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On hypergeometric relations among cubic theta functions

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Abstract: In this paper, an attempt has been made to evaluate certain generalized cubic theta functions and also several identities establish by using cubic theta functions.

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1. Introduction, Notations and Definitions

Ramanujan recorded hundreds of modular equations in his three notebooks. Chapters 19 - 21 in Ramanujans second notebook are almost exclusively devoted to modular equations. Ramanujan used modular equations to evaluate class invariants, certain q continued fractions, theta functions and certain other quotients and products of theta functions. Throughout this paper we shall adopt the following notations and definitions

For any number a and q, real or complex and $|q| < 1$,

$$[\alpha; q]_n = [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}); & n > 0 \\ 1; & n = 0 \end{cases}$$

Accordingly, we have

$$[\alpha; q]_\infty = \prod_{r=0}^{\infty}(1-\alpha q^r)$$

Also

$$[a_1, a_2, ..., a_r; q]_n = [a_1; q]_n[a_2; q]_n...[a_r; q]_n.$$

and

$$[a; q]_{-n} = \frac{q^{n(n+1)/2}}{(-a)^n[q/a; q]_n}$$

Motivated with the Jacobi's identity,

$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q) \tag{1.1}$$

Borweins in 1991 discovered an elegant cubic analogue of this identity. Borweins defined following functions which are called cubic theta functions;

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{1.2}$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \left(\omega = e^{2\pi i/3}\right), \tag{1.3}$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{\left(m+\frac{1}{3}\right)^2+\left(m+\frac{1}{3}\right)\left(n+\frac{1}{3}\right)+\left(n+\frac{1}{3}\right)^2}. \tag{1.4}$$

Borweins proved that

$$a^3(q) = b^3(q) + c^3(q). \tag{1.5}$$

They also discovered following results;

$$a(q) = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right) \tag{1.6}$$

and

$$a(q) = \Phi(q)\,\Phi\left(q^3\right) + 4q\Psi\left(q^3\right)\Psi\left(q^6\right). \tag{1.7}$$

$$b(q) = \frac{1}{2}\left[3a\left(q^3\right) - a(q)\right], \tag{1.8}$$

$$c(q) = \frac{1}{2}\left[a\left(q^{1/3}\right) - a(q)\right]. \tag{1.9}$$

Borweins established following parametric representations for a(q), b(q) and c(q).

If $m = \dfrac{z_1}{z_3}$, then

$$a(q) = \sqrt{z_1 z_3}\,\frac{m^2+6m-3}{4m}, \tag{1.10}$$

$$b(q) = \sqrt{z_1 z_3}\,\frac{(3-m)\left(9-m^2\right)^{1/3}}{4m^{2/3}}, \tag{1.11}$$

and

$$c(q) = \sqrt{z_1 z_3}\,\frac{3(m+1)\left(m^2+1\right)^{1/3}}{4m}. \tag{1.12}$$

From these parametric representations of a(q), b(q) and c(q) it is easy to prove,

$$a^3(q) = b^3(q) + c^3(q).$$

From (1.8) and (1.9) it is easy to establish,

$$a\left(q^3\right) = b(q) + c\left(q^3\right) \tag{1.13}$$

and

$$a(q) - b(q) = 3c\left(q^3\right). \tag{1.14}$$

2. Main Results

Hypergeometric transformation

Hypergeometric transformation can be used to find hypergeometric representation of the cubic theta functions. Let us consider the following transformation,

$$\begin{aligned} &{}_2F_1\left[c, c+\frac{1}{3}; \frac{3c+1}{2}; 1-\left(\frac{1-x}{1+2x}\right)^3\right] \\ &= (1+2x)^{3c}\, {}_2F_1\left[c, c+\frac{1}{3}; \frac{3c+5}{6}; x^3\right], \end{aligned} \tag{2.1}$$

which is due to Ramanujan.

Putting $c = \dfrac{1}{3}$ and $\dfrac{1-x}{1+2x} = \dfrac{b(q)}{c(q)}$ we get,

$$\begin{aligned} {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q)}{a^3(q)}\right] &= \frac{3a(q)}{a(q)+2b(q)}\times \\ &\times {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{a(q)-b(q)}{a(q)+2b(q)}\right]. \end{aligned} \tag{2.2}$$

Now, using (1.8) and (1.14) we get,

$${}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q)}{a^3(q)}\right] = \frac{a(q)}{a(q^3)}{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{c(q^3)}{a(q^3)}\right]$$

or

$${}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q)}{a^3(q)}\right] = \frac{a(q)}{a(q^3)}{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b(q^3)}{a(q^3)}\right] \tag{2.3}$$

Iterating it m times and putting $n = 3^m$ we get,

$${}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q)}{a^3(q)}\right] = \frac{a(q)}{a(q^n)}{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b(q^n)}{a(q^n)}\right]. \tag{2.4}$$

As $n \to \infty$ we have

$$a(q) = {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q)}{a^3(q)}\right]. \tag{2.5}$$

If we put x for $\dfrac{1-x}{1+2x}$ and $c = \dfrac{1}{3}$ in (2.1) we find

$$ {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right] = \left(\frac{3}{1+2x}\right) {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-x}{1+2x}\right)^3\right]. \tag{2.6}$$

Now putting $\dfrac{1-x}{1+2x} = \dfrac{b(q)}{a(q)}$ in (2.6) we have

$$ {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q)}{a^3(q)}\right] = \frac{a(q)}{3a(q^3)} {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q^3)}{a^3(q^3)}\right] \tag{2.7}$$

Repeating the process m times and writing $n = 3^m$ we get

$$ {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q)}{a^3(q)}\right] = \frac{a(q)}{na(q^n)} {}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q^n)}{a^3(q^n)}\right]. \tag{2.8}$$

Dividing (2.4) by (2.8), multiplying both sides by $-\dfrac{2\pi}{\sqrt{3}}$ and taking the exponential we obtain,

$$\exp.\left[-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q)}{a^3(q)}\right]}{{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q)}{a^3(q)}\right]}\right]$$

$$= \left\{\exp.\left[-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{b^3(q^n)}{a^3(q^n)}\right]}{{}_2F_1\left[\frac{1}{3}, \frac{2}{3}; 1; \frac{b^3(q^n)}{a^3(q^n)}\right]}\right]\right\}^n. \tag{2.9}$$

We can write as

$$F\left[\frac{b^3(q)}{a^3(q)}\right] = F^n\left[\frac{b^3(q^n)}{a^3(q^n)}\right]. \tag{2.10}$$

3. Evaluation of cubic theta functions

Since

$$m = \frac{z_1}{z_3} = \frac{\Phi^2(q)}{\Phi^2(q^3)} \tag{3.1}$$

and for $q = e^{-\pi}$

$$\Phi\left(e^{-\pi}\right) = \frac{\pi^{1/4}}{\Gamma\left(3/4\right)} = a \tag{3.2}$$

and

$$m = \frac{\Phi\left(e^{-\pi}\right)}{\Phi\left(e^{-3\pi}\right)} = \sqrt[4]{6\sqrt{3}-9}. \tag{3.3}$$

Thus we have

$$\Phi\left(e^{-3\pi}\right) = \frac{a}{\sqrt[4]{6\sqrt{3}-9}}. \tag{3.4}$$

Hence for $q = e^{-\pi}$

$$z_1 z_3 = \frac{a^2}{\sqrt[4]{6\sqrt{3}-9}}. \tag{3.5}$$

From (1.10), (1.11) and (1.12) we have

$$a\left(e^{-\pi}\right) = \frac{a^2}{\sqrt[4]{6\sqrt{3}-9}} \frac{3^{1/4}}{2\sqrt{2}} \left\{3\sqrt{2\sqrt{3}} - \sqrt{3} + 1\right\}, \tag{3.6}$$

$$b\left(e^{-\pi}\right) = \frac{a^2}{2\sqrt[4]{6\sqrt{3}-9}} \frac{3 - \sqrt{6\sqrt{3}-9}}{\left\{2\left(\sqrt{3}-1\right)\right\}^{1/3}} \tag{3.7}$$

and

$$c\left(e^{-\pi}\right) = \frac{a^2\left(6\sqrt{3}-9\right)^{1/4}}{2\sqrt[4]{6\sqrt{3}-9}} \left\{1 + \sqrt{6\sqrt{3}-9}\right\}. \tag{3.8}$$

4. Generalized cubic theta functions

Hirschhorn, Garvan and Borwein introduced the functions,

$$a\left(q, z\right) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} z^{n-m} \tag{4.1}$$

$$b\left(q, z\right) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \omega^{n-m} z^{m}, \tag{4.2}$$

$$\left(\omega = e^{2\pi i/3}\right)$$

and

$$c\left(q, z\right) = q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} z^{n-m}. \tag{4.3}$$

As $z \to 1$,

$a(q,1) = a(q), b(q,1) = b(q)$ and $c(q,1) = c(q)$.

In the definition of c(q,z) there is a multiplier $q^{1/3}$ which is not in the definition of Hirschhorn, Garvan and Borwein. It is because of the definition of c(q) taken in this paper.

Hirschhorn, Garvan and Borwein gave a number of elegant identities involving $a(q,z), b(q,z)$ and $c(q,z)$.

Some of these identities are given below.

$$a(q,z) = \left(2+z+z^{-1}\right)\frac{[q;q]_\infty [q^2;q^2]_\infty^2}{[-q^3;q^3]_\infty [q^6;q^6]_\infty}[-zq,-q/z;q]_\infty^2$$

$$-\left(1+z+z^{-1}\right)\frac{[q^2;q^2]_\infty [q^3;q^3]_\infty [z^3q^3,q^3/z^3;q^3]_\infty}{[-q^3;q^3]_\infty^3 [zq,q/z;q]_\infty}. \tag{4.4}$$

$$a(q,z) = \frac{1}{3}\left(1+z+z^{-1}\right)\left[1+6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right]\times$$

$$\times\frac{[q^2;q^2]_\infty^2 [z^3q^3,q^3/z^3;q^3]_\infty}{[q^3;q^3]_\infty^2 [zq,q/z;q]_\infty}$$

$$+\frac{1}{3}\left(2-z-z^{-1}\right)\frac{[q;q]_\infty^5}{[q^3;q^3]_\infty^3}[zq,q/z;q]_\infty^2. \tag{4.5}$$

$$b(q,z) = [q;q]_\infty \left[q^3;q^3\right]_\infty \frac{[zq,z/q;q]_\infty}{[zq^3,q^3/z;q^3]_\infty}. \tag{4.6}$$

$$c(q,z) = q^{1/3}\left(1+z+z^{-1}\right)[q;q]_\infty \left[q^3;q^3\right]_\infty \frac{[z^3q^3,q^3/z^3;q^3]_\infty}{[zq,z/q;q]_\infty}. \tag{4.7}$$

5. Evaluation of generalized cubic theta functions

(a) For $z=\omega$, (4.4), (4.6) and (4.7) yields,

$$a(q,\omega) = \frac{[q;q]_\infty [q^2;q^2]_\infty^2}{[-q^3;q^3]_\infty [q^6;q^6]_\infty}\prod_{i=1}^{\infty}\left(1-q^i+q^{2i}\right)^2, \tag{5.1}$$

$$b(q,\omega) = [q;q]_\infty \left[q^3;q^3\right]_\infty \prod_{i=1}^{\infty}\left\{\frac{(1+q^i+q^{2i})}{(1+q^{3i}+q^{6i})}\right\}, \tag{5.2}$$

$$c(q,\omega) = 0. \tag{5.3}$$

(b) Taking $z=-\omega$ in (4.4), (4.6) and (4.7) we get,

$$a\,(q,-\omega)=3\frac{[q;q]_\infty\,[q^2;q^2]_\infty^2}{[-q^3;q^3]_\infty\,[q^6;q^6]_\infty}\prod_{i=1}^{\infty}\left(1+q^i+q^{2i}\right)^2,$$

$$-2\frac{[q^2;q^2]_\infty\,[q^3;q^3]_\infty}{[-q^3;q^3]_\infty}\left\{\prod_{i=1}^{\infty}\left(1-q^i+q^{2i}\right)^2\right\}^{-1}, \tag{5.4}$$

$$b\,(q,-\omega)=[q;q]_\infty\left[q^3;q^3\right]_\infty\prod_{i=1}^{\infty}\left\{\frac{(1-q^i+q^{2i})}{(1-q^{3i}+q^{6i})}\right\}, \tag{5.5}$$

$$c\,(q,-\omega)=2q^{1/3}\frac{[q;q]_\infty\,[q^3;q^3]_\infty\,[-q^3;q^3]_\infty^2}{\prod_{i=1}^{\infty}\left(1-q^i+q^{2i}\right)}. \tag{5.6}$$

(c) For $z=-1$, in (4.4), (4.6) and (4.7) yield,

$$a\,(q,-1)=\frac{[q^2;q^2]_\infty\,[q^3;q^3]_\infty}{[-q^3;q^3]_\infty\,[-q;q]_\infty^2}, \tag{5.7}$$

$$b\,(q,-1)=\frac{[q;q]_\infty\,[q^3;q^3]_\infty\,[-q;q]_\infty^2}{[-q^3;q^3]_\infty^2}, \tag{5.8}$$

$$c\,(q,-1)=-q^{1/3}\frac{[q;q]_\infty\,[q^3;q^3]_\infty\,[-q^3;q^3]_\infty^2}{[-q^2;q^2]_\infty^2}. \tag{5.9}$$

(d) For $z=1$, in (4.5), (4.6) and (4.7) yield,

$$a\,(q)=\left[1+6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right], \tag{5.10}$$

$$b\,(q)=\frac{[q;q]_\infty^3}{[q^3;q^3]_\infty}, \tag{5.11}$$

$$c\,(q)=3q^{1/3}\frac{[q^3;q^3]_\infty^3}{[q;q]_\infty}. \tag{5.12}$$

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