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PROPERTIES OF THE WRONSKIAN FOR THE SOLUTION OF A MATRIX DIFFERENTIAL EQUATION

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Abstract: In this research paper, an eigenvalue problem related to a matrix differential operator is considered. Problems, existence theorems are also discussed in this research paper. In this article the properties of Wronskian for the solution of a Matrix Differential Equation are proved which are useful in finding further results with the expansion of eigenvalue related to the problem.

Keywords and Phrases: Eigen value, Matrix Differential Equation, Matrix Differential Operator, Boundary value problem, Wronskian properties.

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1. Introduction

In this paper firstly we consider some differential equations then we check that it is a boundary value problem or not. If the problem is boundary value problem, then we check for existence and uniqueness theorem. Firstly, N.K. Chakravarty considered a pair of differential equations.

$$\frac{d^2v}{dx^2} + pu + qv - u = 0 \tag{1.1}$$

$$\frac{d^2u}{dx^2} + qu + rv - v = 0 \tag{1.2}$$

which is equivalent to $(L - \lambda)u = 0$.

Here $L = \begin{pmatrix} p(x) & \frac{d^2}{dx^2} + q(x) \\ \frac{d^2}{dx^2} + q(x) & r(x) \end{pmatrix}$, $u = \begin{pmatrix} u \\ - \\ v \end{pmatrix} = \{u, v\}$ & λ is a variable parameter and $p(x)$, $q(x)$, $r(x)$ are real value functions of x, continuous in (a, b).

For boundary value problem, conditions vectors are

$$\Phi_1 + \Phi_1(a/x, \lambda) = \{x_1(a/x, \lambda), y_2(a/x, \lambda)\} = \{x_1, y_1\}, \ (i = 1, 2)$$

and

$$\Phi_j + \Phi_j(b/x, \lambda) = \{x_j(b/x, \lambda), y_j(b/x, \lambda)\} = \{x_j, y_j\}, \ (j = 3, 4)$$

Therefore, the Wronskian $w(\lambda)$ has been defined by

$$w(\lambda) = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1 & y_2 & y_3 & y_4 \\ x_1' & x_2' & x_3' & x_4' \end{vmatrix} \tag{1.3}$$

2. Problem Statement

The differential equation which is considered in the problem is given below,

$$-\frac{d}{dx}\left(P_0 \frac{du}{dx}\right) + pu + rv = \lambda(F_{11}u + F_{11}v), \tag{2.1}$$

$$i\frac{dv}{dx} + qv + ru = -\lambda(F_{11}u + F_{11}v)$$

where,

(i) P_0 is a real valued function of u, having continuous derivatives of the first order in $a \leq x \leq b$.

(ii) P, q, r are all real valued function of u continuous in $a \leq x \leq s$.

(iii) $P_0(x) > 0$ for $a \leq x \leq b$.

(iv) F_{11}, F_{12}, F_{21}, F_{22} are real valued continuous functions of x such that the matrix $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ is symmetric and positive define for $a \leq x \leq b$, and parameter, real or complex.

(v) For a given vector

$$\Phi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.2}$$

and the operator

$$L = \begin{pmatrix} -\frac{d}{dx}\left(P_0\frac{d}{dx} + P\right) & r \\ r & i\frac{d}{dx} + q \end{pmatrix}, \tag{2.3}$$

Therefore, the equation (1.1) reduces into

$$L\Phi = -\lambda F \Phi$$

We impose the Following boundary condition on $\Phi = \begin{pmatrix} u \\ v \end{pmatrix}$

$$u'(a) = 0 \tag{2.5}$$

$$v(a) = v(b) \tag{2.6}$$

$$u'(b) = 0 \tag{2.7}$$

Now, the equation (2.4) together with equation (2.5), (2.6) and (2.7) becomes a boundary value problem.

3. Existence and Uniqueness Theorem

Let $P_0(x)$, $p(x)$, $q(x)$, $r(x)$ and F satisfy equation (1.1) and let λ, B, C be three constants not all vanishing. Simultaneously, then the differential equation (1.4) has a unique solution $\Phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ which satisfies,

$$u(\alpha) = \lambda_1 u(\alpha) = \beta_1 v(\alpha) = C \tag{3.1}$$

where $a \le \alpha \le b$, the accent denoting differentiation with respect to x. Also for each x in the closed interval ax, a, b, (x), (x), $U'(x)$, $V'(x)$ and $u''(x)$ are all integral functions of the complex variable λ.

This is a traditional theorem and can be proved almost similarly as it has been proved in Bhagat.

4. Definition and Properties of the Wronskian

We shall investigate some properties of the Wronskian of the solutions of the differential equation (2.1). Here, we firstly we define the Wronskian of the vectors. Let $\Phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $\Phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, $\Phi_3 = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}$ are the three vectors which are the

solution of equation (1.4).
Now the functional determinant of the vectors are given below.

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1' & u_2' & u_3' \end{vmatrix} \tag{4.1}$$

Equation (4.1) is called the Wronskian of the three vectors Φ_1, Φ_2, Φ_3. Where, $\Phi_1 = \Phi_1(x, \lambda)$ $u_1 = u_1(x, \lambda)$ etc and $x \in [a, b]$.
Now equation (4.1) may be denoted as

$$W_x(\Phi_1, \Phi_2, \Phi_3) = w(\Phi_1, \Phi_2, \Phi_3)(x) = w$$

$$\Rightarrow w(\Phi_1, \Phi_2, \Phi_3)(x) = w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1' & u_2' & u_3' \end{vmatrix}$$

Where, the suffix x denotes the particular value of the Wronskian at x.

4.1. Properties of the Wronskian

Theorem 1. *$w(\Phi_1, \Phi_2, \Phi_3)$ is not independent of x.*
Proof.

$$W = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1' & u_2' & u_3' \end{vmatrix}$$

$$\Rightarrow \frac{dw}{dx} = \begin{vmatrix} u_1' & u_2' & u_3' \\ v_1 & v_2 & v_3 \\ u_1' & u_2' & u_3' \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1' & v_2' & v_3' \\ u_1' & u_2' & u_3' \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ u_1'' & u_2'' & u_3'' \end{vmatrix}$$

Putting the value of $v_1'.v_2'.v_3' : u_1''.u_2''.u_3''$ are using $R_2 \to R_2 - rR_1$ in the second determinant and $R_3 \to R_3 - rR_2$.
In the third determinant we get

$$\frac{dw}{dx} = \left[i \left(q + \lambda.F_{22} \neq \frac{P_0'}{P_0} \right) \right] w$$

$$= W' \neq 0...w \quad \text{and} \quad i(q + \lambda.F_{22}) \neq \frac{P_0'}{P_0}$$

Which shows that Wronskian is not independent of x.

Theorem 2. *$P_0(x)Wx(\Phi_1, \Phi_2, \Phi_3)(x, \lambda)x \times P_0(x)Wx(\bar{\Phi_1}, \bar{\Phi_2}, \bar{\Phi_3})(x\bar{\lambda})x$ is independent of x.*

Proof. From theorem 1, we have

$$\frac{w'}{w} = i(q + \lambda.F_{22}) - \frac{P_0'}{P_0}$$

Integrating the above equation then we get

$$\begin{aligned} \int_{x_0}^{x} \frac{w'}{w} dx =& i \int_{x_0}^{x} (q + \lambda.F_{22}) dx - \int_{x_0}^{x} \frac{P_0'}{P_0} dx \quad (4.3) \\ \Rightarrow \log \left(\frac{w_x P_0(x)}{wx_0.P_0(x_0)} \right) =& i \int_{x_0}^{x} (q + \lambda.F_{22}) dx \\ \Rightarrow P_0(x) w_x =& w_{x_0} P_0(x_0) e^i \int_{x_0}^{x} (q + \lambda.F_{22}) dx \\ \Rightarrow P_0(x) w_x (\bar{\Phi_1}(x, \lambda)).(\bar{\Phi_2}(x, \lambda)).(\bar{\Phi_3}(x, \lambda)) =& P_0(x) w_{x_0} e^i \int_{x_0}^{x} (q + \lambda.F_{22}) dx \end{aligned}$$

Similarly

$$P_0(x) w_x (\bar{\Phi_1}(x, \bar{\lambda})).(\bar{\Phi_2}(x, \bar{\lambda})).(\bar{\Phi_3}(x, \bar{\lambda})) = P_0(x_0) \bar{w} x_0 e^{-i} \int_{x_0}^{x} (q + \lambda.F_{22}) dx$$

Hence

$$\begin{aligned} & P_0(x) w(\Phi_1, \Phi_2, \Phi_3(x, \lambda)).P_0(x) \bar{w}(\bar{\Phi_1}, \bar{\Phi_2}, \bar{\Phi_3}(x, \bar{\lambda})) \\ & = P_0(x_0) wx_0 \times P_0(x_0) w\bar{x}_0 \\ & = |P_0(x_0) wx_0|^2. \end{aligned}$$

Therefore, $|P_0(x_0)wx_0|^2$ is constant and independent of x. Hence its proof.

5. Conclusions

Here, a partial differential equation problem is taken. The solution of the equation is very difficult. We convert the partial differential equation into ordinary differential equation for finding the solution. For finding the solution of differential equation (2.1) by using the properties of Wronskian. Here $\Phi_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $\Phi_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, $\Phi_3 = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}$ are the three vectors which are the solution of equation (2.4).

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