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CERTAIN TRANSFORMATIONS INVOLVING POLY-BASIC HYPERGEOMETRIC SERIES

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Abstract: In this paper, making use of Bailey transform and certain known summation formulas an attempt has been made to establish transformation formulas for poly-basic hypergeometric series.

Keywords and Phrases: Summation formula, transformation formula, Bailey transform, poly-basic hypergeometric series.

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1. Introduction, Notations and Definitions

For real or complex $q(|q| < 1)$, the q-shifted factorial is defined by,

$$[\alpha; q]_n = \begin{cases} 1, & \text{if} \quad n = 0 \\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}), & \text{if} \quad n = 1, 2, 3, ... \end{cases} . \tag{1.1}$$

Also,

$$[\alpha; q]_\infty = \prod_{n=0}^{\infty}(1 - \alpha q^n). \tag{1.2}$$

A basic hypergeometric function is defined as,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n q^{\lambda n(n-1)/2}}{[q, b_1, b_2, ..., b_s; q]_n}, \tag{1.3}$$

where $[a_1, a_2, ..., a_r; q]_n = [a_1; q]_n[a_2; q]_n...[a_r; q]_n$.
The series (1.3) converges absolutely for all values of z if $\lambda > 0$ and for $|z| < 1$ if $\lambda = 0$.
A poly-basic hypergeometric series is defined as

$$\Phi\left[\begin{array}{l} a_1, a_2, ..., a_r : c_{1,1}, ..., c_{1,r_1}; ...; c_{m,1}, ..., c_{m,r_m}; q, q_1, ..., q_m; z \\ b_1, b_2, ..., b_s : d_{1,1}, ..., d_{1,s_1}; ...; d_{m,1}, ..., d_{m,s_m} \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n} \prod_{j=1}^{m} \frac{[c_{j,1}, ..., c_{j,r_j}; q_j]_n}{[d_{j,1}, ..., d_{j,s_j}; q_j]_n}. \tag{1.4}$$

The well known Bailey's transform states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.5}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.6}$$

where α_r, u_r, v_r and δ_r are functions of r alone and the series for γ_n is convergent, then subject to the convergence,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.7}$$

In this paper, we shall make use of Bailey's transform (1.5) - (1.7) in order to establish transformations for poly-basic hypergeometric series.
We shall make use of following known summations in our analysis,

$${}_4\Phi_3\left[\begin{array}{l} q, q^a : q_1^b; zq^{a+1}q_1^{b+1}; q, q_1, qq_1; z \\ zq^{a+1} : zq_1^{b+1}; zq^a q_1^b \end{array}\right]_N = \frac{(1-zq^a)(1-zq_1^b)}{(1-z)(1-zq^a q_1^b)}$$

$$-\frac{[q^a; q]_{N+1}[q_1^b; q_1]_{N+1} z^{N+1}}{(1-z)(1-zq^a q_1^b)[zq^{a+1}; q]_N [zq_1^{b+1}; q_1]_N}. \tag{1.8}$$

[R.P. Agarwal 1; 7.12 (2)]

As $N \to \infty$

$${}_4\Phi_3\left[\begin{array}{l} q, q^a : q_1^b; zq^{a+1}q_1^{b+1}; q, q_1, qq_1; z \\ zq^{a+1} : zq_1^{b+1}; zq^a q_1^b \end{array}\right] = \frac{(1-zq^a)(1-zq_1^b)}{(1-z)(1-zq^a q_1^b)}. \tag{1.9}$$

$$\sum_{k=0}^{n} \frac{(1-ap^k q^k)[a; p]_k [c; q]_k c^{-k}}{(1-a)[q; q]_k [ap/c; p]_k} = \frac{[ap; p]_n [cq; q]_n}{[q; q]_n [ap/c; p]_n c^n}. \tag{1.10}$$

[Gasper and Rahman 2; (3.6.8), p. 71]

$$\sum_{k=0}^{n}\frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a/bc;q]_k}{(1-a)(1-b)[q,aq/b;q]_k[ap/c,bcp;p]_k}=\frac{[ap,bp;p]_n[cq,aq/bc;q]_n}{[q,aq/b;q]_n[ap/c,bcp;p]_n}. \tag{1.11}$$

[Gasper and Rahman 2; (3.6.7), p. 71]

2. Main Results

Taking $u_r=v_r=1$ and $\delta_r=\dfrac{[q^a;q]_r[q_1^b;q_1]_r[xq^{a+1}q_1^{b+1};qq_1]_rx^r}{[xq^{a+1};q]_r[xq_1^{b+1};q_1]_r[xq^aq_1^b;qq_1]_r}$ in (1.6) we get

$$\gamma_n=\frac{(1-xq^a)(1-xq_1^b)[q^a;q]_n[q_1^b;q_1]_nx^n}{(1-x)(1-xq^aq_1^b)[xq^a;q]_n[xq_1^b;q_1]_n}. \tag{2.1}$$

Thus from (1.5) and (1.7) we have
If

$$\beta_n=\sum_{r=0}^{n}\alpha_r. \tag{2.2}$$

then, under suitable convergence conditions we find,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\sum_{n=0}^{\infty}\frac{[q^a;q]_n[q_1^b;q_1]_nx^n}{[xq^a;q]_n[xq_1^b;q_1]_n}\alpha_n$$

$$=\sum_{n=0}^{\infty}\frac{[q^a;q]_n[q_1^b;q_1]_n[xq^{a+1}q_1^{b+1};qq_1]_nx^n}{[xq^{a+1};q]_n[xq_1^{b+1};q_1]_n[xq^aq_1^b;qq_1]_n}\beta_n. \tag{2.3}$$

Now, we shall make use of (2.3) in order to establish transformations for q-hypergeometric series.

(i) Choosing $\alpha_r=\dfrac{[p^\alpha;p]_r[p_1^\beta;p_1]_r[zp^{\alpha+1}p_1^{\beta+1};pp_1]_rz^r}{[zp^{\alpha+1};p]_r[zp_1^{\beta+1};p_1]_r[zp^\alpha p_1^\beta;pp_1]_r}$ in (2.2) we have from (1.8),

$$\beta_n=\frac{(1-zp^\alpha)(1-zp_1^\beta)}{(1-z)(1-zp^\alpha p_1^\beta)}-\frac{[p^\alpha;p]_{n+1}[p_1^\beta;p_1]_{n+1}z^{n+1}}{(1-z)(1-zp^\alpha p_1^\beta)[zp^{\alpha+1};p]_n[zp_1^{\beta+1};p]_n}. \tag{2.4}$$

Putting these values of α_n and β_n in (2.3) we get,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\,\Phi\left[\begin{matrix}q,q^a:q_1^b;p^\alpha;p_1^\beta;zp^{\alpha+1}p_1^{\beta+1};q,q_1,p,p_1,pp_1;xz\\ xq^a:xq_1^b;zp^{\alpha+1};zp_1^{\beta+1};zp^\alpha p_1^\beta\end{matrix}\right]$$

$$= \frac{(1-zp^{\alpha})(1-zp_1^{\beta})(1-xq^a)(1-xq_1^b)}{(1-x)(1-z)(1-xq^aq_1^b)(1-zp^{\alpha}p_1^{\beta})} - \frac{(1-p^{\alpha})(1-p_1^{\beta})z}{(1-z)(1-zp^{\alpha}p_1^{\beta})}$$

$$\Phi\left[\begin{array}{l} q, q^a : q_1^b; xq^{a+1}q_1^{b+1}; p^{\alpha+1}; p_1^{\beta+1}; q, q_1, qq_1, p, p_1; xz \\ xq^{a+1} : xq_1^{b+1}; xq^aq_1^b; zp^{\alpha+1}; zp_1^{\beta+1} \end{array}\right]. \tag{2.5}$$

(ii) Next, choosing $\alpha_r = \dfrac{(1-\alpha p^r p_1^r)[\alpha; p]_r[\beta; p_1]_r\beta^{-r}}{(1-\alpha)[p_1; p_1]_r[\alpha p/\beta; p]_r}$ in (2.2) and using (1.10) we get

$$\beta_n = \frac{[\alpha p; p]_n[\beta p_1; p_1]_n}{[p_1; p_1]_n[\alpha p/\beta; p]_n\beta^n}.$$

Putting the values of α_n and β_n in (2.3) we get,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\Phi\left[\begin{array}{l} \beta : \alpha; \alpha pp_1; q^a; q_1^b; p_1, p, pp_1, q, q_1; x/\beta \\ - : \alpha p/\beta; \alpha; xq^a; xq_1^b \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} \beta p_1 : \alpha p; q^a; q_1^b; xq^{a+1}q_1^{b+1}; p_1, p, q, q_1, qq_1; x/\beta \\ - : \alpha p/\beta; xq^{a+1}; xq_1^{b+1}; xq^aq_1^b \end{array}\right]. \tag{2.6}$$

(iii) Again, choosing $\alpha_r = \dfrac{(1-\alpha p^r p_1^r)(1-\beta p^r p_1^{-r})[\alpha, \beta; p]_r[\gamma, \alpha/\beta\gamma; p_1]_r}{(1-\alpha)(1-\beta)[p_1, \alpha p_1/\beta; p_1]_r[\alpha p/\gamma, \beta\gamma p; p]_r}$ in (2.2) and using (1.11) we get,

$$\beta_n = \frac{[\alpha p, \beta p; p]_n[\gamma p_1, \alpha p_1/\beta\gamma; p_1]_n}{[p_1, \alpha p_1/\beta; p_1]_n[\alpha p/\gamma, \beta\gamma p; p]_n}.$$

Putting these values in (2.3) we obtain,

$$\frac{(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\Phi\left[\begin{array}{l} \gamma, \alpha/\beta\gamma : \alpha, \beta; \alpha pp_1; \beta p/p_1; q^a; q_1^b; p_1, p, pp_1, p/p_1, qq_1; x \\ \alpha p_1/\beta : \alpha p/\gamma, \beta\gamma p; \alpha; \beta; xq^a; xq_1^b \end{array}\right]$$

$$= \Phi\left[\begin{array}{l} \gamma p_1, \alpha p_1/\beta\gamma : \alpha p, \beta p; q^a; q_1^b; xq^{a+1}q_1^{b+1}; p_1, p, q, q_1, qq_1; x \\ \alpha p_1/\beta : \alpha p/\gamma, \beta\gamma p; xq^{a+1}; xq_1^{b+1}; xq^aq_1^b \end{array}\right]. \tag{2.7}$$

(iv) Lastly, taking $\alpha_r = z^r$ in (2.2) we get

$$\beta_n = \frac{1-z^{n+1}}{1-z}$$

Putting these values of α_n and β_n in (2.3) we get,

$$\frac{(1-z)(1-xq^a)(1-xq_1^b)}{(1-x)(1-xq^aq_1^b)}\Phi\left[\begin{array}{l} q, q^a : q_1^b; q, q_1; zx \\ xq^a : xq_1^b \end{array}\right]$$

$$= \Phi \left[\begin{array}{l} q, q^a : q_1^b; xq^{a+1}q_1^{b+1}; q, q_1, qq_1; x \\ xq^{a+1} : xq_1^{b+1}; xq^a q_1^b \end{array} \right]$$

$$-z\Phi \left[\begin{array}{l} q, q^a : q_1^b; xq^{a+1}q_1^{b+1}; q, q_1, qq_1; zx \\ xq^{a+1} : xq_1^{b+1}; xq^a q_1^b \end{array} \right]. \tag{2.8}$$

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[1] Agarwal, R.P., Generlized Hypergeometric Series and its Applications to the theory of Combinatorial Analysis and partition theory, (Unpublished monograph).

[2] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, 1991.