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APPLICATION OF CONJUGATE BAILEY PAIR AND CONJUGATE WP-BAILEY PAIR TO ESTABLISH SUMMATIONS AND q- SERIES IDENTITIES

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Abstract: In this paper, making use of Bailey pair, conjugate Bailey pair, WP-Bailey pair and conjugate WP-Bailey pair, four theorems have been established which yield certain double series identities as special cases.

Keywords and Phrases: Bailey pair, WP-Bailey pair, conjugate Bailey pair, conjugate WP-Bailey pair, summation formula and identities.

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1. Introduction, Notations and Definitions

The widely-investigated transform, which was discovered by Bailey in 1947, is being used ever since then in order to obtain various ordinary hypergeometric series and q-hypergeometric series identities as well as the Rogers-Ramanujan type identities. As an application of Bailey transform, Bailey himself introduced the Bailey pair which has been further generalized as the well-poised (or WP-) Bailey pair by Andrews [2]. Making use of the Bailey pair and WP-Bailey pair, many researchers derived some new transformation formulas and identities between basic hypergeometric series and new single-sum and double-sum identities of the Rogers-Ramanujan-Slater type. Recently, Srivastava *et al.* [9] and [10]) and Singh *et al.* [5] used the Bailey pair, the WP-Bailey pair and the derived WP-Bailey pair in order to establish many useful transformations and q-series identities.

In this paper, we shall adopt following notations and definitions. The q-rising factorial is defined (for $|q| < 1$) by

$$(a;q)_0 = 1 \qquad \text{and} \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \qquad (n = 1, 2, 3, \cdots).$$

We also write

$$(a_1, a_2, \cdots, a_r; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

A basic (or q-) hypergeometric series is defined by (see [7, p. 347, Eq. 9.4 (272)]; see also [4] and [6])

$$ {}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; \\ & q; z \\ b_1, b_2, \cdots, b_s; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_n}{(q, b_1, b_2, \cdots, b_s; q)_n} \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n.$$

The q-series in above converges for all values of z if $r < 1 + s$ and for $|z| < 1$ if $r = 1 + s$.

We now state the Bailey transform as follows (see, for details, [2]; see also [3]):

The Bailey Transform. *If*

$$\beta_n = \sum_{r=0}^{n} \alpha_r \, u_{n-r} \, v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r \, v_{r+2n}, \tag{1.2}$$

then, under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \, \gamma_n = \sum_{n=0}^{\infty} \beta_n \, \delta_n, \tag{1.3}$$

where u_r, v_r, α_r and δ_r are arbitrarily chosen sequences of r alone.

In application of Bailey's transform (1.1) to (1.3) Bailey chose

$$u_r = \frac{1}{(q;q)_r} \qquad \text{and} \qquad v_r = \frac{1}{(aq;q)_r}.$$

With this choice Bailey transform takes the following new shape:

If

$$\begin{aligned} \beta_n &= \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \\ &= \frac{1}{(a,aq;q)_n} \sum_{r=0}^{n} \frac{(-1)^n (q^{-n};q)_r \, q^{nr+r} \alpha_r}{q^{r(r+1)/2}(aq^{1+n};q)_r} \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \gamma_n &= \sum_{r=0}^{n} \frac{\delta_{r+n}}{(q;q)_r (aq;q)_{r+2n}} \\ &= \frac{1}{(aq;q)_{2n}} \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r (aq^{1+2n};q)_r}, \end{aligned} \tag{1.5}$$

then, under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *are arbitrarily chosen sequences of* n *alone.*

A pair of sequences $\langle\alpha_n, \beta_n\rangle$ satisfying (1.4) is called a Bailey pair. On the other hand, a pair of sequences $\langle\gamma_n, \delta_n\rangle$ satisfying (1.5) is called a conjugate Bailey pair with respect to the parameter a.

Following the above-cited Andrew's work [2], a WP-Bailey pair relative to the parameter a is a pair of sequences $\langle\alpha_n(a, k; q), \beta_n(a, k; q)\rangle$ which are constrained by

$$\beta_n(a,k;q) = \sum_{r=0}^{n} \frac{\left(\frac{k}{a};q\right)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\alpha_r(a,k;q)$$
$$= \frac{\left(\frac{k}{a},k;q\right)_n}{(q,aq;q)_n}\sum_{n=0}^{n}\frac{(kq^n,q^{-n};q)_r\left(\frac{aq}{k}\right)^r}{\left(\frac{aq^{1-n}}{k},aq^{1+n};q\right)_r}\alpha_r(a,k;q). \tag{1.7}$$

Indeed, when $k \to 0$, a WP-Bailey pair reduces to the classical Bailey pair given in (1.4). Bailey's definition of a conjugate Bailey pair can now be extended to define a conjugate WP-Bailey pair relative to the parameter a to be a pair of sequences $\langle\gamma_n(a, k; q), \delta_n(a, k; q)\rangle$ such that

$$\gamma_n(a,k;q) = \sum_{r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_r(k;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n}(a,k;q)$$
$$= \frac{(k;q)_{2n}}{(aq;q)_{2n}}\sum_{n=0}^{n}\frac{(kq^{2n};q)_r\left(\frac{k}{a};q\right)_r}{\left(aq^{1+2n};q\right)_r(q;q)_r}\delta_{r+n}(a,k;q). \tag{1.8}$$

Thus, analogous to the Bailey transform, we have the following result.

If $\langle\alpha_n(a, k; q), \beta_n(a, k; q)\rangle$ *is a WP-Bailey pair and* $\langle\gamma_n(a, k; q), \delta_n(a, k; q)\rangle$ *is a conjugate WP-Bailey pair relative to the parameter* a*, then, under suitable convergence conditions,*

$$\sum_{n=0}^{\infty}\alpha_n(a,k;q)\gamma_n(a,k;q) = \sum_{n=0}^{\infty}\beta_n(a,k;q), \delta_n(a,k;q). \tag{1.9}$$

In our present investigation, we shall make use of following summation formulas:

$${}_2\Phi_1\left[\begin{matrix} a,b; & \\ & q;\dfrac{c}{ab} \\ c; & \end{matrix}\right] = \frac{\left(\dfrac{c}{a},\dfrac{c}{b};q\right)_\infty}{\left(c,\dfrac{c}{ab};q\right)_\infty} \qquad \left(\left|\frac{c}{ab}\right|<1\right). \tag{1.10}$$

(see [4, Appendix II, Eq. (II. 8)])

$$
{}_2\Phi_1\left[\begin{matrix} a, b; \\ cq; \end{matrix}\; q; \frac{c}{ab}\right] = \frac{\left(\frac{cq}{a}, \frac{cq}{b}; q\right)_\infty}{\left(cq, \frac{cq}{ab}; q\right)_\infty}\left(\frac{ab(1+c)-c(a+b)}{ab-c}\right) \qquad \left(\left|\frac{c}{ab}\right|<1\right). \tag{1.11}
$$

(see [11, Eq. (1.4)])

$$
{}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d; \end{matrix}\; q; \frac{aq}{bcd}\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \tag{1.12}
$$

(see [4, Appendix II, Eq. (II.20)])

Finally, we recall the following series identity:

$$
\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Omega(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Omega(n+r,r), \tag{1.13}
$$

provided that each member exists.

(see [8, p. 100, Lemma 1, Eq. 2.1 (2)])

2. Main Results

In this section we establish following four theorems which shall be used in next section.

Theorem 1. *If $\alpha_n(a)$ is an arbitrary sequence such that*

$$
\alpha_0(a) = 1
$$

then,

$$
\sum_{n,r=0}^{\infty}\frac{(\rho_1,\rho_2;q)_{n+r}}{(q;q)_r\,(aq;q)_{n+2r}}\left(\frac{aq}{\rho_1\rho_2}\right)^{n+r}\alpha_r(a)
$$

$$
= \frac{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_\infty}{\left(aq, \frac{aq}{\rho_1\rho_2}; q\right)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^{n}\alpha_n(a). \tag{2.1}
$$

Proof. Taking

$$
\delta_r = (\rho_1,\rho_2;q)_r\left(\frac{aq}{\rho_1\rho_2}\right)^{r}
$$

in (1.5), summing the series by making use of (1.10) we get after some simplifications,

$$\gamma_n = \frac{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_\infty}{\left(aq, \frac{aq}{\rho_1\rho_2}; q\right)_\infty} \frac{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n}{\left(aq, \frac{aq}{\rho_1\rho_2}; q\right)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n.$$

Putting these values of γ_n and δ_n in (1.6), again putting the series value of β_n from (1.4) in (1.6) and applying the lemma (1.13) we get (2.1).

Theorem 2. *If $\alpha_n(a,k)$ is an arbitrary sequence such that*

$$\alpha_0(a,k) = 1$$

then

$$\sum_{n,r=0}^{\infty} \left(\frac{1-kq^{2n+2r}}{1-k}\right) \frac{(\rho_1,\rho_2;q)_{n+r} \left(\frac{k}{a};q\right)_n (k;q)_{n+2r}}{\left(\frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_{n+r} (q;q)_n (aq;q)_{n+2r}} \left(\frac{aq}{\rho_1\rho_2}\right)^{n+r} \alpha_r(a,k)$$
$$= \frac{\left(kq, \frac{kq}{\rho_1\rho_2}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_\infty}{\left(aq, \frac{aq}{\rho_1\rho_2}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_\infty} \sum_{n,r=0}^{\infty} \frac{(\rho_1,\rho_2;q)_n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a,k). \quad (2.2)$$

Proof. Taking

$$\delta_r(a,k) = \frac{(q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2; q)_r}{\left(\sqrt{k}, -\sqrt{k}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_r} \left(\frac{aq}{\rho_1\rho_2}\right)^r$$

in (1.8), summing the series by making use of (1.12) we get after some simplifications,

$$\gamma_n(a,k) = \frac{\left(kq, \frac{kq}{\rho_1\rho_2}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_\infty}{\left(aq, \frac{aq}{\rho_1\rho_2}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_\infty} \frac{(\rho_1,\rho_2;q)_n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n.$$

Now, putting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ in (1.9), again putting the series value of $\beta_n(a,k)$ from (1.7) in (1.9) and applying the lemma (1.13) we get (2.2).

Theorem 3. *If $\alpha_n(a,k)$ is an arbitrary sequence such that*

$$\alpha_0(a,k) = 1$$

then

$$\sum_{n,r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_n (k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}} \left(\frac{a^2q}{k^2}\right)^{n+r} \alpha_r(a,k)$$

$$= \frac{\left(\frac{aq}{k}, \frac{a^2q}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(kq;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \alpha_n(a,k). \qquad (2.3)$$

Proof. Taking

$$\delta_r(a,k) = \left(\frac{a^2q}{k^2}\right)^r$$

in (1.8) and summing the series by making use of (1.10) we get,

$$\gamma_n(a,k) = \frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} \left(\frac{a^2q}{k^2}\right)^n \frac{\left(\frac{aq}{k}, \frac{a^2q}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty}.$$

Putting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ in (1.9), again putting the series value of $\beta_n(a,k)$ from (1.7) in (1.9) and applying the lemma (1.13) we get (2.3).

Theorem 4. *If $\alpha_n(a,k)$ is an arbitrary sequence such that*

$$\alpha_0(a,k) = 1$$

then

$$\sum_{n,r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_n (k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}} \left(\frac{a^2}{k^2}\right)^{n+r} \alpha_r(a,k)$$

$$= \frac{\left(\frac{aq}{k}, \frac{a^2q}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \left(\frac{k}{k+a}\right) \sum_{n=0}^{\infty} \frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}} (1+aq^{2n}) \left(\frac{a^2}{k^2}\right)^n \alpha_n(a,k). \qquad (2.4)$$

Proof. Taking

$$\delta_r(a,k) = \left(\frac{a^2}{k^2}\right)^r$$

in (1.8) and summing the series by using (1.11) we find,

$$\gamma_n(a,k) = \frac{\left(\frac{aq}{k}, \frac{aq^2}{k};q\right)_\infty}{\left(aq, \frac{a^2q}{k^2};q\right)_\infty} \left(\frac{k}{k+a}\right) \frac{(k;q)_{2n}(1+aq^{2n})}{(a^2q/k;q)_{2n}} \left(\frac{a^2}{k^2}\right)^n.$$

Putting these values of $\gamma_n(a,k)$ and $\delta_n(a,k)$ in (1.9) and series value of $\beta_n(a,k)$ from (1.7) in (1.9) and applying the lemma (1.13) we get (2.4).

3. Special Cases

In this section we shall deduce certain interesting double series identities from theorems established in previous section.

Remark 1. If we choose $\rho_1, \rho_2 \to \infty$ in (2.1), we get

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2} a^{n+r} \alpha_r(a)}{(q;q)_r (aq;q)_{n+2r}} = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} q^{n^2} a^n \alpha_n(a). \tag{3.1}$$

Upon substituting these values of $\alpha_n = \frac{1}{(q;q)_n}$ in (3.1), we derive the following summation formula

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r} a^{n+r}}{(q;q)_r^2 (aq;q)_{n+2r}} = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n}. \tag{3.2}$$

Taking $\alpha_n = \frac{q^n}{(q;q)_n}$ in (3.1) we have

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r} a^{n+r}}{(q;q)_r^2 (aq;q)_{n+2r}} = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n} a^n}{(q;q)_n}. \tag{3.3}$$

From (3.1) and (3.2) we have,

$$\frac{\displaystyle\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r} a^{n+r}}{(q;q)_r^2 (aq;q)_{n+2r}}}{\displaystyle\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2} a^{n+r} \alpha_r(a)}{(q;q)_r (aq;q)_{n+2r}}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n} a^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n}}$$

$$= \frac{1}{1+} \frac{aq}{1+} \frac{aq^2}{1+} \frac{aq^3}{1+\ldots} \tag{3.4}$$

For $a = 1$, (3.2) yields

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_r^2 (q;q)_{n+2r}} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q)_\infty (q,q^4;q^5)_\infty}. \tag{3.5}$$

Taking $a = 1$ in (3.3) we get,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r}}{(q;q)_r^2 (q;q)_{n+2r}} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty (q^2,q^3;q^5)_\infty}. \tag{3.6}$$

Taking $a = 1$, $\alpha_n = \frac{q^{-n}}{(q;q)_n}$ in (3.1) we get,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2-r}}{(q;q)_r^2(q;q)_{n+2r}} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q;q)_n}$$

$$= \frac{1}{(q;q)_\infty} \left\{ \frac{1}{(q,q^4;q^5)_\infty} + \frac{1}{(q^2,q^3;q^5)_\infty} \right\}. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we find,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2-r}}{(q;q)_r^2(q;q)_{n+2r}} = \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2}}{(q;q)_r^2(q;q)_{n+2r}} + \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r}}{(q;q)_r^2(q;q)_{n+2r}}. \tag{3.8}$$

Taking $a = q$ in (3.2) we get,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+1)}}{(q;q)_r^2(q;q)_{n+2r+1}} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}. \tag{3.9}$$

For $a = 1$, (3.3) yields

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r}}{(q;q)_r^2(q;q)_{n+2r}} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}. \tag{3.10}$$

Comparing (3.9) and (3.10) we have,

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)(n+r+1)}}{(q;q)_r^2(q;q)_{n+2r+1}} = \sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r}}{(q;q)_r^2(q;q)_{n+2r}}. \tag{3.11}$$

Remark 2. Taking $\rho_1, \rho_2 \to \infty$ in (2.2) we obtain,

$$\sum_{n,r=0}^{\infty} \left(\frac{1-kq^{2n+2r}}{1-k} \right) \frac{q^{(n+r)^2} \left(\frac{k}{a};q\right)_n (k;q)_{n+2r} a^{n+r}}{(q;q)_n (aq;q)_{n+2r}} \alpha_r(a,k)$$

$$= \frac{(kq;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} q^{n^2} a^n \alpha_n(a,k). \tag{3.12}$$

Taking $\alpha_n(a,k) = \frac{1}{(q;q)_n}$ in (3.12) we have

$$\sum_{n,r=0}^{\infty} \left(\frac{1-kq^{2n+2r}}{1-k} \right) \frac{q^{(n+r)^2} \left(\frac{k}{a};q\right)_n (k;q)_{n+2r} a^{n+r}}{(q;q)_n (q;q)_r (aq;q)_{n+2r}}$$

$$= \frac{(kq;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2} a^n}{(q;q)_n}. \tag{3.13}$$

Again, taking $\alpha_n(a,k) = \frac{q^n}{(q;q)_n}$ in (3.12) we have

$$\sum_{n,r=0}^{\infty}\left(\frac{1-kq^{2n+2r}}{1-k}\right)\frac{q^{(n+r)^2+r}\left(\frac{k}{a};q\right)_n (k;q)_{n+2r}a^{n+r}}{(q;q)_n(q;q)_r(aq;q)_{n+2r}}$$

$$= \frac{(kq;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}. \tag{3.14}$$

From (3.13) and (3.14) we have,

$$\frac{\displaystyle\sum_{n,r=0}^{\infty}\left(\frac{1-kq^{2n+2r}}{1-k}\right)\frac{q^{(n+r)^2+r}\left(\frac{k}{a};q\right)_n (k;q)_{n+2r}a^{n+r}}{(q;q)_n(q;q)_r(aq;q)_{n+2r}}}{\displaystyle\sum_{n,r=0}^{\infty}\left(\frac{1-kq^{2n+2r}}{1-k}\right)\frac{q^{(n+r)^2}\left(\frac{k}{a};q\right)_n (k;q)_{n+2r}a^{n+r}}{(q;q)_n(q;q)_r(aq;q)_{n+2r}}}$$

$$= \frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}a^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}a^n}{(q;q)_n}} = \frac{1}{1+}\frac{aq}{1+}\frac{aq^2}{1+}\frac{aq^3}{1+\ldots}. \tag{3.15}$$

For $a = 1$, (3.13) yields

$$\sum_{n,r=0}^{\infty}\left(\frac{1-kq^{2n+2r}}{1-k}\right)\frac{q^{(n+r)^2}(k;q)_n (k;q)_{n+2r}}{(q;q)_n(q;q)_r(q;q)_{n+2r}}$$

$$= \frac{(kq;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{(kq;q)_\infty}{(q;q)_\infty}\frac{1}{(q,q^4;q^5)_\infty}. \tag{3.16}$$

For $a = 1$, (3.14) gives

$$\sum_{n,r=0}^{\infty}\left(\frac{1-kq^{2n+2r}}{1-k}\right)\frac{q^{(n+r)^2+r}(k;q)_n (k;q)_{n+2r}}{(q;q)_n(q;q)_r(q;q)_{n+2r}}$$

$$= \frac{(kq;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{(kq;q)_\infty}{(q;q)_\infty}\frac{1}{(q^2,q^3;q^5)_\infty}. \tag{3.17}$$

For $k = q$, (3.16) yields

$$\sum_{n,r=0}^{\infty}(1-q^{2n+2r+1})\frac{q^{(n+r)^2}}{(q;q)_r} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{3.18}$$

For $k = q$, (3.17) yields

$$\sum_{n,r=0}^{\infty}(1-q^{2n+2r+1})\frac{q^{(n+r)^2+r}}{(q;q)_r} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{3.19}$$

From (3.18) and (3.19) we have

$$\frac{\sum_{n,r=0}^{\infty}(1-q^{2n+2r+1})\frac{q^{(n+r)^2+r}}{(q;q)_r}}{\sum_{n,r=0}^{\infty}(1-q^{2n+2r+1})\frac{q^{(n+r)^2}}{(q;q)_r}} = \frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}}$$

$$\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots}. \tag{3.20}$$

Remark 3. Upon setting

$$\alpha_n(a,k) = \frac{(a^2q/k;q)_{2n}(z;q)_n}{(k;q)_{2n}(q;q)_n}$$

in (2.3), we get

$$\sum_{n,r=0}^{\infty}\frac{\left(\frac{k}{a};q\right)_n(k;q)_{n+2r}\left(\frac{a^2q}{k};q\right)_{2r}(z;q)_r}{(q;q)_n(aq;q)_{n+2r}(k;q)_{2r}(q;q)_r}\left(\frac{a^2q}{k^2}\right)^{n+r}$$

$$= \frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(z;q)_n}{(q;q)_n}\left(\frac{a^2q}{k^2}\right)^n$$

$$= \frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\frac{\left(\frac{a^2zq}{k^2};q\right)_\infty}{\left(\frac{a^2q}{k^2};q\right)_\infty}. \tag{3.21}$$

As $k\to\infty$, (2.3) yields

$$\sum_{n,r=0}^{\infty}\frac{q^{n^2+2r^2+2nr}a^{2r}a^n\alpha_r(a,k)}{(q;q)_n(aq;q)_{n+2r}}$$

$$= \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2}a^{2n}\alpha_n(a,k). \tag{3.22}$$

Choosing

$$\alpha_n(a,k) = \frac{1}{(q^2, a^2q^2; q^2)_n}$$

in (3.22) we get

$$\sum_{n,r=0}^{\infty} \frac{q^{n^2+2r(r+n)}a^{n+2r}}{(q;q)_n(aq;q)_{n+2r}(q^2;q^2)_n(a^2q^2;q^2)_n}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{a^{2n}q^{2n^2}}{(q^2;q^2)_n(a^2q^2;q^2)_n}.$$
$$= \frac{1}{(aq;q)_\infty(a^2q^2;q^2)_\infty}. \tag{3.23}$$

Remark 4. By letting

$$\alpha_n = \frac{(a^2q/k;q)_{2n}(z;q)_n}{(k;q)_{2n}(q;q)_n}$$

in (2.4), we deduce that

$$\sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}(a^2q/k;q)_{2r}(z;q)_r}{(q;q)_n(aq;q)_{n+2r}(k;q)_{2r}(q;q)_r} \left(\frac{a^2}{k^2}\right)^{n+r}$$
$$= \frac{\left(\frac{aq}{k}, \frac{a^2q}{k}; q\right)_\infty}{\left(aq, \frac{a^2q}{k^2}; q\right)_\infty} \frac{k}{k+a} \left\{ \frac{(a^2z/k^2;q)_\infty}{(a^2/q^2;q)_\infty} + a\frac{(a^2zq^2/k^2;q)_\infty}{(a^2q^2/k^2;q)_\infty} \right\} \tag{3.24}$$

As $k \to \infty$, (2.4) yields

$$\sum_{n,r=0}^{\infty} \frac{q^{n(n-1)/2}q^{(n+2r)(n+2r-1)/2}a^{n+2r}}{(q;q)_n(aq;q)_{n+2r}} \alpha_r(a,k)$$
$$= \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} q^{n(2n-1)}a^{2n}(1+aq^{2n})\alpha_n(a,k)$$
$$= \frac{1}{(aq;q)_\infty} \left\{ \sum_{n=0}^{\infty} q^{2n^2-n}a^{2n}\alpha_n(a,k) + a\sum_{n=0}^{\infty} q^{2n^2+n}a^{2n}\alpha_n(a,k) \right\}. \tag{3.25}$$

Now choosing

$$\alpha_n(a,k) = \frac{q^n}{(q^2;q^2)_n(a^2q^2;q^2)_n}$$

in (3.25) we find,

$$(aq;q)_\infty \sum_{n,r=0}^{\infty} \frac{q^{n^2-n+2r^2+2nr} a^{n+2r}}{(q;q)_n (aq;q)_{n+2r} (q^2;q^2)_r (a^2q^2;q^2)_r}$$
$$= \frac{1}{(a^2q^2;q^2)_\infty} + \sum_{n=0}^{\infty} \frac{q^{2n^2+2n} a^{2n+1}}{(q^2;q^2)_n (a^2q^2;q^2)_n}. \tag{3.26}$$

5. Concluding Remarks and Observations

In our present investigation, which is motivated essentially by the earlier works of Andrews [2], Srivastava *et al.* ([9] and [10]) and Singh *et al.* [5], we have successfully applied the widely-studied Bailey transform, the Bailey pair of sequences and the well-poised (or WP-) Bailey pair of sequences in order to derive many useful summation formulas for q-hypergeometric series as well as bi-basic hypergeometric series. We have also presented several other related q-series identities. Our results are stated here as Theorems 1, 2, 3 and 4 and Remarks 1 to 4.

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