South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 1 (2020), pp. 135-142

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

REMARKS ON THE PAPER "$\alpha-\psi$-GERAGHTY CONTRACTION TYPE MAPPINGS AND SOME RELATED FIXED POINT RESULTS"

K. Anthony Singh

Department of Mathematics,
D. M. College of Science, Imphal, Manipur, INDIA

E-mail : anthonykumam@manipuruniv.ac.in

(**Received:** *Jul. 07, 2019* **Accepted:** *Jan. 14, 2020* **Published:** *Apr. 30, 2020)*

Abstract: Recently, Erdal Karapinar ([8] Karapinar E., $\alpha-\psi-$Geraghty contraction type mappings and some related fixed point results, Filomat, 28:1 37-48 (2014)) introduced the notion of generalized $\alpha-\psi-$Geraghty contraction type mappings in the setting of a metric space and proved the existence and uniqueness of a fixed point of such mappings. But we observe that the condition (H1) for the uniqueness of the fixed point of such mappings as proposed by Karapinar is not enough. The aim of this paper is to attempt to rectify it by proposing a slightly stronger condition in place of condition (H1).

Keywords and Phrases: Metric space, fixed point, $\alpha-$admissible map, triangular $\alpha-$admissible map, generalized $\alpha-\psi-$ Geraghty contraction type mapping.

2010 Mathematics Subject Classification: Primary: 46T99; Secondary: 47H10, 54H25, 46J10, 46J15.

1. Introduction and Preliminaries

The Banach contraction principle is one of the most important and fundamental results in fixed point theory. The study of fixed point problems is indeed a powerful tool in nonlinear analysis and the techniques of fixed point theory have very useful applications in many disciplines. Due to this, several authors have improved, generalized and extended this basic result of Banach by defining new contractive conditions and replacing the metric space by more general abstract spaces. Among

such results, the works of Geraghty [4], Amini-Harandi and Emami [1], Caballero et al. [2], Gordji et al. [5], Samet et al.[10] and Karapinar and Samet [7] can be mentioned. Very recently, Cho et al. [3] defined the concept of $\alpha-$Geraghty contraction type maps in the setting of a metric space starting from the definition of generalized $\alpha-$Geraghty contraction type maps and proved the existence and uniqueness of a fixed point of such maps in the context of a complete metric space. Further as generalizations of the type of maps defined by Cho et al. [3], Erdal Karapinar [8] introduced the concept of generalized $\alpha-\psi-$Geraghty contraction type maps and $\alpha-\psi-$Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al.[3].
First, we recall some basic definitions and related results on the topic in the literature.
Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

By using such a function, Geraghty proved the following interesting result.

Theorem 1.1. [4] *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \le \beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.

Definition 1.2. [10] *Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function. Then T is said to be $\alpha-$admissible if $\alpha(x,y) \ge 1$ implies $\alpha(Tx,Ty) \ge 1$.*

Definition 1.3. [6] *A map $T : X \to X$ is said to be triangular $\alpha-$admissible if*

(T1) T is $\alpha-$admissible,

(T2) $\alpha(x,z) \ge 1$ and $\alpha(z,y) \ge 1$ imply $\alpha(x,y) \ge 1$.

Lemma 1.4. [6] *Let $T : X \to X$ be a triangular $\alpha-$admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \ge 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

2. Remarks on some results of [8]

Erdal Karapinar [8] defined the following class of auxiliary functions.
Let Ψ denote the class of functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;

(b) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$;

(c) ψ is continuous;

(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

Karapinar [8] also introduced the following contraction and proved the following interesting results i.e. Theorem 2.2., Theorem 2.4. and Theorem 2.5.

Definition 2.1. [8] *Let (X, d) be a metric space and $\alpha : X \times X \to R$ be a function. A map $T : X \to X$ is called a generalized $\alpha-\psi-$Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\psi \in \Psi$

Theorem 2.2. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

1. *T is a generalized $\alpha - \psi-$Geraghty contraction type map,*
2. *T is triangular $\alpha-$ admissible,*
3. *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,*
4. *T is continuous.*

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.
***The above Theorem 2.2. is indeed Theorem 2.4.(page 38) in the paper[8] by Karapinar.**

Definition 2.3. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is $\alpha-$regular if the following condition is satisfied:*
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$for all k.

Theorem 2.4. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to R$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

1. *T is a generalized $\alpha-\psi-$Geraghty contraction type map,*

2. *T is triangular $\alpha-$ admissible,*

3. *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,*

4. *$\{x_n\}$ is $\alpha-$regular.*

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.
***The above Theorem 2.4. is indeed Theorem 2.6.(page 40) in the paper[8] by Karapinar.**

For the uniqueness of a fixed point of a generalized $\alpha-\psi-$Geraghty contraction type mapping, Karapinar[8] considered the following hypothesis.
(H1) For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Here $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.5. [8] *Adding condition (H1) to the hypotheses of Theorem 2.2.(resp. Theorem 2.4.), we obtain that x^* is the unique fixed point of T.*
***The above Theorem 2.5. is indeed Theorem 2.7.(page 41) in the paper[8] by Karapinar. Here we reproduce the exact proof of the Theorem given by Karapinar in [8](page 41).**
Proof. Due to Theorem 2.4. (resp. Theorem 2.6.), we have a fixed point, say $x^* \in X$. Let $y^* \in X$ be another fixed point of T.
Then, by assumption, there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \quad \text{and} \quad \alpha(y^*, z) \geq 1. \tag{15}$$

Since T is $\alpha-$admissible, from (1), we have

$$\alpha(x^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geq 1, \quad \text{for all } \ n.$$

Hence we have

$$\begin{aligned} d(x^*, T^n z) &\leq \alpha(x^*, T^{n-1}z) d(Tx^*, TT^{n-1}z) \\ &\leq \beta(d(x^*, T^{n-1}z)) d(x^*, T^{n-1}z) \\ &< d(x^*, T^{n-1}z) \end{aligned} \tag{16}$$

for all $n \in \mathbb{N}$.
Thus the sequence $\{d(x^*, T^n z)\}$ is nonincreasing, and there exists $u \geq 0$ such that

$\lim_{n\to\infty} d(x^*, T^n z) = u$.
From (16) we have

$$\frac{d(x^*, T^n z)}{d(x^*, T^{n-1} z)} \leq \beta(d(x^*, T^{n-1} z))$$

and thus $\lim_{n\to\infty} \beta(d(x^*, T^n z)) = 1$. Hence $\lim_{n\to\infty} d(x^*, T^n z) = 0$ which yields $\lim_{n\to\infty} T^n z = x^*$.

Similarly, we have $\lim_{n\to\infty} T^n z = y^*$. Thus, we have $x^* = y^*$.

Remark 1. The above inequality (16) is indeed not the contraction condition of a generalized $\alpha-\psi-$Geraghty contraction type map but a much stronger condition. Indeed the contraction condition for the mapping T is:

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\psi \in \Psi$.
But the contraction condition used in (16) is indeed a stronger condition:

$$\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

With the original contraction condition for T as in the Theorems 2.2. and 2.4., the inequality (16) could be

$$\begin{aligned} \psi(d(x^*, T^n z)) &\leq \alpha(x^*, T^{n-1} z)\psi(d(Tx^*, TT^{n-1} z)) \\ &\leq \beta(\psi(M(x^*, T^{n-1} z)))\psi(M(x^*, T^{n-1} z)) \\ &< \psi(M(x^*, T^{n-1} z)), \end{aligned} \tag{16}$$

and

$$\begin{aligned} M(x^*, T^{n-1} z) &= \max\{d(x^*, T^{n-1} z), d(x^*, Tx^*), d(T^{n-1} z, TT^{n-1} z)\} \\ &= \max\{d(x^*, T^{n-1} z), d(x^*, x^*), d(T^{n-1} z, T^n z)\} \\ &= \max\{d(x^*, T^{n-1} z), d(T^{n-1} z, T^n z)\} \end{aligned}$$

These conditions are not sufficient enough to always imply $d(x^*, T^n z) < d(x^*, T^{n-1} z)$ and $\lim_{n\to\infty} T^n z = x^*$ (as was arrived at in the proof by Karapinar [8]).
Here our observation is that the condition (H1) of Karapinar is not enough to prove the uniqueness of a fixed point of a generalized $\alpha-\psi-$Geraghty contraction type map. We propose the following condition (H1*) which is slightly stronger than condition (H1) and prove the uniqueness of a fixed point of such maps using the

new condition.
(H1*) For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1, \alpha(y,z) \geq 1$ and $\alpha(z,Tz) \geq 1$. Here $Fix(T)$ denotes the set of fixed points of T.
With this new condition, we modify Theorem 2.5. as Theorem 2.5.* and prove it.

Theorem 2.5.* *Adding condition (H1*) to the hypotheses of Theorem 2.2.(resp. Theorem 2.4.), we obtain that* x^* *is the unique fixed point of* T.
Proof: Due to Theorem 2.2. (resp. Theorem 2.4.), we have a fixed point, say $x^* \in X$. Let $y^* \in X$ be another fixed point of T.
Then, by (H1*), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1, \alpha(y^*, z) \geq 1 \text{ and } \alpha(z, Tz) \geq 1.$$

Since T is $\alpha-$admissible, from (3), we have

$$\alpha(x^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geq 1, \quad \text{for all } n \in \mathbb{N}.$$

Then we have

$$\begin{aligned} \psi(d(x^*, T^{n+1}z)) &\leq \alpha(x^*, T^n z)\psi(d(Tx^*, TT^n z)) \\ &\leq \beta(\psi(M(x^*, T^n z)))\psi(M(x^*, T^n z)), \forall n \in \mathbb{N}. \end{aligned}$$

Here we have

$$\begin{aligned} M(x^*, T^n z) &= \max\{d(x^*, T^n z), d(x^*, Tx^*), d(T^n z, TT^n z)\} \\ &= \max\{d(x^*, T^n z), d(x^*, x^*), d(T^n z, T^{n+1}z)\} \\ &= \max\{d(x^*, T^n z), d(T^n z, T^{n+1}z)\} \end{aligned}$$

By Theorem 2.2. (or Theorem 2.4.) we deduce that the sequence $\{T^n z\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ in the above equality, we get $\lim_{n\to\infty} M(x^*, T^n z) = d(x^*, z^*)$. Let us suppose that $z^* \neq x^*$. Then we have

$$\frac{\psi(d(x^*, T^{n+1}z))}{\psi(M(x^*, T^n z))} \leq \beta(\psi(M(x^*, T^n z))) < 1.$$

Taking limit $n \to \infty$, we get $\lim_{n\to\infty} \beta(\psi(M(x^*, T^n z))) = 1$. Therefore, we have $\lim_{n\to\infty} \psi(M(x^*, T^n z)) = 0$. This implies $\lim_{n\to\infty} M(x^*, T^n z) = 0$ i.e. $d(x^*, z^*) = 0$, which is a contradiction. Therefore we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus we have $y^* = x^*$. Hence x^* is the unique fixed point of T.

Remark 2. In the example (Example 3.10. to illustrate Theorem 3.2. as in the paper [8], page 45), we observe that it is Theorem 3.3. not Theorem 3.2. which is being illustrated. It is also stated that condition (iv) of Theorem 3.2. (Theorem 3.3.) is satisfied with $x_n = T^n x_1 = \frac{1}{3^n}$. But we observe that condition (iv) of Theorem 3.2.(Theorem 3.3.) is to be satisfied not by only a particular sequence like $x_n = T^n x_1 = \frac{1}{3^n}$ but by any sequence $\{x_n\}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Here we observe that the satisfying of condition (iv) of Theorem 3.2.(Theorem 3.3.) in the setting of Example 3.10. of [8] can be stated as "if $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then we must have $x_n \in [0,1] \forall n \in \mathbb{N}$ and also $x \in [0,1]$. Therefore, by definition of the function α we must have $\alpha(x_n, x) \geq 1$. Hence, condition (iv) of Theorem 3.2. (Theorem 3.3.) is satisfied".

References

[1] Amini-Harandi A., Emami H., A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., 72 2238-2242 (2010).

[2] Caballero J., Harjani J., Sadarangani K., A best proximity point theorem for Geraghty- contractions, Fixed Point Theory Appl., 2012 Article ID 231 (2012).

[3] Cho S.H., Bae J.S., Karapinar E., Fixed point theorems for $\alpha-$Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl., 2013 Article ID 329 (2013).

[4] Geraghty M., On contractive mappings, Proc. Am. Math. Soc., 40 604-608 (1973).

[5] Gordji M.E., Ramezami M., Cho Y.J., Pirbavafa S., A generalization of Geraghty's Theorem in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl., 2012 Article ID 74 (2012).

[6] Karapinar E., Kumam P., Salimi P., On $\alpha-\psi-$Meir-Keeler contractive mappings, Fixed Point Theory Appl., 2013 Article ID 94 (2013).

[7] Karapinar E., Samet B., Generalized $\alpha - \psi-$contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal., 2012 Article ID 793486 (2012).

[8] Karapinar E., $\alpha-\psi-$Geraghty contraction type mappings and some related fixed point results, Filomat, 28:1 37-48 (2014).

[9] Rhoades B.E., A comparison of various definitions of contractive mappings, Trans. Am. Math. Soc., 226 257-290 (1977).

[10] Samet B., Vetro C., Vetro P., Fixed point theorems for $\alpha-\psi-$contractive mappings, Nonlinear Anal., 75 2154-2165 (2012).