfloor$ and $G_2 = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$ with $G_2 \neq P_{n_1}$ then $G = G_1 \cup G_2$ is not an edge vertex prime graph.

3. Concluding Remark

Here, we have derived some new results related to edge vertex prime labeling technique. And we have also derived necessary condition to be an edge vertex prime graph. To explore some new edge vertex prime graph is an open problem. To find sufficient condition for the same is also an open problem.

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