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A HYPERGEOMETRIC GENERATING FUNCTION INFLUENCED BY THE WORK OF BURCHNALL, KRALL-FRINK AND RAINVILLE

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Abstract: In this paper,we obtain a hypergeometric generating relation associated with Srivastava-Daoust double hypergeometric function using a closed form of reduction formula and series rearrangement technique.Some results of Burchnall, Krall-Frink and Rainville are also obtained as special cases of our result.

Keywords and Phrases: Srivastava-Daoust double hypergeometric function; Series rearrangement technique;Reduction formula;Rainville polynomials;Generalized Bessel polynomials.

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1. Introduction and Preliminaries

In this paper, we shall use the following notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}$; $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$; $\mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

Pochhammer symbol

The Pochhammer symbol $(\alpha)_p (\alpha, p \in \mathbb{C})$[11, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [16, p.23, Eq.(22) and Eq.(23)] is defined by

$$(\alpha)_p = \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)}$$

$$(\alpha)_p = \begin{cases} 1 & ;(p=0;\alpha\in\mathbb{C}\backslash\{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ;(p=n\in\mathbb{N};\alpha\in\mathbb{C}), \\ \frac{(-1)^n k!}{(k-n)!} & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;0\leq n\leq k), \\ 0 & ;(\alpha=-k;p=n;n,k\in\mathbb{N}_0;n>k), \\ \frac{(-1)^n}{(1-\alpha)_n} & ;(p=-n;n\in\mathbb{N};\alpha\in\mathbb{C}\backslash\mathbb{Z}). \end{cases}$$

It being understood conventionally that$(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{ccc}(\alpha_p); & & \\ & & z \\ (\beta_q); & & \end{array}\right] = {}_pF_q\left[\begin{array}{ccc}\alpha_1,\alpha_2,\ldots,\alpha_p; & & \\ & & z \\ \beta_1,\beta_2,\ldots,\beta_q; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ; \quad j=1,2,\ldots,q,$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z|<\infty$, if $p\leq q$,

(ii) converges for $|z|<1$, if $p=q+1$,

(iii) diverges for all z, $z\neq 0$, if $p>q+1$'

(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\omega)>0$

(v) converges conditionally for $|z|=1$ $(z\neq 1)$, if $p=q+1$ and $-1<\Re(\omega)\leqq 0$,

(vi) diverges for $|z|=1$, if $p=q+1$ and $\Re(\omega)\leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.

Double hypergeometric function of Srivastava-Daoust

In an earlier paper [13], Srivastava and Daoust defined a generalization of the Kampé de Fériet function [3, p. 150] by means of the double hypergeometric series (see also [13, p. 199] and [15]).

$$S^{A:\ B;\ B'}_{C:\ D;\ D'}\begin{pmatrix}[(a_A)\ :\ \vartheta,\ \varphi]:[(b_B):\psi];[(b'_{B'}):\psi']; & \\ & x,\ y \\ [(c_C)\ :\ \delta,\ \varepsilon]:[(d_D):\eta];[(d'_{D'}):\eta']; & \end{pmatrix}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma[a_j+m\vartheta_j+n\varphi_j]\prod_{j=1}^{B}\Gamma[b_j+m\psi_j]\prod_{j=1}^{B'}\Gamma[b'_j+n\psi'_j]}{\prod_{j=1}^{C}\Gamma[c_j+m\delta_j+n\varepsilon_j]\prod_{j=1}^{D}\Gamma[d_j+m\eta_j]\prod_{j=1}^{D'}\Gamma[d'_j+n\eta'_j]}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.3}$$

$$=\frac{\prod_{j=1}^{A}\Gamma[a_j]\prod_{j=1}^{B}\Gamma[b_j]\prod_{j=1}^{B'}\Gamma[b'_j]}{\prod_{j=1}^{C}\Gamma[c_j]\prod_{j=1}^{D}\Gamma[d_j]\prod_{j=1}^{D'}\Gamma[d'_j]}\times \tag{1.4}$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m\vartheta_j+n\varphi_j}\prod_{j=1}^{B}(b_j)_{m\psi_j}\prod_{j=1}^{B'}(b'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(c_j)_{m\delta_j+n\varepsilon_j}\prod_{j=1}^{D}(d_j)_{m\eta_j}\prod_{j=1}^{D'}(d'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.5}$$

$$=\frac{\prod_{j=1}^{A}\Gamma[a_j]\prod_{j=1}^{B}\Gamma[b_j]\prod_{j=1}^{B'}\Gamma[b'_j]}{\prod_{j=1}^{C}\Gamma[c_j]\prod_{j=1}^{D}\Gamma[d_j]\prod_{j=1}^{D'}\Gamma[d'_j]}\times \tag{1.6}$$

$$\times F^{A:\ B;\ B'}_{C:\ D;\ D'}\begin{pmatrix}[(a_A)\ :\ \vartheta,\ \varphi]:[(b_B):\psi];[(b'_{B'}):\psi']; & \\ & x,\ y \\ [(c_C)\ :\ \delta,\ \varepsilon]:[(d_D):\eta];[(d'_{D'}):\eta']; & \end{pmatrix}$$

$$=\sum_{m,n=0}^{\infty}\Omega_{m,n}\ x^m y^n, \tag{1.7}$$

where the coefficients

$$\left\{\begin{array}{c}\vartheta_1,...,\vartheta_A;\ \varphi_1,...,\varphi_A;\ \psi_1,...,\psi_B;\ \psi'_1,...,\psi'_{B'};\ \delta_1,...,\delta_C;\\ \varepsilon_1,...,\varepsilon_C;\ \eta_1,...,\eta_D;\ \eta'_1,...,\eta'_{D'}\end{array}\right. \tag{1.8}$$

are real and positive.

Indeed it is easy to observe that when $y\to 0$, S $\begin{pmatrix}x\\y\end{pmatrix}$ reduces to the generalized hypergeometric series introduced by Wright [17,18] and when the positive real constraints in (1.8) are all taken as unity, it would equal

$$\frac{\prod_{j=1}^{A}\Gamma[a_j]\ \prod_{j=1}^{B}\Gamma[b_j]\ \prod_{j=1}^{B'}\Gamma[b'_j]}{\prod_{j=1}^{C}\Gamma[c_j]\ \prod_{j=1}^{D}\Gamma[d_j]\ \prod_{j=1}^{D'}\Gamma[d'_j]}\ F\left[\begin{matrix}(a_A):(b_B);(b'_{B'}); \\ \\ (c_C):(d_D);(d'_{D'});\end{matrix}\quad x,\ y\right], \tag{1.9}$$

where $F[x,y]$ denotes Kampé de Fériet's double hypergeometric function in the contracted notation of Burchnall and Chaundy [5, p. 112] in preference, for the sake of generality and elegance, to the one used by Kampé de Fériet himself [3, p. 150].

$$E_1=\left(\mu_1^{1+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{B}\psi_j}\right)\ \frac{\prod_{j=1}^{C}(\mu_1\delta_j+\mu_2\varepsilon_j)^{\delta_j}\ \prod_{j=1}^{D}(\eta_j)^{\eta_j}}{\prod_{j=1}^{A}(\mu_1\vartheta_j+\mu_2\varphi_j)^{\vartheta_j}\ \prod_{j=1}^{B}(\psi_j)^{\psi_j}}, \tag{1.10}$$

$$E_2=\left(\mu_2^{1+\sum_{j=1}^{D'}\eta'_j-\sum_{j=1}^{B'}\psi'_j}\right)\ \frac{\prod_{j=1}^{C}(\mu_1\delta_j+\mu_2\varepsilon_j)^{\varepsilon_j}\ \prod_{j=1}^{D'}(\eta'_j)^{\eta'_j}}{\prod_{j=1}^{A}(\mu_1\vartheta_j+\mu_2\varphi_j)^{\varphi_j}\ \prod_{j=1}^{B'}(\psi'_j)^{\psi'_j}}. \tag{1.11}$$

$$\Delta_1=1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\eta_j-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B}\psi_j, \tag{1.12}$$

$$\Delta_2=1+\sum_{j=1}^{C}\varepsilon_j+\sum_{j=1}^{D'}\eta'_j-\sum_{j=1}^{A}\varphi_j-\sum_{j=1}^{B'}\psi'_j. \tag{1.13}$$

Case I. The double power series in (1.3) converges for all complex values of x and y when $\Delta_1>0$ and $\Delta_2>0$.

Case II. The double power series in (1.3) is convergent when $\Delta_1=0,\ \Delta_2=0,\ |x|<\rho_1,\ |y|<\rho_2$ where

$$\rho_1=\min_{\mu_1,\mu_2>0}(E_1),\ \rho_2=\min_{\mu_1,\mu_2>0}(E_2).$$

Case III. The double power series in (1.3) would diverge except when, trivially, $x=y=0$ when $\Delta_1<0$ and $\Delta_2<0$.

Further analysis of Case II

When

$$\varphi_j=\vartheta_j,\ 1\le j\le A, \tag{1.14}$$

$$\varepsilon_j=\delta_j,\ 1\le j\le C, \tag{1.15}$$

$$G_1 = \frac{\prod_{j=1}^{C}(\delta_j)^{\delta_j}\ \prod_{j=1}^{D}(\eta_j)^{\eta_j}}{\prod_{j=1}^{A}(\vartheta_j)^{\vartheta_j}\ \prod_{j=1}^{B}(\psi_j)^{\psi_j}}, \tag{1.16}$$

$$G_2 = \frac{\prod_{j=1}^{C}(\delta_j)^{\delta_j}\ \prod_{j=1}^{D'}(\eta'_j)^{\eta'_j}}{\prod_{j=1}^{A}(\vartheta_j)^{\vartheta_j}\ \prod_{j=1}^{B'}(\psi'_j)^{\psi'_j}}, \tag{1.17}$$

$$\varDelta = 1 + \sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D}\eta_j - \sum_{j=1}^{A}\vartheta_j - \sum_{j=1}^{B}\psi_j, \tag{1.18}$$

$$\bigoplus = 1 + \sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D'}\eta'_j - \sum_{j=1}^{A}\vartheta_j - \sum_{j=1}^{B'}\psi'_j \tag{1.19}$$

and

$$\omega = \sum_{j=1}^{A}\vartheta_j - \sum_{j=1}^{C}\delta_j. \tag{1.20}$$

Case II(a). Then double power series in (1.3) will converge when $\varDelta = 0$; $\bigoplus = 0$; $\omega > 0$ and

$$\left(\frac{|x|}{G_1}\right)^{\frac{1}{\omega}} + \left(\frac{|y|}{G_2}\right)^{\frac{1}{\omega}} < 1, \tag{1.21}$$

Case II(b). Then double power series in (1.3) will converge when $\varDelta = 0$; $\bigoplus = 0$; $\omega \leq 0$ and

$$\max\left(\frac{|x|}{G_1}, \frac{|y|}{G_2}\right) < 1. \tag{1.22}$$

Series rearrangement technique is based upon certain interchanges of the order of a double (or multiple) summation. Several hypergeometric generating relations have been established using series rearrangement technique.

Here, we consider some well known results.

Cauchy's double series identity [16, p.100]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\Phi(m-n,n), \tag{1.23}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}\Phi(m,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+2n,n), \tag{1.24}$$

provided that the associated double series are absolutely convergent.
Reduction formula [11, p. 70, Q.No.10]

$$(1-z)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1} = {}_2F_1\left[\begin{array}{cc}\gamma,\gamma+\frac{1}{2}; & \\ & z\\ 2\gamma; & \end{array}\right] \tag{1.25}$$

and

$$\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1} = {}_2F_1\left[\begin{array}{cc}\gamma,\gamma-\frac{1}{2}; & \\ & z\\ 2\gamma; & \end{array}\right]. \tag{1.26}$$

Simple Bessel polynomials of Krall and Frink, [7, p.101, Eq.(3)]

$$y_n(x) = {}_2F_0\left[\begin{array}{cc}-n,n+1; & \\ & -\frac{x}{2}\\ - \quad ; & \end{array}\right] \tag{1.27}$$

and

$$y_{-n}(x)=y_{n-1}(x), y_0(x)=1; n\in\mathbb{N} \quad .$$

Krall and Frink [7, p.106, Eq.(25)]

$$\sum_{n=0}^{\infty}\frac{y_{n-1}(x)t^n}{n!}=\exp\left(\frac{1-\sqrt{(1-2xt)}}{x}\right). \tag{1.28}$$

Krall and Frink [7, p.106]

$$\sum_{n=0}^{\infty}y_n(x)t^n=(1-t)^{(-1)}\ {}_2F_0\left[\begin{array}{cc}1,\frac{1}{2}; & \\ & \frac{2xt}{(1-t)^2}\\ -; & \end{array}\right]. \tag{1.29}$$

Generalized Bessel polynomials of Krall and Frink [7, p.108, Eq.(34)]

$$y_n(x,a,b) = {}_2F_0\left[\begin{array}{cc}-n,a-1+n; & \\ & -\frac{x}{b}\\ - \quad ; & \end{array}\right], \tag{1.30}$$

when $a = b = 2$ in (1.30) we get simple Bessel polynomial

$$y_n(x, 2, 2) = y_n(x). \tag{1.31}$$

$$e^z = 1 + z \;\; {}_1F_1\left[\begin{array}{cc} 1; & \\ & z \\ 2; & \end{array}\right]. \tag{1.32}$$

Rainville polynomials [11, p.294, Eq.(3)]

$$\varphi_n(c, x) = \frac{(c)_n}{n!} \;\; {}_2F_0\left[\begin{array}{cc} -n, \; c+n; & \\ & x \\ - \quad ; & \end{array}\right], \tag{1.33}$$

When x is replaced by $-\frac{x}{2}$ and $c = 1$ in (1.33), we get

$$\varphi_n(1, -\frac{x}{2}) = y_n(x), \tag{1.34}$$

When x is replaced by $-\frac{x}{b}$ and $c = (a-1)$ in (1.33), we get

$$\varphi_n(a-1, -\frac{x}{b}) = \frac{(a-1)_n}{n!} y_n(x, a, b). \tag{1.35}$$

Linear generating function

Two functions $F(x, t)$ and $G(x, t)$ of two independent variables x and t are called generating functions of the sets $\{f_n(x)\}$ and $\{g_n(x)\}$ respectively, if it is possible to represent $F(x, t)$ and $G(x, t)$ in the following series expansions of t

$$F(x, t) = \sum_{n=0}^{\infty} b_n \; f_n(x) t^n \quad ; t \neq 0, \tag{1.36}$$

$$G(x, t) = \sum_{n=-\infty}^{+\infty} c_n \; g_n(x) t^n \quad ; t \neq 0, \tag{1.37}$$

where the coefficients b_n and c_n are independent of x and t and may contain some parameters related with $f_n(x)$, $g_n(x)$ respectively.

Motivated by the work of Burchnall [5], Krall-Frink [7], McBride [9],Erdélyi et al. [6, Ch.19, pp.245-278], Magnus et al.[8], Srivastava [12, 14, 16] and Agarwal [1, 2], we obtain a generating relation involving Srivastava-Daoust double hypergeometric functions.

The present article is organized as follows. In section 2, we obtain a generating relation. In section 3, we have given the proof of hypergeometric generating relation using series rearrangement technique and quadratic transformation.In section 4,we discuss some special cases.

2. Main Hypergeometric Generating Relations

When the values of parameters and arguments leading to the results which do not make sense are tacitly excluded. Then

$$F^{A+D+G:K;P}_{B+E+H:L;Q}\left(\begin{array}{c}[(a_A):2,2],[(d_D):1,2],[(g_G):1,1]\mathbf{:}[(k_K):1];[(p_P):1];\\ \\ [(b_B):2,2],[(e_E):1,2],[(h_H):1,1]\mathbf{:}[(\ell_L):1];[(q_Q):1];\end{array}\beta t, yt\right)$$

$$=\sum_{m=0}^{\infty}\frac{\prod_{i=1}^{A}(\frac{a_i}{2})_m\ \prod_{i=1}^{A}(\frac{1+a_i}{2})_m\ \prod_{i=1}^{D}(d_i)_m\ \ \prod_{i=1}^{G}(g_i)_m\ \ \prod_{i=1}^{K}(k_i)_m\beta^m\ 2^{(2A-2B)m}}{\prod_{i=1}^{B}(\frac{b_i}{2})_m\ \prod_{i=1}^{B}(\frac{1+b_i}{2})_m\ \prod_{i=1}^{E}(e_i)_m\prod_{i=1}^{H}(h_i)_m\prod_{i=1}^{L}(\ell_i)_m m!}\times$$

$$\times\ {}_{1+D+P+L}F_{E+Q+K}\left[\begin{array}{c}-m,\ ,(d_D)+m,\ (p_P),\ 1-(\ell_L)-m;\\ \\ (e_E)+m,\ (q_Q),\ 1-(k_K)-m;\end{array}(-1)^{K+L+1}\ \frac{y}{\beta}\right]t^m \quad (2.1)$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{A}(\frac{a_i}{2})_n\prod_{i=1}^{A}(\frac{1+a_i}{2})_n\ \prod_{i=1}^{D}(\frac{d_i}{2})_n\prod_{i=1}^{D}(\frac{1+d_i}{2})_n\ \prod_{i=1}^{G}(g_i)_n\prod_{i=1}^{P}(p_i)_n y^n 2^{(2A+2D+2B+2E)n}}{\prod_{i=1}^{B}(\frac{b_i}{2})_n\prod_{i=1}^{B}(\frac{1+b_i}{2})_n\ \prod_{i=1}^{E}(\frac{e_i}{2})_n\prod_{i=1}^{E}(\frac{1+e_i}{2})_n\ \prod_{i=1}^{H}(h_i)_n\prod_{i=1}^{Q}(q_i)_n n!}$$

$${}_{1+E+K+Q}F_{D+L+P}\left[\begin{array}{c}-n,1-(e_E)-2n,(k_k),1-(q_Q)-n;\\ \\ 1-(d_D)-2n,(\ell_L),1-(p_P)-n;\end{array}(-1)^{(D+P+E+Q+1)}\frac{\beta}{y}\right]t^n. \quad (2.2)$$

Convergence conditions

Suppose

$$\Delta_1 = 1+2B+E+H+L-2A-D-G-K, \quad (2.3)$$

$$\Delta_2 = 1+2B+2E+H+Q-2A-2D-G-P, \quad (2.4)$$

(i)When $\Delta_1 > 0,\ \Delta_2 > 0$, then double series in left hand side of equations (2.1) and (2.2) is convergent for all finite (real/complex) values of β, y and t,

(ii)When $\Delta_1 = \Delta_2 = 0$, then double series in left hand side of equations (2.1) and (2.2) is convergent for suitably constrained values of $|\beta t|, |yt|$,

provided that in each hypergeometric function, denominator parameters are neither zero nor negative integers.

3. Proof of Main Generating Relations

Let

$$\Phi = F^{A+D+G:K;P}_{B+E+H:L;Q}\left(\begin{array}{c} [(a_A):2,2],[(d_D):1,2],[(g_G):1,1]\text{:}[(k_K):1];[(p_P):1]; \\ \\ [(b_B):2,2],[(e_E):1,2],[(h_H):1,1]\text{:}[(\ell_L):1];[(q_Q):1]; \end{array} \beta t, yt\right),$$

Then

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_A)]_{2m+2n}[(d_D)]_{m+2n}[(g_G)]_{m+n}[(k_K)]_m[(p_P)]_n\beta^m y^n t^{m+n}}{[(b_B)]_{2m+2n}[(e_E)]_{m+2n}[(h_H)]_{m+n}[(\ell_L)]_m[(q_Q)]_n m!n!},$$

where $[(a_A)]_m = (a_1)_m(a_2)_m...(a_A)_m = \prod_{i=1}^{A}(a_i)_m$.

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{2m+2n}...(a_A)_{2m+2n}(d_1)_{m+2n}...(d_D)_{m+2n}(g_1)_{m+n}...(g_G)_{m+n}}{(b_1)_{2m+2n}...(b_B)_{2m+2n}(e_1)_{m+2n}...(e_E)_{m+2n}(h_1)_{m+n}...(h_H)_{m+n}}\times$$

$$\times\frac{(k_1)_m...(k_K)_m(p_1)_n...(p_P)_n\beta^m y^n t^{m+n}}{(\ell_1)_m...(\ell_L)_m(q_1)_n...(q_Q)_n m!n!}. \tag{3.1}$$

Replacing m by $m-n$ in equation (3.1) and applying double series identity(1.23), we get

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_1)_{2m}...(a_A)_{2m}(d_1)_{m+n}...(d_D)_{m+n}(g_1)_m...(g_G)_m(k_1)_{m-n}...(k_K)_{m-n}}{(b_1)_{2m}...(b_B)_{2m}(e_1)_{m+n}...(e_E)_{m+n}(h_1)_m...(h_H)_m(\ell_1)_{m-n}...(\ell_L)_{m-n}}\times$$

$$\times\frac{(p_1)_n...(p_P)_n(-m)_n\beta^{m-n}y^n t^m}{(q_1)_n...(q_Q)_n(-1)^n\ m!n!}$$

$$= \sum_{m=0}^{\infty}\frac{(a_1)_{2m}...(a_A)_{2m}(d_1)_m...(d_D)_m(g_1)_m...(g_G)_m(k_1)_m...(k_K)_m\beta^m}{(b_1)_{2m}...(b_B)_{2m}(e_1)_m...(e_E)_m(h_1)_m...(h_H)_m(\ell_1)_m...(\ell_L)_m m!}\times$$

$$\times\left(\sum_{n=0}^{m}\frac{(-m)_n(d_1+m)_n...(d_D+m)_n(k_1+m)_{-n}...(k_K+m)_{-n}(p_1)_n...(p_P)_n y^n}{(e_1+m)_n...(e_E+m)_n(\ell_1+m)_{-n}...(\ell_L+m)_{-n}(q_1)_n...(q_Q)_n(-1)^n n!\beta^n}\right)t^m$$

$$= \sum_{m=0}^{\infty}\frac{2^{2m}(\frac{a_1}{2})_m(\frac{1+a_1}{2})_m...2^{2m}(\frac{a_A}{2})_m(\frac{1+a_A}{2})_m(d_1)_m...(d_D)_m(g_1)_m...(g_G)_m(k_1)_m...(k_K)_m\beta^m}{2^{2m}(\frac{b_1}{2})_m(\frac{1+b_1}{2})_m...2^{2m}(\frac{b_B}{2})_m(\frac{1+b_B}{2})_m(e_1)_m...(e_E)_m(h_1)_m...(h_H)_m(\ell_1)_m...(\ell_L)_m m!}\times$$

$$\times\left(\sum_{n=0}^{m}\frac{(-m)_n(d_1+m)_n...(d_D+m)_n(-1)^n...(-1)^n(p_1)_n...(p_P)_n}{(e_1+m)_n...(e_E+m)_n(q_1)_n...(q_Q)_n}\times\right.$$

$$\left.\times\frac{(1-\ell_1-m)_n...(1-\ell_L-m)_n y^n}{(1-k_1-m)_n...(1-k_K-m)_n(-1)^n...(-1)^n(-1)^n\beta^n n!}\right)t^m$$

$$= \sum_{m=0}^{\infty}\frac{(\frac{a_1}{2})_m...(\frac{a_A}{2})_m(\frac{1+a_1}{2})_m...(\frac{1+a_A}{2})_m(d_1)_m...(d_D)_m(g_1)_m...(g_G)_m}{(\frac{b_1}{2})_m...(\frac{b_B}{2})_m(\frac{1+b_1}{2})_m...(\frac{1+b_B}{2})_m(e_1)_m...(e_E)_m(h_1)_m...(h_H)_m}\times$$

$$\times \frac{(k_1)_m...(k_K)_m \beta^m 2^{(2A-2B)m}}{(\ell_1)_m...(\ell_L)_m m!} \times$$

$$\times \left(\sum_{n=0}^{m} \frac{(-m)_n (d_1+m)_n...(d_D+m)_n (p_1)_n...(p_P)_n}{(e_1+m)_n...(e_E+m)_n (q_1)_n...(q_Q)_n} \times \right.$$

$$\left. \times \frac{(1-\ell_1-m)_n...(1-\ell_L-m)_n (-1)^{(K+L+1)n} y^n}{(1-k_1-m)_n...(1-k_K-m)_n \beta^n n!} \right) t^m.$$

Now using the definition(1.1) of generalized hypergeometric function of one variable, we get the main generating relation(2.1).
Similarly, when we replace n by $n-m$ in equation (3.1) and after simplification, we get right hand side of (2.2).

4. Special Cases

In generating relation (2.1)put $A = B = 0$, we get

$$F^{D+G:K;P}_{E+H:L;Q} \left(\begin{array}{c} [(d_D):1,2],[(g_G):1,1]\textbf{:}\ [(k_K):1]\ ;\ [(p_P):1]; \\ \\ [(e_E):1,2],[(h_H):1,1]\textbf{:}\ [(\ell_L):1]\ ;\ [(q_Q):1]; \end{array} \beta t, \quad yt \right)$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_m \ \prod_{i=1}^{G}(g_i)_m \ \ \prod_{i=1}^{K}(k_i)_m \beta^m}{\prod_{i=1}^{E}(e_i)_m \ \prod_{i=1}^{H}(h_i)_m \prod_{i=1}^{L}(\ell_i)_m m!} \ {}_{1+D+L+P}F_{E+K+Q} \left[\begin{array}{c} -m, (d_D)+m, \\ \\ (e_E)+m, \end{array} \right.$$

$$\left. \begin{array}{c} 1-(\ell_L)-m, \ \ (p_P); \\ \\ 1-(k_K)-m, \ \ (q_Q); \end{array} (-1)^{(K+L+1)} \frac{y}{\beta} \right] t^m. \tag{4.1}$$

In generating relation (4.1)put $\beta = \frac{b}{2}$, $y = \frac{x}{2}$, $D = H = 1, d_1 = (d-1), h_1 = (d-1), E = G = K = P = L = Q = 0$ and using Reduction formula (1.25), expansion of exponential function and definition of generalized Bessel's polynomials (1.30),after rationalization we get a correct form of generating relation of Burchnall [4, p.67, Eq.(26)]

$$\sum_{m=0}^{\infty} \frac{(\frac{b}{2})^m y_m(x,d,b) t^m}{m!} = (1-2xt)^{-\frac{1}{2}} \left(\frac{1+\sqrt{(1-2xt)}}{2} \right)^{2-d} \exp\left[\frac{b}{2x} \left\{ 1-\sqrt{(1-2xt)} \right\} \right]. \tag{4.2}$$

When $d = b = 2$ in generating relation(4.2) and using a result (1.31), we get

$$\sum_{m=0}^{\infty} \frac{y_m(x) t^m}{m!} = (1-2xt)^{-\frac{1}{2}} \exp\left(\frac{1-\sqrt{(1-2xt)}}{x} \right). \tag{4.3}$$

In generating relation (4.1)put $\beta = 1$ and $y = -x$, we get

$$\mathbf{F}^{D+G:K;P}_{E+H:L;Q} \left(\begin{array}{c} [(d_D):1,2],[(g_G):1,1]\textbf{:}\ [(k_K):1]\ ;\ [(p_P):1]; \\ \\ [(e_E):1,2],[(h_H):1,1]\textbf{:}\ [(\ell_L):1]\ ;\ [(q_Q):1]; \end{array} t, \quad -xt \right)$$

.

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_m \ \prod_{i=1}^{G}(g_i)_m \ \ \prod_{i=1}^{K}(k_i)_m}{\prod_{i=1}^{E}(e_i)_m \ \prod_{i=1}^{H}(h_i)_m \prod_{i=1}^{L}(\ell_i)_m m!} \ {}_{1+D+L+P}F_{E+K+Q}\left[\begin{array}{c} -m, (d_D)+m, \\ \\ (e_E)+m, \end{array}\begin{array}{c} 1-(\ell_L)-m, \ \ (p_P); \\ \\ 1-(k_K)-m, (q_Q); \end{array} (-1)^{(K+L)}x\right] t^m. \tag{4.4}$$

In generating relation (4.4), put $D = H = 1, d_1 = h_1 = c$, $G = E = P = Q = 0$ and when $K \leq L$, we get a known result of Rainville [10, p.104,Eq.(3 and 4)],

$$(1+4xt)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{(1+4xt)}}\right)^{c-1} \ {}_KF_L\left[\begin{array}{c} (k_K); \\ \\ (\ell_L); \end{array} \frac{2t}{(1+\sqrt{(1+4xt)}}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(c+n)_n(-x)^n}{n!} \ {}_{K+1}F_{L+1}\left[\begin{array}{c} -n, (k_K); \\ \\ 1-c-2n, (\ell_L); \end{array} -\tfrac{1}{x}\right] t^n \tag{4.5}$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{K}(k_i)_m}{\prod_{i=1}^{L}(\ell_i)_m m!} \ {}_{2+L}F_K\left[\begin{array}{c} -m, c+m, 1-(\ell_L)-m; \\ \\ 1-(k_K)-m; \end{array} (-1)^{K+L}x\right] t^m. \tag{4.6}$$

In generating relation (4.6),put $K = L = 0$ and applying the definition of Rainville's polynomial $\varphi_m(c,x)$(1.33), we get a known result of Rainville [11, p.296,Eq.(15)], see also [4].

$$(1+4xt)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{(1+4xt)}}\right)^{c-1} \exp\left(\frac{2t}{1+\sqrt{(1+4xt)}}\right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{(c)_m} \varphi_m(c,x)t^m. \tag{4.7}$$

In generating relation (4.7) put $c = 1$, we get another known result of Krall and Frink [7].

5.Conclusion

In this paper, we obtained a generating relation by using reduction formula and series rearrangement technique. We conclude our work with these words that the result obtained above is remarkable that can lead to innumerable generating functions and generating relations. Besides, presented generating relation is supposed to find various applications in

Numerical Analysis, Statistics and Linear Programming. The given generating relation may be very functional to non-professionalists who are fascinated in Applied Mathematics and the Theory of Probability.

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