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A STUDY OF SOME INTEGRAL TRANSFORMS ON Q FUNCTION

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Abstract: The present paper deals with the new generalization of a Q function using generalized Mittag-Leffler function .The Mittag-Leffler function arises usually in the solution of fractional order differential equation and fractional order integral equations. Various Integral Transforms such as Laplace transform, Fourier transform, Euler beta transform and Whittaker transform with their several special cases are obtained to illustrate our main results.

Keywords and Phrases: Mittag-Leffler function; Generalized Mittag-Leffler function; Laplace transform; Fourier transform; Euler beta transform; Whittaker transform.

2010 Mathematics Subject Classification: 33E12, 44A10, 26A33.

1. Introduction

During the last twenty years Mittag-Leffler function has come into eminence after it was discovered by Mittag-Leffler who was a Swedish mathematician. And now it has a vast use in solving the problems related to physics, biology, engineering, and earth sciences and more. Mittag-Leffler function has applications in applied problems, such as fluid flow, rheology and diffusive transport akin to diffusion, electric networks, probability, and statistical distribution theory.

Besides this, The Mittag-Leffler function appears in the solution of certain boundary value problems involving fractional Integral-differential equations of Volterra type. When physical phenomena deviate from exponential behavior then the M-L type functions play the major role being generalization of exponential function. Different generalization of the Mittag-Leffler function and its properties has been investigated. Furthermore, various integral transforms have discussed.

2. Preliminary Results, Notations and Terminology

In 1903 the Swedish mathematician Gosta Mittag-Leffler [2] had introduced the function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

where $z \in C$, $Re(\alpha) > 0$ and 'Γ'' represents well known Gamma function. This whole function is known as Mittag-Leffler function.
The direct generalization of the Mittag-Leffler function was introduced by Wiman [9] in 1905 in his work on zeroes of function equation

$$E_{\alpha,\beta}(z) = \sum \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{2.1}$$

where $\alpha, \beta \in C$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.
Prabhakar [8] in 1971 and stated like:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum \frac{(\gamma)_n z^n}{n!\Gamma(\alpha n + \beta)} \tag{2.2}$$

For which $\alpha, \beta, \gamma \in C$, real part of $\alpha > 0$, real part of $\beta > 0$ and real part of $\gamma = 0$.
Pochhammer symbol defined as:

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k)(\gamma + 3k)...(\gamma + (n-1)k), \quad \text{where } \gamma \in C, k \in R, n \in N$$

The function $E^{\gamma,q}_{\alpha,\beta}(x)$ was introduced in 2007 by Shukla and Prajapati [5], Salim (2010) which is defined for $\alpha, \beta, \gamma \in C$, $R < (\alpha) > 0$, $R(\beta) > 0$, $R(\gamma) > 0$ and $q \in (0, 1) \cup N$ as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n}^{\infty} \frac{(\gamma)_{qn} z^n}{n!\Gamma(\alpha n + \beta)} \tag{2.3}$$

where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalised Pochhammer symbol.
Further the most generalisation is investigated Khan [11] and defined as follows

$$E^{\mu,\delta,\gamma,q}_{\alpha,\beta,\nu,\sigma,\lambda,\rho}(z) = \sum_{n}^{\infty} \frac{(\mu)_{\delta n}(\gamma)_{qn} z^n}{(\lambda)_{\rho n}(\nu)_{\sigma n}\Gamma(\alpha n + \beta)} \tag{2.4}$$

The generalisation of Mittag-Leffler function in [7] (2.7) denoted by $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x)$ and is defined by $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) = Q^{\gamma,q,r}_{\alpha,\beta,\delta}\ (a_1, a_2, ..., a_r, b_1, b_2, ..., b_r, x)$

$$Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)(\delta)_{qs}\Gamma(\alpha s + \beta)} x^s \tag{2.5}$$

where $\alpha, \beta, \gamma, \delta, a_i, b_i \in C$, $\min\{Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0\}$ and $q \in (0, 1) \cup N$

$$(\gamma)_{qs} = \frac{\Gamma(\gamma + qs)}{\Gamma(\gamma)} \quad \text{and} \quad (\delta)_{qs} = \frac{\Gamma(\delta + qs)}{\gamma(\gamma)}$$

Further we can generalize Mittag-Leffler function $Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}$ as

$$Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(x) = Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(a_1, a_2, ..., a_r, b_1, b_2, ..., b_r, x)$$

$$Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(x) = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)(\nu)_{\sigma s}(\delta)_{ps}\Gamma(\alpha s + \beta)} x^s \tag{2.6}$$

where $\alpha, \beta, \nu, \sigma, \mu, \rho, \gamma, \delta, p, a_i, b_i \in C$, $\min\{Re(\alpha) > 0,\ Re(\beta) > 0,\ Re(\gamma) > 0\ Re(\nu) > 0\ Re(\mu) > 0\ Re(\sigma) > 0\ Re(\rho) > 0\}$ and $q \in (0, 1) \cup N$

$$(\gamma)_{qs} = \frac{\Gamma(\gamma + qs)}{\Gamma(\gamma)}, (\mu)_{ps} = \frac{\Gamma(\mu + ps)}{\gamma(\mu)}, (\nu)_{\sigma s} = \frac{\Gamma(\nu + \sigma s)}{\gamma(\nu)}, (\delta)_{ps} = \frac{\Gamma(\delta + ps)}{\gamma(\delta)}$$

Special Cases

1. If we put $\mu, \rho, \nu, \sigma, p = 1$ in (2.6) it will reduce to

$$Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)(\delta)_{qs}\Gamma(\alpha s + \beta)} x^s,$$

which is the generalisation of Mittag-Leffler function in Mohammed [7] (2.7).

2. When there is no upper and lower parameter, we get

$$Q^{\mu,\rho,\gamma,q,0}_{\alpha,\beta,\nu,\sigma,\delta,p}(x) = Q^{\mu,\rho,\gamma,q,0}_{\alpha,\beta,\nu,\sigma,\delta,p}(- - -, - - -, x)$$

$$= \sum_{s=0}^{\infty} \frac{(\mu)_{\rho s}(\gamma)_{qs} x^s}{\Gamma(\alpha s + \beta)(\nu)_{\sigma s}(\delta)_{ps}} = E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(x)$$

Which is the generalization of the Mittag-Leffler in Khan [11] (2.7)
If we take $\mu, \rho, \nu, \sigma, p = 1$ it will reduce to Khan [11] (2.9)

$$Q^{1,1,\gamma,q,0}_{\alpha,\beta,1,1,\delta,1}(x) = Q^{1,1,\gamma,q,0}_{\alpha,\beta,1,1,\delta,1}(- - -, - - -, x) = \sum_{s=0}^{\infty} \frac{(\gamma)_{qs} x^s}{\Gamma(\alpha s + \beta)(\delta)_{qs}} = E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, q = 1$ then

$$Q^{1,1,\gamma,1,0}_{\alpha,\beta,1,1,\delta,1}(x) = Q^{1,1,\gamma,1,0}_{\alpha,\beta,1,1,\delta,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{(\gamma)_s x^s}{\Gamma(\alpha s+\beta)(\delta)_s} = E^{\gamma,\delta}_{\alpha,\beta}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, \delta = 1$ then

$$Q^{1,1,\gamma,q,0}_{\alpha,\beta,1,1,1,1}(x) = Q^{1,1,\gamma,q,0}_{\alpha,\beta,1,1,1,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{(\gamma)_{qs} x^s}{\Gamma(\alpha s+\beta)} = E^{\gamma,q}_{\alpha,\beta}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, q, \delta = 1$ then

$$Q^{1,1,\gamma,1,0}_{\alpha,\beta,1,1,1,1}(x) = Q^{1,1,\gamma,1,0}_{\alpha,\beta,1,1,1,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{(\gamma)_s x^s}{\Gamma(\alpha s+\beta)} = E^{\gamma}_{\alpha,\beta}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, q, \delta, \gamma = 1$ then

$$Q^{1,1,1,1,0}_{\alpha,\beta,1,1,1,1}(x) = Q^{1,1,1,1,0}_{\alpha,\beta,1,1,1,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{x^s}{\Gamma(\alpha s+\beta)} = E_{\alpha,\beta}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, q, \delta, \gamma, \beta = 1$ then

$$Q^{1,1,1,1,0}_{\alpha,1,1,1,1,1}(x) = Q^{1,1,1,1,0}_{\alpha,1,1,1,1,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{x^s}{\Gamma(\alpha s+1)} = E_{\alpha}(x)$$

If we take $\mu, \rho, \nu, \sigma, p, q, \delta, \gamma, \beta, \alpha = 1$ then

$$Q^{1,1,1,1,0}_{1,1,1,1,1,1}(x) = Q^{1,1,1,1,0}_{1,1,1,1,1,1}(---,---,x) = \sum_{s=0}^{\infty} \frac{x^s}{\Gamma(s+1)} = E_1(x) = e^x$$

Laplace Transform. The Laplace transform of integrable function $f(t)$ is defined by

$$L(f)(t) = \int_0^{\infty} e^{-ut} f(t) dt \tag{2.7}$$

Euler beta function. The Beta function is usually defined by

$$B[f(t); a, b] = \int_0^1 t^{a-1}(1-t)^{b-1} f(t) dt \tag{2.8}$$

Fourier Transform. The infinite Fourier transform $f(t)$ is defined as

$$F(f)(t) = \int_{-\infty}^{\infty} e^{iut} f(t)dt \tag{2.9}$$

Fourier Sine and Cosine Transform. If $f(t)$ is a real or complex valued variable t defined for all real numbers, then the two sided Fourier Sine is defined by integral

$$F_s(f)(t) = \int_{-\infty}^{\infty} \sin(2\pi ut) f(t)dt \tag{2.10(a)}$$

and Cosine transform is defined by integral

$$F_c(f)(t) = \int_{-\infty}^{\infty} \cos(2\pi ut) f(t)dt \tag{2.10(b)}$$

Whittaker Transform. If $f(t)$ is a real or complex valued variable t defined for all real numbers, then the Whittaker transform is defined by integral

$$\int_0^{\infty} e^{-\frac{t}{2}} t^{v-1} w_{\lambda,\mu}(t)dt = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)} \tag{2.11}$$

Where $Re(\mu \pm v) > -\frac{1}{2}$ and $w_{\lambda,\mu}(t)$ is Whittaker confluent hypergeometric function.

3. Main Results

3.1. Laplace Transform

Theorem. *For any* $t, \alpha, \beta, v, \sigma, \rho, \delta, \gamma, a_i, b_i \in C$ $\min\{Re(\alpha), Re(\beta), Re(v), Re(\sigma),$ $Re(\rho), Re(\gamma), Re(\delta)\} > 0$*, then the Laplace transform of generalized Mittag-Leffler function is*

$$L\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \frac{s!}{u} Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)(u^{-1})$$

Proof. Using (2.6)

$$Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t) = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs} t^s}{\prod_{n=1}^{r} \beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}$$

and by (2.7), using both we get

$$L\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \int_0^{\infty} e^{-ut}\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right]dt$$

$$= \int_0^{\infty} e^{-ut} \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs} t^s}{\prod_{n=1}^{r} \beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} dt$$

$$= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \int_0^{\infty} e^{-ut} t^s dt$$

$$= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \frac{s!}{u^{s+1}}$$

$$= \frac{s!}{u} \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} u^{-s}$$

$$L\left[Q_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q,r}(t)\right] = \frac{s!}{u} Q_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q,r}(t)(u^{-1}) \tag{3.1}$$

Corollary. *For $\mu = \rho = \nu = \sigma = p = 1$, (3.1) reduces the result obtained by Sontakke [4].*

$$L\left[Q_{\alpha,\beta,\delta}^{\gamma,q,r}(t)\right] = \frac{s!}{u} Q_{\alpha,\beta,\delta}^{\gamma,q,r}(t)(u^{-1})$$

3.2. Euler Beta Transform

Theorem. *For any $t, \alpha, \beta, v, \sigma, \rho, \delta, \gamma, a_i, b_i \in C$ $\min\{Re(\alpha), Re(\beta), Re(v), Re(\sigma), Re(\rho), Re(\gamma), Re(\delta)\} > 0$, then the Euler Beta transform of generalized Mittag-Leffler function is*

$$B\left[Q_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q,r}(t), a, b\right] = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \beta(s+a, b)$$

Proof. Using (2.6) and (2.8)

$$B\left[Q_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q,r}(t), a, b\right] = \int_0^1 t^{a-1}(1-t)^{b-1} Q_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q,r}(t) dt$$

$$= \int_0^1 t^{a-1}(1-t)^{b-1} \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} t^s dt$$

$$= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \int_0^1 t^{s+a-1}(1-t)^{b-1} dt$$

$$= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \beta(s+a, b) \tag{3.2}$$

Corollary. *For $\mu = \nu = \sigma = \lambda = 1$, (3.2) reduces to Sontakke [4]*

$$B\left[Q_{\alpha,\beta,\delta}^{\gamma,q,r}(t), a, b\right] = \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(b_n, s)(\gamma)_{qs}}{\prod_{n=1}^{r} \beta(a_n, s)\Gamma(\alpha s+\beta)(\delta)_{qs}} \beta(s+a, b)$$

3.3. Fourier Transform

Theorem. *For any* $t, \alpha, \beta, v, \sigma, \rho, \delta, \gamma, a_i, b_i \in C$ $\min\{Re(\alpha), Re(\beta), Re(v), Re(\sigma), Re(\rho), Re(\gamma), Re(\delta)\} > 0$, *then the Fourier transform of generalized Mittag-Leffler function is*

$$F\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}(-1)^{-n}i^{-n-1}u^{-(n+1)}$$

Proof. Using (2.6) and (2.9)

$$\begin{aligned}
F\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] &= \int_{-\infty}^{\infty} e^{iuz} Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)dt \\
&= \int_{-\infty}^{\infty} e^{iut} \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} t^s dt \\
&= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}} \int_{-\infty}^{\infty} e^{iut}t^s dt
\end{aligned}$$

Substitute $iut = -z$ and $iudt = -dz$

$$\begin{aligned}
&= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}(-1)^s i^{-s-1}u^{-s-1}\int_{-\infty}^{\infty} e^t t^s dt \\
&= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}(-1)^s i^{-s-1}u^{-s-1}\Gamma(s+1) \\
&= \sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}(-1)^n i^{-n-1}u^{-(n+1)} \qquad (3.3)
\end{aligned}$$

Corollary. *For* $\mu = \nu = \sigma = \lambda = 1$, *(3.3) reduces to Sontakke [4]*

$$F\left[Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t)\right] = \sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\delta)_{qs}}(-1)^{-n}i^{-n-1}u^{-(n+1)}$$

3.4. Fourier Sine and Cosine Transform

Theorem. *For any* $t, \alpha, \beta, v, \sigma, \rho, \delta, \gamma, a_i, b_i \in C$ $\min\{Re(\alpha), Re(\beta), Re(v), Re(\sigma), Re(\rho), Re(\gamma), Re(\delta)\} > 0$, *then the Fourier Sine and Cosine Transform of generalized Mittag-Leffler function is*

$$F_c\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = Real\{\sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\}$$

and

$$F_s\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = Imag\{\sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\}$$

Proof. From the definition (2.6) and (2.10)

$$F_s\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \int_{-\infty}^{\infty}\sin(2\pi ut)\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}t^s dt$$
$$= \sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\int_{-\infty}^{\infty}\sin(2\pi ut)t^s dt$$

and

$$F_c\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \int_{-\infty}^{\infty}\cos(2\pi ut)\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}t^s dt$$
$$= \sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\int_{-\infty}^{\infty}\cos(2\pi ut)t^s dt$$

Since, $e^{-i2\pi ut} = \cos(2\pi ut) - i\sin(2\pi ut)$

$$F\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\int_{-\infty}^{\infty}e^{-2\pi ut}t^s dt$$

On substituting $i2\pi ut = z$ and $i2\pi udt = dz$

$$F\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = \sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\int_{-\infty}^{\infty}e^{-z}z^s dt$$
$$= \sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\Gamma(s+1)$$
$$= \sum_{s=0}^{\infty}\frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}$$

$$F_c\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = Real\{\sum_{s=0}^{\infty} \frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\} \tag{3.4.1}$$

and

$$F_s\left[Q^{\mu,\rho,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\delta,p}(t)\right] = Imag\{\sum_{s=0}^{\infty}\frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}\} \tag{3.4.2}$$

Corollary. *For $\mu=\nu=\sigma=\lambda=1$, (3.4.1) and (3.4.2) reduces to Sontakke [4]*

$$F_c\left[Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t)\right] = Real\{\sum_{s=0}^{\infty}\frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\delta)_{ps}}\}$$

and

$$F_s\left[Q^{\gamma,q,r}_{\alpha,\beta,\delta}(t)\right] = Imag\{\sum_{s=0}^{\infty}\frac{s!\prod_{n=1}^{r}\beta(b_n,s)(\gamma)_{qs}(-1)^{-s}(i2\pi)^{-s-1}u^{-(s+1)}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\delta)_{ps}}\}$$

3.5. Whittaker Transform.

Theorem. *For any $t,\alpha,\beta,v,\sigma,\rho,\delta,\gamma,a_i,b_i\in C$ Whittaker Transform of generalized Mittag-Leffler function is*

$$\int_0^{\infty} e^{\frac{-\phi t}{2}}t^{\zeta-1}w_{\lambda,\psi}(\phi t)Q^{\mu,\delta,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\lambda,\rho}(\omega t^{\eta}) = (\phi)^{-\zeta}\frac{\Gamma\nu\Gamma\delta}{\Gamma\mu\Gamma\gamma}$$

$$\times\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)\Gamma(\mu+\delta s)\Gamma(\gamma+qs)}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\lambda+\rho s)\Gamma(\nu+\sigma s)\Gamma(\beta+\alpha s)}\frac{\Gamma\left(\frac{1}{2}\pm\psi+\zeta+s\eta\right)\left[\frac{\omega}{\phi^{\eta}}\right]^s}{\Gamma(1-\lambda+\zeta+s\eta)s!}$$

Proof. Using (2.6) and (2.11) Substituting $\phi t=v$ in left hand side and $dt=\frac{1}{\phi}dv$

$$=\int_0^{\infty} e^{\frac{-v}{2}}\left(\frac{v}{\phi}\right)^{\zeta-1}w_{\lambda,\psi}(v)Q^{\mu,\delta,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\lambda,\rho}\omega^s\left(\frac{v}{\phi}\right)^{\eta s}\frac{1}{\phi}dv$$

$$=\int_0^{\infty} e^{\frac{-v}{2}}\left(\frac{v}{\phi}\right)^{\zeta-1}w_{\lambda,\psi}(v)\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)(\mu)_{\rho s}(\gamma)_{qs}}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\alpha s+\beta)(\nu)_{\sigma s}(\delta)_{ps}}(\omega)^s\left(\frac{v}{\phi}\right)^{\eta s}\frac{1}{\phi}dv$$

$$=(\phi)^{-\zeta}\frac{\Gamma\nu\Gamma\delta}{\Gamma\mu\Gamma\gamma}\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)\Gamma(\mu+\delta s)\Gamma(\gamma+qs)}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\lambda+\rho s)\Gamma(\nu+\sigma s)\Gamma(\beta+\alpha s)}$$

$$\times\left[\frac{\omega}{\phi^{\eta}}\right]^s\int_0^{\infty}(v)^{s(\eta+\zeta-1)}e^{\frac{-v}{2}}w_{\lambda,\psi}(v)dv$$

Since $\int_0^{\infty} e^{-\frac{t}{2}}(v)^{n-1}w_{\lambda,\psi}(v)dv = \frac{\Gamma\left(\frac{1}{2}+\psi+n\right)\Gamma\left(\frac{1}{2}-\psi+n\right)}{\Gamma(1-\lambda+n)}$

$$=(\phi)^{-\zeta}\frac{\Gamma\nu\Gamma\delta}{\Gamma\mu\Gamma\gamma}\sum_{s=0}^{\infty}\frac{\prod_{n=1}^{r}\beta(b_n,s)\Gamma(\mu+\delta s)\Gamma(\gamma+qs)}{\prod_{n=1}^{r}\beta(a_n,s)\Gamma(\lambda+\rho s)\Gamma(\nu+\sigma s)\Gamma(\beta+\alpha s)}$$

$$\times \frac{\Gamma\left(\frac{1}{2} \pm \psi + \zeta + s\eta\right)\left[\frac{\omega}{\phi^\eta}\right]^s}{\Gamma(1-\lambda+\zeta+s\eta)s!}$$

Corollary. *When there is no upper and lower parameter, we get*

$$\int_0^\infty e^{\frac{-\phi t}{2}} t^{\zeta-1} w_{\lambda,\psi}(\phi t) Q^{\mu,\delta,\gamma,q,r}_{\alpha,\beta,\nu,\sigma,\lambda,\rho}(\omega t^\eta) = (\phi)^{-\zeta} \frac{\Gamma\nu\Gamma\delta}{\Gamma\mu\Gamma\gamma}$$

$$\times \sum_{s=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, s)\Gamma(\mu+\delta s)\Gamma(\gamma+qs)}{\prod_{n=1}^r \beta(a_n, s)\Gamma(\lambda+\rho s)\Gamma(\nu+\sigma s)\Gamma(\beta+\alpha s)} \frac{\Gamma\left(\frac{1}{2} \pm \psi + \zeta + s\eta\right)\left[\frac{\omega}{\phi^\eta}\right]^s}{\Gamma(1-\lambda+\zeta+s\eta)s!}$$

$$= \int_0^\infty e^{\frac{-\phi t}{2}} t^{\zeta-1} w_{\lambda,\psi}(\phi t) E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(\omega t^\eta)$$

Which is the Whittaker transform of Mittag-Leffler function $E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}$ *in Khan [11].*

4. Conclusion

The new Q Function for the generalized Mittag-Leffler function is introduced by evaluating some integral transforms such as Euler beta transform, Laplace transform, Fourier transform, Fourier sine and cosine transform, and the Whittaker transform.

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