South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 1 (2020), pp. 15-36

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

GENERATORS IDEMPOTENT IN SEMI-SIMPLE RING FC_{16p^n}, FOR THE IDEALS CORRESPONDING TO THE MINIMAL CYCLIC CODES OF LENGTH $16p^n$ AND THE CODES

Vishvajit Singh, Manju Pruthi* and Jagbir Singh*

Department of Mathematics,
I.G. University, Meerpur, Rewari, INDIA

E-mail : vishvajit73.sheoran@gmail.com

*Department of Mathematics,
Maharshi Dayanand University, Rohtak, INDIA

E-mail : manju.pruthi@yahoo.com, ahlawatjagbir@gmail.com

(**Received:** *Jul. 19, 2019* **Accepted:** *Jan. 14, 2020* **Published:** *Apr. 30, 2020)*

Abstract: In semi-simple ring $R_{16p^n} \equiv \frac{GF(q)[x]}{<x^{16p^n}-1>}$, where p is prime and q is some prime power (of type $16k+1$), n is a positive integer, subject to order of q modulo p^n is $\frac{\phi(p^n)}{2}$, expression for primitive idempotents are obtained. Generating polynomials, dimensions and minimum distance bounds for the cyclic codes generated by these idempotents are also calculated.

Keywords and Phrases: Cyclotomic cosets, primitive idempotents, generating polynomials, minimum distance.

2010 Mathematics Subject Classification: 11T71, 11T55, 22D20.

1. Introduction

The group algebra FC_{16p^n}, F is field of order q and C_{16p^n} is cyclic group of order $16p^n$ such that $g.c.d.(q, 16p) = 1$, is semi-simple having finite cardinality of collection of primitive idempotents which equals the cardinality of collection of q-cyclotomic cosets modulo $16p^n$ [11]. The primitive idempotents of minimal cyclic codes of length m in case, when order of q modulo m is $\phi(m)$ for $m = 2, 4, p^n, 2p^n$ were computed in [6, 9]. The primitive idempotents of length p^n with order of q

modulo p^n is $\frac{\phi(p^n)}{2}$ were obtained in [10] and minimal quadratic residue codes of length p^n in [7]. Cyclic codes of length $2p^n$ over F, where order of q modulo $2p^n$ is $\frac{\phi(2p^n)}{2}$ were discussed in [8]. Minimal cyclic codes of length $p^n q$, where p and q are distinct odd primes were derived in [1, 3]. Further, when order of q modulo p^n is $\phi(p^n)$, the minimal cyclic codes of length $8p^n$ were discussed in [4, 5]. Irreducible cyclic codes of length $4p^n$ and $8p^n$, where $q \equiv 3(mod\ 8)$ and $p/q-1$ were obtained in [2].

In present paper, we obtained cyclic codes of length $16p^n$ over F where q is some prime power of the form $16k+1$ and order of q modulo p^n is $\frac{\phi(p^n)}{2}$. We considered the case for p being a prime of the form $8k+1$. However in other cases whenever p is a prime of the form $8k+3$, $8k+5$ and $8k+7$ the expression for the idempotents can be obtained by using some permutation on the set A. The q-cyclotomic cosets modulo p^n are obtained in Section 2 and termed those as Ω_{tp^n} for $0 \leq t \leq 15$ and Ω_{ap^i} for $a \in A = \{1, 2, 4, 8, 16, \lambda, 2\lambda, 4\lambda, \mu, 2\mu, \nu, 2\nu, \eta, \xi, \rho, \chi\}$. Primitive idempotents corresponding to Ω_{tp^n} for $0 \leq t \leq 15$ are obtained in Section 3. In Section 4, we derived the expression of primitive idempotents corresponding to Ω_{tp^i} fot $t = 8$, 16, $8g$, $16g$ and those for $t = 2, 4, 2\lambda, 4\lambda, 2\mu, 2\nu, 2g, 4g, 2\lambda g, 4\lambda g, 2\mu g, 2\nu g$ in Section 5. The remaining expressions are obtained in Section 6. Section 7, consists of generating polynomials and dimensions for the corresponding cyclic codes of length $16p^n$. The minimum distance or the bounds for minimum distance of these codes are obtained in Section 8. At the end, an example is discussed to illustrate the various parameters for these codes in Section 9.

2. Cyclotomic Cosets

Let $S = \{1, 2, ..., 16p^n\}$. For $a, b \in S$, consider $a \sim b$ iff $a \equiv bq^i(mod\ 16p^n)$ for some integer $i \geq 0$. This is an equivalence relation on S. The equivalence classes due to this relation are called q-cyclotomic cosets modulo $16p^n$. The q-cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, ..., sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s(mod\ 16p^n)$.

Lemma 2.1. [8] *If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then the order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $0 \leq i \leq n-1$.*

Lemma 2.2. *If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then for $0 \leq i \leq n-1$, order of q modulo $2p^{n-i}$, $4p^{n-i}$, $8p^{n-i}$ and $16p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$.*

Proof. Since $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, therefore by Lemma 2.1, order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $1 \leq i \leq n-1$. Hence

$$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ p^{n-i}) \tag{2.1}$$

Since q is of the form $16k+1$, therefore $q \equiv 1(mod\ 2)$. Hence, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2)$. As $gcd(2, p^{n-i}) = 1$ and order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, so $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2p^{n-i})$. This implies that $\frac{\phi(p^{n-i})}{2}$ is the smallest integer for which (2.1) holds. Hence order of q modulo $2p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$. Similar, result holds for $4p^{n-i}$, $8p^{n-i}$ and $16p^{n-i}$.

Lemma 2.3. *For* $0 \le i \le n-1$ *and* $0 \le k \le \frac{\phi(p^{n-i})}{2} - 1$, $t \not\equiv q^k(mod\ 16p^{n-i})$, *where* $t = \lambda,\ 2\lambda,\ 4\lambda,\ \mu\ ,\ 2\mu\ ,\nu,\ 2\nu,\ \eta,\ \xi,\ \rho,\ \chi$ *and* $\lambda = 1+2p^n\ ,\ \mu = 1+4p^n$, $\nu = 1+6p^n$, $\eta = 1+8p^n$, $\xi = 1+10p^n$, $\rho = 1+12p^n$, $\chi = 1+14p^n$.
Proof. Proof can be obtained by using Lemma 2.1 and Lemma 2.2.

Lemma 2.4. *Let* p *be an odd prime then there exist an integer* g, $1 < g < 16p$ *and is primitive root modulo* p, *further when* p *is of the form* $4k+1$ *then order of* g *modulo* 4, *modulo* 8 *is* 2 *and modulo* 16 *is* 4 *and when* p *is of the form* $4k+3$ *then order of* g *modulo* 4, *modulo* 8 *and modulo* 16 *is* 2. *Also, if* q *is any prime or prime power and* $g.c.d.(p,q) = 1$, *then* $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$.
Proof. Consider the complete residue system, $S_p = \{0,1,2,3,...,p-1\}$ modulo p, $S_2 = \{0,1\}$ modulo 2 and $S_{2p} = \{0,1,2,3,...,2p-1\}$, modulo $2p$. Since $gcd(2,p) = 1$. So there exist an integer $v \in S_p$ such that $2v - p = 1$. Let a be any primitive root modulo p in S_p. For $p \equiv 1(mod8)$, let $g \equiv \{2av + tp + 6ap\}(mod16p)$ where t is a prime of the form $16k_1 + 3$ implies $g \equiv \{2av + tp + 6ap\}(modp) \equiv 2av(modp) \equiv a(1+p)(modp) \equiv a(modp)$. Hence g is primitive root modulo p. Now $g \equiv \{2av + tp + 6ap\}(mod16) \equiv \{a(1+p) + tp + 6ap\}(mod16) \equiv \{a(1+8k+1) + (16k_1+3)(8k+1) + 6a(8k+1)\}(mod16)$ as p is of the form $8k+1$ and t is of the form $16k_1+3$. Clearly, $g \equiv 3(mod2) \equiv 1(mod2)$. Hence order of g modulo 2 is 1, again $g \equiv \{a(1+p) + tp + 6ap\}(mod4) \equiv \{a(1+8k+1) + (16k_1+3)(8k+1) + 6a(8k+1)\}(mod4) \equiv \{8(ak+a+3k)+3\}(mod4) \equiv 3(mod4)$ so $g^2 \equiv 1(mod4)$. Hence order of g modulo 4 is 2. Similarly, $g^2 \equiv 1(mod8)$ and $g^4 \equiv 1(mod16)$. So order of g modulo 8 is 2 and modulo 16 is 4. Now when $p \equiv 3(mod8)$, let $g \equiv \{2av + tp + 4ap\}(mod16p)$, where t is a prime of the form $16k_1+5$, implies g is primitive root modulo p and order of g modulo 4, modulo 8 and modulo 16 is 2. Again when $p \equiv 5(mod8)$, let $g \equiv \{4av + tp + 4ap\}(mod16p)$, where t is a prime of the form $16k_1+7$, implies g is primitive root modulo p and order of g modulo 4, modulo 8 is 2 and modulo 16 is 4. Again $p \equiv 7(mod8)$, let $g \equiv \{4av + tp + 8ap\}(mod16p)$, where t is a prime of the form $16k_1+1$, implies g is primitive root modulo p and order of g modulo 4, modulo 8 and modulo 16 is 2. Let $g \in \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$, so $g = q^i$ for some $1 \le i \le \frac{\phi(p)}{2} - 1$ equivalently $o(g) = o(q^i)$ as order of g modulo $16p$ is $\frac{\phi(p)}{2}$, so $o(q^i) \le \frac{\phi(p)}{2}$ modulo $16p$.

This implies $o(g) \leq \frac{\phi(p)}{2}$ modulo $16p$, but order of g modulo $16p$ is $\phi(p)$. Hence $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$.

Lemma 2.5. *If $p \equiv 1(mod8)$ for given p, there exist a fixed integer g satisfying $gcd(g, 2pq) = 1$, $1 < g < 16p$, $g \not\equiv q^k(mod16p)$ where $0 \leq k \leq \frac{\phi(p)}{2} - 1$ such that for $0 \leq j \leq n-1$, the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo p^{n-j} and the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}$, $g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\lambda, \lambda q, \lambda q^2, ..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}$, $\lambda g, \lambda gq, \lambda gq^2, ..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}$, μ, $\mu q, ..., \mu q^{\frac{\phi(p^{n-j})}{2}-1}$, $\mu g, \mu gq, \mu gq^2, ..., \mu gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\nu, \nu q, ..., \nu q^{\frac{\phi(p^{n-j})}{2}-1}$, $\nu g, \nu gq$, $\nu gq^2, ..., \nu gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\eta, \eta q, ..., \eta q^{\frac{\phi(p^{n-j})}{2}-1}$, $\eta g, \eta gq, ..., \eta gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\xi, \xi q, \xi q^2, ...$, $\xi q^{\frac{\phi(p^{n-j})}{2}-1}$, $\xi g, \xi gq, \xi gq^2, ..., \xi gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\rho, \rho q, \rho q^2, ..., \rho q^{\frac{\phi(p^{n-j})}{2}-1}$, $\rho g, \rho gq, \rho gq^2$, $..., \rho gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\chi, \chi q, \chi q^2, ..., \chi q^{\frac{\phi(p^{n-j})}{2}-1}$, $\chi g, \chi gq, \chi gq^2, ..., \chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ from a reduced residue system modulo $16p^{n-j}$.*

Proof. By Lemma 2.1, order of q modulo p is $\frac{\phi(p)}{2}$. Therefore $\{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$ are incongruent modulo p. But there are exactly $\phi(p)$ numbers in this reduced residue system modulo p. Therefore there exist a number g satisfying $gcd(g, 2pq) = 1$, $1 < g < 16p$, $g \not\equiv q^k(modp)$ for $0 \leq k \leq \frac{\phi(p)}{2} - 1$. Then the set $\{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}$, $g, gq, gq^2, ..., gq^{\frac{\phi(p)}{2}-1}\}$, forms a reduced residue system modulo p. Since for $0 \leq k \leq \frac{\phi(p)}{2} - 1, g \not\equiv q^k(modp)$, it follows that $g \not\equiv q^k(modp^{n-j})$, for $0 \leq k \leq \frac{\phi(p^{n-j})}{2} - 1$. Hence the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo p^{n-j}, $0 \leq j \leq n-1$. Similarly result holds to show that the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}, \lambda, \lambda q, \lambda q^2, ..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}, \lambda g, \lambda gq$, $\lambda gq^2, ..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}, \mu, \mu q, \mu q^2, ..., \mu q^{\frac{\phi(p^{n-j})}{2}-1}, \mu g, \mu gq, ..., \mu gq^{\frac{\phi(p^{n-j})}{2}-1}, \nu, \nu q, \nu q^2$, $..., \nu q^{\frac{\phi(p^{n-j})}{2}-1}$, $\nu g, \nu gq, \nu gq^2, ..., \nu gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\eta, \eta q, \eta q^2, ..., \eta q^{\frac{\phi(p^{n-j})}{2}-1}$, $\eta g, \eta gq$, $\eta gq^2, ..., \eta gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\xi, \xi q, ..., \xi q^{\frac{\phi(p^{n-j})}{2}-1}$, $\xi g, \xi gq, \xi gq^2, ..., \xi gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\rho, \rho q, ...$, $\rho q^{\frac{\phi(p^{n-j})}{2}-1}$, $\rho g, \rho gq, \rho gq^2, ..., \rho gq^{\frac{\phi(p^{n-j})}{2}-1}$, $\chi, \chi q, \chi q^2, ..., \chi q^{\frac{\phi(p^{n-j})}{2}-1}$, $\chi g, \chi gq, \chi gq^2$, $..., \chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $16p^{n-j}$.

Theorem 2.6. *There are $(32n + 16)$ q-cyclotomic cosets modulo $16p^n$ given by $\Omega_{ap^n} = \{ap^n\}$ for $0 \leq a \leq 15$ and for $0 \leq i \leq n-1$ $\Omega_{tp^i} = \{tp^i, tp^iq, ..., tp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$, $\Omega_{tgp^i} = \{tgp^i, tgp^iq, tgp^iq^2, ..., tgp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$, where $t = 1, 2, 4, 8, 16$ defined in Lemma 2.3 and g is as defined in Lemma 2.4.*

Proof. By definition, it is obvious that $\Omega_0 = \{0\}$. Since q is of the form $16k + 1$, so $q \equiv 1(mod16)$, so $ap^nq \equiv ap^n(mod16p^n)$ and hence $\Omega_{ap^n} = \{ap^n\}$

By Lemma 2.2, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod 16p^{n-i})$ equivalently, $tp^i q^{\frac{\phi(p^{n-i})}{2}} \equiv tp^i(mod 16p^n)$, so $\Omega_{tp^i} = \{tp^i, tp^iq, tp^iq^2, ..., tp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$. Similarly, $\Omega_{tgp^i} = \{tgp^i, tgp^iq, tgp^iq^2, ..., tgp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$. Obviously, $|\Omega_{ap^n}| = 1$ for every a and $|\Omega_{tp^i}| = |\Omega_{tgp^i}| = \frac{\phi(p^{n-i})}{2}$, so $\sum_{i=0}^{n-1}|\Omega_{tp^i}| = \sum_{i=0}^{n-1}|\Omega_{tgp^i}| = \frac{\phi(p^n)}{2} + \frac{\phi(p^{n-1})}{2} + \frac{\phi(p^{n-2})}{2} + ... + \frac{\phi(p)}{2} = \frac{p^n-p^{n-1}}{2} + \frac{p^{n-1}-p^{n-2}}{2} + \frac{p^{n-2}-p^{n-3}}{2} + ... + \frac{p-1}{2} = \frac{p^n-1}{2}$. Hence $\sum_{a=0}^{15}|\Omega_{ap^n}| + \sum_{t\in A}[\sum_{i=0}^{n-1}\{|\Omega_{tp^i}| + |\Omega_{tgp^i}|\}] = 16 + \frac{32(p^n-1)}{2} = 16 + 16(p^n-1) = 16p^n$ it follows that these are the only distinct q-cyclotomic cosets modulo $16p^n$ in this case.

3. Primitive Idempotents Corresponding to Ω_{tp^n}, $0 \le t \le 15$

Throughout this paper, we consider α to be $16p^nth$ root of unity in some extension field of F. Let M_s be the minimal ideal in $R_{16p^n} = \frac{F[x]}{<x^{16p^n}-1>} \equiv FC_{16p^n}$, generated by $\frac{(x^{16p^n}-1)}{m_s(x)}$, where $m_s(x)$ is the minimal polynomial for α^s, $s \in \Omega_s$. We denote $P_s(x)$, the primitive idempotent in R_{16p^n}, corresponding to the minimal ideal M_s, given by $P_s(x) = \frac{1}{16p^n}\sum_{t=0}^{16p^n-1}\epsilon_i^s x^s$ where $\epsilon_i^s = \sum_{s\in\Omega_s}\alpha^{-is}$ and $\overline{C}_s = \sum_{s\in\Omega_s}x^s$. Then,

$$P_s(x) = \frac{1}{16p^n}[\sum_{t=0}^{15}\epsilon_{tp^n}^s\overline{C}_{tp^n} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\epsilon_{ap^i}^s\overline{C}_{ap^i} + \sum_{a\in A}\epsilon_{agp^i}^s\overline{C}_{agp^i}\}][4] \tag{3.1}$$

Lemma 3.1. *For cyclotomic cosets Ω_{tp^n}, $1 \le t \le 15$, (i) $\Omega_{tp^n} = -\Omega_{(16-t)p^n}$.*

Proof. Since $\Omega_{tp^n} = \{tp^n\}$, therefore $\{(16-t)p^nq + tp^n\} = p^n\{(16-t)q + t\} = p^n\{(16-t)(16k+1) + t\} \equiv 0(mod 16p^n)$. Hence the result holds.

Theorem 3.2. *The explicit expression for the primitive idempotents P_{tp^n}, $0 \le t, w \le 15$ in R_{16p^n} are given by*

$$P_{wp^n}(x) = \frac{1}{16p^n}[\{\sum_{t=0}^{15}\alpha^{(16-w)tp^{2n}}\overline{C}_{tp^n}\} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\alpha^{(16-w)ap^{n+i}}\overline{C}_{ap^i} + \sum_{a\in A}\alpha^{(16-w)agp^{n+i}}\overline{C}_{agp^i}\}].$$

Proof. By definition, we have $\epsilon_i^s = \sum_{s\epsilon\Omega_s}\alpha^{-is}$ and $\overline{C}_s = \sum_{s\epsilon\Omega_s}x^s$ where α is $16p^n$th root of unity. For $s = 0$, since $\epsilon_k^0 = 1$ for all $0 \le k \le 16p^n - 1$, so $P_0(x) = \frac{1}{16p^n}[\sum_{t=0}^{15}\overline{C}_{tp^n} + \sum_{i=0}^{n-1}\{\sum_{a\in A}\overline{C}_{ap^i} + \sum_{a\in A}\overline{C}_{agp^i}\}]$.

Due to Lemma 3.1, $\Omega_{p^n} = -\Omega_{15p^n}$ and so $\epsilon_k^{p^n} = \sum_{s\epsilon\Omega_{p^n}} \alpha^{-ks} = \alpha^{-p^n k} = \alpha^{15p^n k}$

$\epsilon_0^{p^n} = -\epsilon_{8p^n}^{p^n} = 1,$ $\epsilon_{p^n}^{p^n} = -\epsilon_{9p^n}^{p^n} = -\alpha^{7p^{2n}},$ $\epsilon_{2p^n}^{p^n} = -\epsilon_{10p^n}^{p^n} = -\alpha^{6p^{2n}},$

$\epsilon_{3p^n}^{p^n} = -\epsilon_{11p^n}^{p^n} = -\alpha^{5p^{2n}},$ $\epsilon_{4p^n}^{p^n} = -\epsilon_{12p^n}^{p^n} = -\alpha^{4p^{2n}},$ $\epsilon_{5p^n}^{p^n} = -\epsilon_{13p^n}^{p^n} = -\alpha^{3p^{2n}},$
$\epsilon_{6p^n}^{p^n} = -\epsilon_{14p^n}^{p^n} = -\alpha^{2p^{2n}},$ $\epsilon_{7p^n}^{p^n} = -\epsilon_{15p^n}^{p^n} = -\alpha^{p^{2n}},$ $\epsilon_{p^i}^{p^n} = -\epsilon_{\eta p^i}^{p^n} = -\alpha^{7p^{n+i}},$
$\epsilon_{2p^i}^{p^n} = -\epsilon_{2\mu p^i}^{p^n} = -\alpha^{6p^{n+i}},$ $\epsilon_{4p^i}^{p^n} = -\epsilon_{4\lambda p^i}^{p^n} = -\alpha^{4p^{n+i}},$ $\epsilon_{8p^i}^{p^n} = -\epsilon_{16p^i}^{p^n} = -1,$
$\epsilon_{\lambda p^i}^{p^n} = -\alpha^{7\lambda p^{n+i}},$ $\epsilon_{2\lambda p^i}^{p^n} = -\alpha^{6\lambda p^{n+i}},$ $\epsilon_{\mu p^i}^{p^n} = -\alpha^{7\mu p^{n+i}},$
$\epsilon_{\nu p^i}^{p^n} = -\alpha^{7\nu p^{n+i}},$ $\epsilon_{2\nu p^i}^{p^n} = -\alpha^{6\nu p^{n+i}},$ $\epsilon_{\xi p^i}^{p^n} = -\alpha^{7\xi p^{n+i}},$
$\epsilon_{\rho p^i}^{p^n} = -\alpha^{7\rho p^{n+i}},$ $\epsilon_{\chi p^i}^{p^n} = -\alpha^{7\chi p^{n+i}},$ $\epsilon_{gp^i}^{p^n} = -\epsilon_{\eta gp^i}^{p^n} = -\alpha^{7gp^{n+i}},$
$\epsilon_{2gp^i}^{p^n} = -\epsilon_{2\mu gp^i}^{p^n} = -\alpha^{6gp^{n+i}},$ $\epsilon_{4gp^i}^{p^n} = -\epsilon_{4\lambda gp^i}^{p^n} = -\alpha^{4gp^{n+i}},$ $\epsilon_{8gp^i}^{p^n} = -\epsilon_{16gp^i}^{p^n} = -1,$
$\epsilon_{\lambda gp^i}^{p^n} = -\alpha^{7\lambda gp^{n+i}},$ $\epsilon_{2\lambda gp^i}^{p^n} = -\alpha^{6\lambda gp^{n+i}},$ $\epsilon_{\mu gp^i}^{p^n} = -\alpha^{7\mu gp^{n+i}},$
$\epsilon_{\nu gp^i}^{p^n} = -\alpha^{7\nu gp^{n+i}},$ $\epsilon_{2\nu gp^i}^{p^n} = -\alpha^{6\nu gp^{n+i}},$ $\epsilon_{\xi gp^i}^{p^n} = -\alpha^{7\xi gp^{n+i}},$
$\epsilon_{\rho gp^i}^{p^n} = -\alpha^{7\rho gp^{n+i}},$ $\epsilon_{\chi gp^i}^{p^n} = -\alpha^{7\chi gp^{n+i}}.$

Using these in (3.1) the expression for $P_{p^n}(x)$ can be obtained.
Similarly, using Lemma 3.1, $P_{wp^n}(x)$, $2 \le w \le 15$ can be obtained.

4. Primitive Idempotents Corresponding to P_{tp^i}, $t = 8, 16, 8g, 16g$

For $0 \le j \le n-1$, we define: $H_j = p^j \sum_{s\epsilon\Omega_{8gp^j}} \alpha^s$; $I_j = p^j \sum_{s\epsilon\Omega_{16gp^j}} \alpha^s$; $Q_j = p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s$; $R_j = p^j \sum_{s\epsilon\Omega_{16p^j}} \alpha^s$. Since $Q_j^q = (p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s)^q = (p^j)^q \, (\sum_{s\epsilon\Omega_{8p^j}} \alpha^s)^q = (p^j)^q \sum_{s\epsilon\Omega_{8p^j}} \alpha^{qs} = (p^j)^q \sum_{s\epsilon q\Omega_{8p^j}} \alpha^s$. However, $(p^j)^q = p^j$, therefore $Q_j^q = p^j \sum_{s\epsilon q\Omega_{8p^j}} \alpha^s$. Moreover, Ω_{8p^j} is a cyclotomic coset, therefore $q\Omega_{8p^j} = \Omega_{8p^j}$. Hence $Q_j^q = p^j \sum_{s\epsilon\Omega_{8p^j}} \alpha^s = Q_j$ and so $Q_j \in GF(q)$. Similarly $H_j, I_j, R_j \in GF(q)$.

Lemma 4.1. $0 \le j \le n-1$, $I_j + R_j = \begin{cases} -p^{n-1} \; \textit{if } j = n-1 \\ 0 \; \textit{otherwise.} \end{cases}$

Proof. By definition $I_j + R_j = p^j \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} (\delta^{gq^t} + \delta^{q^t})$ where $\delta = \alpha^{16p^j}$.

As the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ is a reduced residue system modulo (p^{n-j}) therefore

$$I_j + R_j = p^j[\sum_{t=0}^{p^{n-j}} \delta^t - \sum_{t=1,p/t}^{p^{n-j}} \delta^t] = p^j[\sum_{t=0}^{p^{n-j}} \delta^t - \sum_{t=1}^{p^{n-j-1}} \delta^{pt}] = \begin{cases} -p^{n-1} \text{ if } j = n-1 \\ 0 \textit{ otherwise.} \end{cases}$$

Lemma 4.2. $0 \le j \le n-1$, $H_j + Q_j = \begin{cases} p^{n-1} \textit{ if } j = n-1 \\ 0 \textit{ otherwise.} \end{cases}$

Proof. This result can be obtained in similar pattern that Lemma 4.1 and using the fact that $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ is a reduced residue system modulo $(2p^{n-j})$.

Lemma 4.3. *For cyclotomic cosets* $\Omega_{p^i}, 0 \le i \le n-1$
(i) $\eta^2\Omega_{p^i} = \eta\Omega_{\eta p^i} = \Omega_{p^i} = \nu^2\Omega_{p^i} = \chi^2\Omega_{p^i}$
(ii) $\mu^2\Omega_{p^i} = \Omega_{\eta p^i} = \eta\Omega_{p^i} = \rho^2\Omega_{p^i} = \lambda^2\Omega_{p^i} = \xi^2\Omega_{p^i}$
(iii) $(2\lambda)^2\Omega_{p^i} = 2\Omega_{2p^i} = 4\Omega_{p^i} = \Omega_{4p^i} = (2\mu)^2\Omega_{p^i} = (2\nu)^2\Omega_{p^i}$
(iv) $(4\lambda)^2\Omega_{p^i} = \Omega_{16p^i} = 2\Omega_{8p^i} = 4\Omega_{4p^i} = 16\Omega_{p^i}$.

Proof. (i) Since $\eta = 1 + 8p^n$. So $\eta^2 = (1 + 8p^n)^2 = 1 + 64p^{2n} + 16p^n \equiv 1(mod 16p^n)$. Hence $\eta^2 \equiv 1(mod 16p^n) \Rightarrow \eta^2\Omega_p i = \Omega p^i$. Now $\eta^2\Omega_p i = \{\eta^2 p^i,$ $\eta^2 p^i q,\ \eta^2 p^i q^2, ...,\ \eta^2 p^i q^{\frac{\phi(p^{n-i})}{2}-1}\} = \eta\{\eta p^i,\ \eta p^i q,\ \eta p^i q^2,\ ...,\ \eta p^i q^{\frac{\phi(p^{n-i})}{2}-1}\} = \eta\Omega_{\eta p^i}$. Now $\nu^2 = (1 + 6p^n)^2 = 1 + 36p^{2n} + 12p^n \equiv 1(mod 16p^n)$ and $\chi^2 = (1 + 14p^n)^2 = 1 + 196p^{2n} + 28p^n = 1(mod 16p^n)$. Hence $\nu^2\Omega_{p^i} = \chi^2\Omega_{p^i} = \Omega_{p^i}$. Similar result holds for remaining.

Lemma 4.4. *For* $1 \le i \le n-1$,
(i) $\Omega_{(1+tp^n)p^i} = -\Omega_{\{1+(14-t)p^n\}p^i}$ *and hence* $\Omega_{(1+tp^n)gp^i} = -\Omega_{\{1+(14-t)p^n\}gp^i}$, *for* $t = 0, 2, 4, 6$
(ii) $\Omega_{(2+tp^n)p^i} = -\Omega_{\{2+(12-t)p^n\}p^i}$ *and hence* $\Omega_{(2+tp^n)gp^i} = -\Omega_{\{2+(12-t)p^n\}gp^i}$, *for* $t = 0, 4$
(iii) $\Omega_{4p^i} = -\Omega_{4\lambda p^i}$ *and hence* $\Omega_{4gp^i} = -\Omega_{4\lambda gp^i}$,
(iv) $\Omega_{tp^i} = -\Omega_{tp^i}$ *and hence* $\Omega_{tgp^i} = -\Omega_{tgp^i}$, *for* $t = 8, 16$.

Proof. Since $\chi = 1 + 14p^n \equiv -1(mod 16)$ and $q^{\frac{\phi(p^n)}{2}} \equiv 1(mod 16)$. Further, $q^{\frac{\phi(p^n)}{4}} \equiv 1(mod 16)$, as $q \equiv 1(mod 16)$, therefore $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod 16)$. Also $q^{\frac{\phi(p^n)}{2}} \equiv 1(mod p^n)$ so $q^{\frac{\phi(p^n)}{4}} \equiv -1(mod p^n)$ and $\chi \equiv 1(mod p^n)$, thus $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod p^n)$. However $(16, p^n) = 1$ thus $\chi q^{\frac{\phi(p^n)}{4}} \equiv -1(mod 16p^n)$ and therefore $-\Omega_{\chi p^i} = \Omega_{p^i}$. Hence $-\Omega_{\chi gp^i} = \Omega_{gp^i}$.
Proof of remaining can be obtained using relations of congruences and similar reasoning as for the relation obtained.

Lemma 4.5. *For* $0 \le i \le n$; $0 \le j \le n-1$,

$$\sum_{s\epsilon\Omega_{p^j}}\alpha^{8gp^is}=\sum_{s\epsilon\Omega_{2p^j}}\alpha^{4gp^is}=\sum_{s\epsilon\Omega_{2p^j}}\alpha^{4\lambda gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{2\lambda gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{2\mu gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{2\nu gp^is}$$
$$=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\lambda gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\mu gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\nu gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\eta gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\xi gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\rho gp^is}$$
$$=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{\chi gp^is}=\sum_{s\epsilon\Omega_{2\lambda p^j}}\alpha^{4\lambda gp^is}$$
$$=\sum_{s\epsilon\Omega_{4\lambda p^j}}\alpha^{2\mu gp^is}=\sum_{s\epsilon\Omega_{4\lambda p^j}}\alpha^{2\nu gp^is}=\begin{cases}-\frac{\phi(p^{n-j})}{2}, & if\ \ i+j\geq n,\\ \frac{1}{p^j}H_{i+j}, & if\ \ i+j\leq n-1,\quad g\neq 1,\\ \frac{1}{p^j}Q_{i+j}, & if\ \ i+j\leq n-1,\quad g=1.\end{cases}$$

Proof. As α is $16p^n$th root of unity in some extension field of $GF(q)$, so

$$\sum_{s\epsilon\Omega_{4\lambda p^j}}\alpha^{2\nu gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\alpha^{8(1+2p^n)(1+6p^n)gp^{i+j}q^t}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\alpha^{8gp^{i+j}q^t}=\sum_{s\epsilon\Omega_{p^j}}\alpha^{8gp^is}$$

If $\beta=\alpha^{8gp^{i+j}}$ then $\sum_{s\epsilon\Omega_{p^j}}\alpha^{8gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}(\alpha^{8gp^{i+j}q^t})=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}(\beta^{q^t})$.

For $i+j\geq n$, β is 16th root of unity, and so $\sum_{s\epsilon\Omega_{p^j}}\alpha^{8gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\alpha^{8gp^{i+j}q^t}=-\frac{\phi(p^{n-j})}{2}$. For $i+j\leq n-1$, β is $16p^{n-i-j}$th root of unity. Then $\beta^{q^l}=\beta^{q^r}$ which is possible when $l\equiv r(mod\frac{\phi(p^{n-i-j})}{2})$, due to Lemma 2.2. So $\sum_{s\epsilon\Omega_{p^j}}\alpha^{8gp^is}=$

$$\frac{\phi(p^{n-j})}{\phi(p^{n-i-j})}\sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1}\beta^{q^t}=\frac{p^{i+j}}{p^j}\sum_{s\epsilon\Omega_{8gp^{i+j}}}\alpha^s=\frac{1}{p^j}H_{i+j}.$$ Similar result hold for Q_{i+j}.

Proof of the Lemma 4.6 can be similarly obtained using definition for I_j and R_j.

Lemma 4.6. *For $0\leq i\leq n$; $0\leq j\leq n-1$*

$$\sum_{s\epsilon\Omega_{p^j}}\alpha^{16gp^is}=\sum_{s\epsilon\Omega_{2p^j}}\alpha^{8gp^is}=\sum_{s\epsilon\Omega_{2p^j}}\alpha^{16gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{4gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{8gp^is}=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{16gp^is}$$
$$=\sum_{s\epsilon\Omega_{4p^j}}\alpha^{4\lambda gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{16gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{2\lambda gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{4\lambda gp^is}=\sum_{s\epsilon\Omega_{8p^j}}\alpha^{2\mu gp^is}=$$
$$\sum_{s\epsilon\Omega_{8p^j}}\alpha^{2\nu gp^is}=\sum_{s\epsilon\Omega_{16p^j}}\alpha^{16gp^is}=\sum_{s\epsilon\Omega_{16p^j}}\alpha^{\lambda gp^is}=\sum_{s\epsilon\Omega_{16p^j}}\alpha^{2\lambda gp^is}=\sum_{s\epsilon\Omega_{16p^j}}\alpha^{4\lambda gp^is}$$

$$= \sum_{s\epsilon\Omega_{16p^j}} \alpha^{\mu g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{2\mu g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{\nu g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{2\nu g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{\eta g p^i s} =$$

$$\sum_{s\epsilon\Omega_{16p^j}} \alpha^{\xi g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{\rho g p^i s} = \sum_{s\epsilon\Omega_{16p^j}} \alpha^{\chi g p^i s} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{4\lambda g p^i s}$$

$$= \begin{cases} \frac{\phi(p^{n-j})}{2}, if\ \ i+j \geq n, \\ \frac{1}{p^j} I_{i+j},\ \ if\ \ i+j \leq n-1, \quad g \neq 1, \\ \frac{1}{p^j} R_{i+j},\ \ if\ \ i+j \leq n-1, \quad g = 1. \end{cases}$$

Theorem 4.7. *For $p \equiv 1(mod 8)$, the expressions for the primitive idempotents corresponding to P_{8p^i}, P_{8gp^i}, P_{16p^i} and P_{16gp^i} in R_{16p^n} are given by*

$$P_{8p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\ \{\sum_{t=0}^{15}(-1)^t \overline{C}_{tp^n}\} + \tfrac{\phi(p^{n-j})}{2} \sum_{i=n-j}^{n-1} \{\sum_{a\in A}(-1)^a \overline{C}_{ap^i} + \sum_{a\in A}(-1)^a \overline{C}_{agp^i}\}$$

$$+ \tfrac{1}{p^j} \sum_{i=0}^{n-j-1} \{Q_{i+j}(\overline{C}_{p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\eta p^i} + \overline{C}_{\xi p^i} + \overline{C}_{\rho p^i} + \overline{C}_{\chi p^i}) + R_{i+j}(\overline{C}_{2p^i} +$$
$$\overline{C}_{4p^i} + \overline{C}_{8p^i} + \overline{C}_{16p^i} + \overline{C}_{2\lambda p^i} + \overline{C}_{4\lambda p^i} + \overline{C}_{2\mu p^i} + \overline{C}_{2\nu p^i}) + H_{i+j}(\overline{C}_{gp^i} + \overline{C}_{\lambda g p^i} + \overline{C}_{\mu g p^i} +$$
$$\overline{C}_{\nu g p^i} + \overline{C}_{\eta g p^i} + \overline{C}_{\xi g p^i} + \overline{C}_{\rho g p^i} + \overline{C}_{\chi g p^i}) + I_{i+j}(\overline{C}_{2gp^i} + \overline{C}_{4gp^i} + \overline{C}_{8gp^i} + \overline{C}_{16gp^i} + \overline{C}_{2\lambda g p^i} +$$
$$\overline{C}_{4\lambda g p^i} + \overline{C}_{2\mu g p^i} + \overline{C}_{2\nu g p^i})\}]\ \ P_{16p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}\overline{C}_{tp^n}\} + \tfrac{\phi(p^{n-j})}{2} \sum_{i=n-j}^{n-1}\{\sum_{a\in A}\overline{C}_{ap^i} +$$

$$\sum_{a\in A}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{R_{i+j}\{\sum_{a\in A}\overline{C}_{ap^i}\} + I_{i+j}\{\sum_{a\in A}\overline{C}_{agp^i}\}\}]$$

where $Q_{n-1} = \frac{1}{2}p^{n-1}(1+\sqrt{p})$, $H_{n-1} = \frac{p^{n-1}}{2}(1-\sqrt{p})$, $I_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}-1)$ and $R_{n-1} = \frac{1}{2}p^{n-1}(-\sqrt{p}-1)$ and for all $j \leq n-2$ $Q_j = H_j = I_{n-3} = R_{n-3} = 0$.

Proof. Since $\Omega_{8p^j} = -\Omega_{8p^j}$, as obtained in Lemma 4.4, so $\epsilon_k^{8p^j} = \sum_{s\epsilon\Omega_{p^j}} \alpha^{-ks} = \sum_{s\epsilon\Omega_{p^j}} \alpha^{8ks}$. Thus $\epsilon_{tp^n}^{8p^j} = (-1)^t \frac{\phi(p^{n-j})}{2}$ for $0 \leq t \leq 15$ and using Lemma 4.3, 4.5 − 4.6, wherever required we obtain

$$\epsilon_{p^i}^{8p^j} = \epsilon_{\lambda p^i}^{8p^j} = \epsilon_{\mu p^i}^{8p^j} = \epsilon_{\nu p^i}^{8p^j} = \epsilon_{\eta p^i}^{8p^j} = \epsilon_{\xi p^i}^{8p^j} = \epsilon_{\rho p^i}^{8p^j} = \epsilon_{\chi p^i}^{8p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2} \text{ if } i+j \geq n \\ \frac{1}{p^j} Q_{i+j} \text{ if } i+j \leq n-1 \end{cases}$$

$$\epsilon_{2p^i}^{8p^j} = \epsilon_{4p^i}^{8p^j} = \epsilon_{8p^i}^{8p^j} = \epsilon_{16p^i}^{8p^j} = \epsilon_{2\lambda p^i}^{8p^j} = \epsilon_{4\lambda p^i}^{8p^j} = \epsilon_{2\mu p^i}^{8p^j} = \epsilon_{2\nu p^i}^{8p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2} \text{ if } i+j \geq n \\ \frac{1}{p^j} R_{i+j} \text{ if } i+j \leq n-1 \end{cases}$$

$$\epsilon_{gp^i}^{8p^j} = \epsilon_{\lambda g p^i}^{8p^j} = \epsilon_{\mu g p^i}^{8p^j} = \epsilon_{\nu g p^i}^{8p^j} = \epsilon_{\eta g p^i}^{8p^j} = \epsilon_{\xi g p^i}^{8p^j} = \epsilon_{\rho g p^i}^{8p^j} = \epsilon_{\chi g p^i}^{8p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2} \text{ if } i+j \geq n \\ \frac{1}{p^j} H_{i+j} \text{ if } i+j \leq n-1 \end{cases}$$

$$\epsilon_{2gp^i}^{8p^j}=\epsilon_{4gp^i}^{8p^j}=\epsilon_{8gp^i}^{8p^j}=\epsilon_{16gp^i}^{8p^j}=\epsilon_{2\lambda gp^i}^{8p^j}=\epsilon_{4\lambda gp^i}^{8p^j}=\epsilon_{2\mu gp^i}^{8p^j}=\epsilon_{2\nu gp^i}^{8p^j}=\begin{cases}\frac{\phi(p^{n-j})}{2} \text{ if } i+j\geq n\\ \frac{1}{p^j}I_{i+j} \text{ if } i+j\leq n-1\end{cases}$$

Using all these in (3.1), the expression for P_{8p^j} is obtained.
Using Lemma 4.3 – 4.6, the expression for P_{16p^j}, P_{8gp^j} and P_{16gp^j} can be derived. However, the expression for $P_{8gp^j}(x)$, $P_{16gp^j}(x)$ can be written by interchanging Q and R by H and I in the expression of P_{8p^j}, P_{16p^j}.
Since $\overline{C}_k=\sum_{s\epsilon\Omega_k}x^s$ and $(\overline{C}_k)_{\alpha^{16p^j}}=\overline{C}_k(\alpha^{16p^j})=\sum_{s\epsilon\Omega_k}(\alpha^{16p^j})^s$. Therefore $\overline{C}_{tp^n}(\alpha^{16p^j})=1$ for $0\leq t\leq 15$. $\overline{C}_{ap^i}(\alpha^{16p^j})=\begin{cases}\frac{\phi(p^{n-j})}{2}, if\ i+j\geq n\\ \frac{1}{p^j}R_{i+j}, if\ i+j\leq n-1.\end{cases}$

and $\overline{C}_{agp^i}(\alpha^{16p^j})=\begin{cases}\frac{\phi(p^{n-j})}{2}, if\ i+j\geq n\\ \frac{1}{p^j}I_{i+j}, if\ i+j\leq n-1.\end{cases}$

Using all these in $P_{16p^j}(\alpha^{16p^j})=1$, to obtain

$$16p^n=\phi(p^{n-j})[8+8\sum_{i=n-j}^{n-1}\phi(p^{n-i})]+\frac{16}{p^j}\sum_{i=0}^{n-j-1}\frac{1}{p^i}(R^2{}_{i+j}+I^2{}_{i+j})$$

which in turn implies $\frac{1}{p^j}\sum_{i=0}^{n-j-1}\frac{1}{p^i}(R^2{}_{i+j}+I^2{}_{i+j})=\frac{p^{n-1}}{2}(p+1)$

In particular for $j=n-1$, $\frac{1}{p^{n-1}}(R^2{}_{n-1}+I^2{}_{n-1})=\frac{p^{n-1}}{2}(p+1)$
Using Lemma 4.1 to obtain $I_{n-1}=\frac{1}{2}p^{n-1}(\sqrt{p}-1)$ and $R_{n-1}=\frac{1}{2}p^{n-1}(-\sqrt{p}-1)$ and so $I_{n-2}=R_{n-2}=I_{n-3}=R_{n-3}=...=0$.
Relations for Q_{i+j} and H_{i+j} can be derived using Lemma 4.2 and the fact that $P_{8p^j}(\alpha^{8p^j})=1$.

5. Primitive Idempotents Corresponding to P_{tp^i}, $t=2,\ 4,\ 2\lambda,\ 4\lambda,\ 2\mu,\ 2\nu,\ 2g,\ 4g,\ 2\lambda g\ ,\ 4\lambda g,\ 2\mu g,\ 2\nu g$

For $0\leq j\leq n-1$, define $C_j=p^j\sum_{s\epsilon\Omega_{2\lambda gp^j}}\alpha^s$, $F_j=p^j\sum_{s\epsilon\Omega_{2gp^j}}\alpha^s$, $G_j=p^j\sum_{s\epsilon\Omega_{4gp^j}}\alpha^s$, $K_j=p^j\sum_{s\epsilon\Omega_{2\lambda p^j}}\alpha^s$, $O_j=p^j\sum_{s\epsilon\Omega_{2p^j}}\alpha^s$, $P_j=p^j\sum_{s\epsilon\Omega_{4p^j}}\alpha^s$. Due to similar procedure as in Section 4, C_j, F_j, G_j, K_j, O_j, $P_j\in GF(q)$.

Lemma 5.1. *For* $0\leq j\leq n-1$, $K_j-\beta^{p^j}O_j=0$*;* $C_j-\beta^{gp^j}F_j=0$ *where* $\beta=\alpha^{4p^n}$.
Proof. Since $\alpha^{2\lambda p^j}=\alpha^{2(1+2p^n)p^j}=\alpha^{2p^j}\beta^{p^j}$. So $K_j=p^j\sum_{s\epsilon\Omega_{\lambda p^i}}\alpha^{2s}=p^j[\alpha^{2\lambda p^i}+\alpha^{2q\lambda p^i}+\alpha^{2q^2\lambda p^i}+...+\alpha^{2q^{\frac{\phi(p^{n-j})}{2}-1}\lambda p^i}]=p^j[\alpha^{2p^i}\beta^{p^i}+\alpha^{2qp^i}\beta^{p^i}+\alpha^{2q^2p^i}\beta^{p^i}+...+$

$\alpha^{2p^iq^{\frac{\phi(p^{n-j})}{2}-1}}\beta^{p^i}] = \beta^{p^i}[p^j \sum_{s\epsilon\Omega_{2p^j}} \alpha^s] = \beta^{p^i}O_j$. Remaining can be obtained on similar lines.

Proof of Lemma 5.2 − 5.4 can be obtained on similar lines as that of Lemma 4.5 and represent C_{i+j}, F_{i+j}, G_{i+j}, K_{i+j}, O_{i+j}, P_{i+j}.

Lemma 5.2. *For* $0 \le i \le n$; $0 \le j \le n-1$ $\sum_{s\epsilon\Omega_{2p^j}} \alpha^{\lambda gp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{\xi gp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\eta gp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\nu gp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\chi gp^is} = \sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{\rho gp^is} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{\nu gp^is} = -\sum_{s\epsilon\Omega_{2p^j}} \alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\mu gp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\rho gp^is} = -\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\xi gp^is}$

$$= -\sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{\eta gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{2\lambda gp^{i+j}}, & if\ i+j \ge n, \quad g \ne 1, \\ \frac{1}{p^j}C_{i+j}, & if\ i+j \le n-1, \ g \ne 1, \\ \frac{\phi(p^{n-j})}{2}\alpha^{2\lambda p^{i+j}}, & if\ i+j \ge n, \quad g = 1, \\ \frac{1}{p^j}K_{i+j}, & if\ i+j \le n-1, \quad g = 1. \end{cases}$$

Lemma 5.3. *For* $0 \le i \le n$; $0 \le j \le n-1$ $\sum_{s\epsilon\Omega_{p^j}} \alpha^{2gp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\lambda gp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\xi gp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\rho gp^is} = \sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\nu gp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{\chi gp^is} =$

$$-\sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{\eta gp^is} = -\sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{\xi gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{2gp^{i+j}}, & if\ i+j \ge n, \quad g \ne 1, \\ \frac{1}{p^j}F_{i+j}, & if\ i+j \le n-1, \ g \ne 1, \\ \frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}}, & if\ i+j \ge n, \quad g = 1, \\ \frac{1}{p^j}O_{i+j}, & if\ i+j \le n-1, \quad g = 1. \end{cases}$$

Lemma 5.4. *For* $0 \le i \le n$; $0 \le j \le n-1$ $\sum_{s\epsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\lambda gp^is} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\nu gp^is} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\xi gp^is} = \sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\chi gp^is} = \sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{2\nu gp^is} = \sum_{s\epsilon\Omega_{2\mu p^j}} \alpha^{2\mu gp^is} = \sum_{s\epsilon\Omega_{2\nu p^j}} \alpha^{2\nu gp^is} = -\sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\mu gp^is} = -\sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\eta gp^is} = -\sum_{s\epsilon\Omega_{4\lambda p^j}} \alpha^{\rho gp^is} = -\sum_{s\epsilon\Omega_{2\lambda p^j}} \alpha^{2\mu gp^is} =$

$$\begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{4gp^{i+j}}, & if\ i+j \ge n, \quad g \ne 1, \\ \frac{1}{p^j}G_{i+j}, & if\ i+j \le n-1, \ g \ne 1, \\ \frac{\phi(p^{n-j})}{2}\alpha^{4p^{i+j}}, & if\ i+j \ge n, \quad g = 1, \\ \frac{1}{p^j}P_{i+j}, & if\ i+j \le n-1, \quad g = 1. \end{cases}$$

Theorem 5.5. *For $p \equiv 1(mod8)$, the expressions for the primitive idempotents corresponding to P_{2p^i}, P_{4p^i}, $P_{2\lambda p^i}$, $P_{4\lambda p^i}$, $P_{2\mu p^i}$, $P_{2\nu p^i}$, P_{2gp^i}, P_{4gp^i}, $P_{2\lambda gp^i}$, $P_{4\lambda gp^i}$, $P_{2\mu gp^i}$ and $P_{2\nu gp^i}$ are given by* $P_{2p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{2\lambda tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{2\lambda ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{2\lambda agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-K_{i+j}\overline{C}_{p^i} - P_{i+j}\overline{C}_{2p^i} + Q_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} - O_{i+j}\overline{C}_{\lambda p^i} + P_{i+j}\overline{C}_{2\lambda p^i} + Q_{i+j}\overline{C}_{4\lambda p^i} + K_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{2\mu p^i} + O_{i+j}\overline{C}_{\nu p^i} + P_{i+j}\overline{C}_{2\nu p^i} - K_{i+j}\overline{C}_{\eta p^i} - O_{i+j}\overline{C}_{\xi p^i} + K_{i+j}\overline{C}_{\rho p^i} + O_{i+j}\overline{C}_{\chi p^i} - C_{i+j}\overline{C}_{gp^i} - G_{i+j}\overline{C}_{2gp^i} + H_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - F_{i+j}\overline{C}_{\lambda gp^i} + G_{i+j}\overline{C}_{2\lambda gp^i} + H_{i+j}\overline{C}_{4\lambda gp^i} + C_{i+j}\overline{C}_{\mu gp^i} - G_{i+j}\overline{C}_{2\mu gp^i} + F_{i+j}\overline{C}_{\nu gp^i} + G_{i+j}\overline{C}_{2\nu gp^i} - C_{i+j}\overline{C}_{\eta gp^i} - F_{i+j}\overline{C}_{\xi gp^i} + C_{i+j}\overline{C}_{\rho gp^i} + F_{i+j}\overline{C}_{\chi gp^i}\}]$

$P_{4p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{4tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{4ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{4agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-P_{i+j}\overline{C}_{p^i} + Q_{i+j}\overline{C}_{2p^i} + R_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + P_{i+j}\overline{C}_{\lambda p^i} + Q_{i+j}\overline{C}_{2\lambda p^i} + R_{i+j}\overline{C}_{4\lambda p^i} - P_{i+j}\overline{C}_{\mu p^i} + Q_{i+j}\overline{C}_{2\mu p^i} + P_{i+j}\overline{C}_{\nu p^i} + Q_{i+j}\overline{C}_{2\nu p^i} - P_{i+j}\overline{C}_{\eta p^i} + P_{i+j}\overline{C}_{\xi p^i} - P_{i+j}\overline{C}_{\rho p^i} + P_{i+j}\overline{C}_{\chi p^i} - G_{i+j}\overline{C}_{gp^i} + H_{i+j}\overline{C}_{2gp^i} + I_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + G_{i+j}\overline{C}_{\lambda gp^i} + H_{i+j}\overline{C}_{2\lambda gp^i} + I_{i+j}\overline{C}_{4\lambda gp^i} - G_{i+j}\overline{C}_{\mu gp^i} + H_{i+j}\overline{C}_{2\mu gp^i} + G_{i+j}\overline{C}_{\nu gp^i} + H_{i+j}\overline{C}_{2\nu gp^i} - G_{i+j}\overline{C}_{\eta gp^i} + G_{i+j}\overline{C}_{\xi gp^i} - G_{i+j}\overline{C}_{\rho gp^i} + G_{i+j}\overline{C}_{\chi gp^i}\}]$

$P_{2\lambda p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{2tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{2ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{2agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-O_{i+j}\overline{C}_{p^i} + P_{i+j}\overline{C}_{2p^i} + Q_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} - K_{i+j}\overline{C}_{\lambda p^i} - P_{i+j}\overline{C}_{2\lambda p^i} + Q_{i+j}\overline{C}_{4\lambda p^i} + O_{i+j}\overline{C}_{\mu p^i} + P_{i+j}\overline{C}_{2\mu p^i} + K_{i+j}\overline{C}_{\nu p^i} - P_{i+j}\overline{C}_{2\nu p^i} - O_{i+j}\overline{C}_{\eta p^i} - K_{i+j}\overline{C}_{\xi p^i} + O_{i+j}\overline{C}_{\rho p^i} + K_{i+j}\overline{C}_{\chi p^i} - F_{i+j}\overline{C}_{gp^i} + G_{i+j}\overline{C}_{2gp^i} + H_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - C_{i+j}\overline{C}_{\lambda gp^i} - G_{i+j}\overline{C}_{2\lambda gp^i} + H_{i+j}\overline{C}_{4\lambda gp^i} + C_{i+j}\overline{C}_{\mu gp^i} + G_{i+j}\overline{C}_{2\mu gp^i} + C_{i+j}\overline{C}_{\nu gp^i} - G_{i+j}\overline{C}_{2\nu gp^i} - F_{i+j}\overline{C}_{\eta gp^i} - C_{i+j}\overline{C}_{\xi gp^i} + F_{i+j}\overline{C}_{\rho gp^i} + C_{i+j}\overline{C}_{\chi gp^i}\}]$

$P_{4\lambda p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}\alpha^{4tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{4ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{4agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{P_{i+j}\overline{C}_{p^i} + Q_{i+j}\overline{C}_{2p^i} + R_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} - P_{i+j}\overline{C}_{\lambda p^i} + Q_{i+j}\overline{C}_{2\lambda p^i} + R_{i+j}\overline{C}_{4\lambda p^i} + P_{i+j}\overline{C}_{\mu p^i} + Q_{i+j}\overline{C}_{2\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} +$

$$Q_{i+j}\overline{C}_{2\nu p^i} + P_{i+j}\overline{C}_{\eta p^i} - P_{i+j}\overline{C}_{\xi p^i} + P_{i+j}\overline{C}_{\rho p^i} - P_{i+j}\overline{C}_{\chi p^i} + G_{i+j}\overline{C}_{gp^i} + H_{i+j}\overline{C}_{2gp^i} + I_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - G_{i+j}\overline{C}_{\lambda gp^i} + H_{i+j}\overline{C}_{2\lambda gp^i} + I_{i+j}\overline{C}_{4\lambda gp^i} + G_{i+j}\overline{C}_{\mu gp^i} + H_{i+j}\overline{C}_{2\mu gp^i} - G_{i+j}\overline{C}_{\nu gp^i} + H_{i+j}\overline{C}_{2\nu gp^i} + G_{i+j}\overline{C}_{\eta gp^i} - G_{i+j}\overline{C}_{\xi gp^i} + G_{i+j}\overline{C}_{\rho gp^i} - G_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{2\mu p^j}(x) = \frac{1}{16p^n}[\frac{\Phi(p^{n-j})}{2}\{\sum_{t=0}^{15}\alpha^{2\lambda tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{2\lambda ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{2\lambda agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{K_{i+j}\overline{C}_{p^i} - P_{i+j}\overline{C}_{2p^i} + Q_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + O_{i+j}\overline{C}_{\lambda p^i} + P_{i+j}\overline{C}_{2\lambda p^i} + Q_{i+j}\overline{C}_{4\lambda p^i} - K_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{2\mu p^i} - O_{i+j}\overline{C}_{\nu p^i} + P_{i+j}\overline{C}_{2\nu p^i} + K_{i+j}\overline{C}_{\eta p^i} + O_{i+j}\overline{C}_{\xi p^i} - K_{i+j}\overline{C}_{\rho p^i} - O_{i+j}\overline{C}_{\chi p^i} + C_{i+j}\overline{C}_{gp^i} - G_{i+j}\overline{C}_{2gp^i} + H_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + F_{i+j}\overline{C}_{\lambda gp^i} + G_{i+j}\overline{C}_{2\lambda gp^i} + H_{i+j}\overline{C}_{4\lambda gp^i} - C_{i+j}\overline{C}_{\mu gp^i} - G_{i+j}\overline{C}_{2\mu gp^i} - F_{i+j}\overline{C}_{\nu gp^i} + G_{i+j}\overline{C}_{2\nu gp^i} + C_{i+j}\overline{C}_{\eta gp^i} + F_{i+j}\overline{C}_{\xi gp^i} - C_{i+j}\overline{C}_{\rho gp^i} - F_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{2\nu p^j}(x) = \frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}\alpha^{2tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{2ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{2agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{O_{i+j}\overline{C}_{p^i} + P_{i+j}\overline{C}_{2p^i} + Q_{i+j}\overline{C}_{4p^i} + R_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + K_{i+j}\overline{C}_{\lambda p^i} - P_{i+j}\overline{C}_{2\lambda p^i} + Q_{i+j}\overline{C}_{4\lambda p^i} - O_{i+j}\overline{C}_{\mu p^i} + P_{i+j}\overline{C}_{2\mu p^i} - K_{i+j}\overline{C}_{\nu p^i} - P_{i+j}\overline{C}_{2\nu p^i} + O_{i+j}\overline{C}_{\eta p^i} + K_{i+j}\overline{C}_{\xi p^i} - O_{i+j}\overline{C}_{\rho p^i} - K_{i+j}\overline{C}_{\chi p^i} + F_{i+j}\overline{C}_{gp^i} + G_{i+j}\overline{C}_{2gp^i} + H_{i+j}\overline{C}_{4gp^i} + I_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + C_{i+j}\overline{C}_{\lambda gp^i} - G_{i+j}\overline{C}_{2\lambda gp^i} + H_{i+j}\overline{C}_{4\lambda gp^i} - F_{i+j}\overline{C}_{\mu gp^i} + G_{i+j}\overline{C}_{2\mu gp^i} - C_{i+j}\overline{C}_{\nu gp^i} - G_{i+j}\overline{C}_{2\nu gp^i} + F_{i+j}\overline{C}_{\eta gp^i} + C_{i+j}\overline{C}_{\xi gp^i} - F_{i+j}\overline{C}_{\rho gp^i} - C_{i+j}\overline{C}_{\chi gp^i}\}].$$

Where C_{i+j}, F_{i+j}, G_{i+j}, K_{i+j}, O_{i+j}, P_{i+j} can be obtained, using Lemma 5.1 *and the following relations,*

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{G_{i+j}^2 + P_{i+j}^2}{p^i}\} = -\frac{p^{n-1}}{2}(p+1),$$

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}P_{i+j}G_{i+j} = \frac{p^{n-1}}{4}(1-p),$$

$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K_{i+j}O_{i+j} + C_{i+j}F_{i+j}}{p^i}\} = -\frac{p^{n-1}}{2}(p+1).$$

$$\frac{4}{p^j}\sum_{i=0}^{n-j-1}\{\frac{K^2{}_{i+j} + C^2{}_{i+j}}{p^i}\} + \frac{4}{p^j}\sum_{i=0}^{n-j-1}\{\frac{F^2{}_{i+j} + O^2{}_{i+j}}{p^i}\} = 0.$$

The expressions for P_{2gp^j}, P_{4gp^j}, $P_{2\lambda gp^j}$, $P_{4\lambda gp^j}$, $P_{2\mu gp^j}$ and $P_{2\nu gp^j}$ can be obtained by replacing P, Q, R, K, O by G, H, I, C, F and $\alpha^{up^{i+j}}$ by $\alpha^{ugp^{i+j}}$ respectively in the expression of P_{2p^j}, P_{4p^j}, $P_{2\lambda p^j}$, $P_{4\lambda p^j}$, $P_{2\mu p^j}$ and $P_{2\nu p^j}$.

Proof. These expressions can be obtained using Lemmas 4.3 – 4.6, 5.2 – 5.4

and similar procedure as in Theorem 4.9. Also the relations can be derived using $P_{4p^j}(\alpha^{4p^i}) = 1$, $P_{4p^j}(\alpha^{4gp^i}) = 0$, $P_{2p^j}(\alpha^{2p^i}) = 1$ and $P_{2p^j}(\alpha^{2\nu p^i}) = 0$ and Lemma 5.1.

6. Primitive Idempotents Corresponding to P_{tp^i}, $t = 1, \lambda, \mu, \nu, \eta, \xi, \rho, \chi, g, \lambda g, \mu g, \nu g, \eta g, \xi g, \rho g, \chi g$

For $0 \le j \le n-1$, define $A_j = p^j \sum_{s\epsilon\Omega_{gp^j}} \alpha^s$; $B_j = p^j \sum_{s\epsilon\Omega_{\lambda gp^j}} \alpha^s$; $D_j = p^j \sum_{s\epsilon\Omega_{\mu gp^j}} \alpha^s$; $E_j = p^j \sum_{s\epsilon\Omega_{\nu gp^j}} \alpha^s$; $J_j = p^j \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^s$; $L_j = p^j \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^s$; $M_j = p^j \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^s$; $N_j = p^j \sum_{s\epsilon\Omega_{p^j}} \alpha^s$. Using similar procedure as in Section 4 to obtain A_j, B_j, D_j, E_j, J_j, L_j, M_j, $N_j \in GF(q)$. Proof of Lemma 6.1 is similar to that of Lemma 5.1.

Lemma 6.1. *For $0 \le j \le n-1$, $D_j - \beta^{gp^j} A_j = 0$; $E_j - \beta^{gp^j} B_j = 0$; $L_j - \beta^{p^j} N_j = 0$; $M_j - \beta^{p^j} J_j = 0$; where $\beta = \alpha^{4p^n}$.*

Proof. Proof of Lemma 6.2 – 6.5 is similar to that of Lemma 4.5.

Lemma 6.2. *For $0 \le i \le n$; $0 \le j \le n-1$*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{gp^i s} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\xi gp^i s} = \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{\rho gp^i s} = \sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{\nu gp^i s} = \sum_{s\epsilon\Omega_{\eta p^j}} \alpha^{\eta gp^i s} = \sum_{s\epsilon\Omega_{\chi p^j}} \alpha^{\chi gp^i s} =$$

$$-\sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{\chi gp^i s} = -\sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\lambda gp^i s} = -\sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{\mu gp^i s} = -\sum_{s\epsilon\Omega_{\xi p^j}} \alpha^{\xi gp^i s} = -\sum_{s\epsilon\Omega_{\rho p^j}} \alpha^{\rho gp^i s}$$

$$= \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{gp^{i+j}}, & if\ i+j \ge n, \quad g \ne 1, \\ \frac{1}{p^j} A_{i+j}, & if\ i+j \le n-1,\ g \ne 1, \\ \frac{\phi(p^{n-j})}{2}\alpha^{p^{i+j}}, & if\ i+j \ge n, \quad g = 1, \\ \frac{1}{p^j} N_{i+j}, & if\ i+j \le n-1,\ g = 1. \end{cases}$$

Lemma 6.3. *For $0 \le i \le n$; $0 \le j \le n-1$*

$$\sum_{s\epsilon\Omega_{p^j}} \alpha^{\lambda gp^i s} = \sum_{s\epsilon\Omega_{\eta p^j}} \alpha^{\xi gp^i s} = \sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{\nu gp^i s}$$

$$= \sum_{s\epsilon\Omega_{\rho p^j}} \alpha^{\chi gp^i s} = -\sum_{s\epsilon\Omega_{p^j}} \alpha^{\xi p^i s} = -\sum_{s\epsilon\Omega_{\nu p^j}} \alpha^{\rho gp^i s} = -\sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\eta gp^i s} = -\sum_{s\epsilon\Omega_{\mu p^j}} \alpha^{\chi gp^i s} =$$

$$\begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{\lambda gp^{i+j}}, & if\ i+j \ge n, \quad g \ne 1, \\ \frac{1}{p^j} B_{i+j}, & if\ i+j \le n-1,\ g \ne 1, \\ \frac{\phi(p^{n-j})}{2}\alpha^{\lambda p^{i+j}}, & if\ i+j \ge n, \quad g = 1, \\ \frac{1}{p^j} J_{i+j}, & if\ i+j \le n-1,\ g = 1. \end{cases}$$

Lemma 6.4. *For $0 \le i \le n$; $0 \le j \le n-1$* $\sum_{s\epsilon\Omega_{p^j}} \alpha^{\mu gp^i s} = \sum_{s\epsilon\Omega_{\lambda p^j}} \alpha^{\nu gp^i s} =$

$$\sum_{s\epsilon\Omega_{\eta p^j}}\alpha^{\rho gp^is} = \sum_{s\epsilon\Omega_{\xi p^j}}\alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{p^j}}\alpha^{\rho gp^is} = -\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{\mu p^j}}\alpha^{\eta gp^is} =$$

$$-\sum_{s\epsilon\Omega_{\nu p^j}}\alpha^{\xi gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{\mu gp^{i+j}}, & if\ i+j\geq n, \quad g\neq 1,\\ \frac{1}{p^j}D_{i+j}, & if\ i+j\leq n-1,\ g\neq 1,\\ \frac{\phi(p^{n-j})}{2}\alpha^{\mu p^{i+j}}, & if\ i+j\geq n, \quad g=1,\\ \frac{1}{p^j}L_{i+j}, & if\ i+j\leq n-1,\ g=1.\end{cases}$$

Lemma 6.5. *For* $0\leq i\leq n;\ 0\leq j\leq n-1$ $\sum_{s\epsilon\Omega_{p^j}}\alpha^{\nu gp^is} = \sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{\rho gp^is} =$

$$\sum_{s\epsilon\Omega_{\mu p^j}}\alpha^{\xi gp^is} = \sum_{s\epsilon\Omega_{\eta p^j}}\alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{p^j}}\alpha^{\chi gp^is} = -\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{\mu gp^is} = -\sum_{s\epsilon\Omega_{\nu p^j}}\alpha^{\eta gp^is} =$$

$$-\sum_{s\epsilon\Omega_{\xi p^j}}\alpha^{\rho gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{\nu gp^{i+j}}, & if\ i+j\geq n, \quad g\neq 1,\\ \frac{1}{p^j}E_{i+j}, & if\ i+j\leq n-1,\ g\neq 1,\\ \frac{\phi(p^{n-j})}{2}\alpha^{\nu p^{i+j}}, & if\ i+j\geq n, \quad g=1,\\ \frac{1}{p^j}M_{i+j}, & if\ i+j\leq n-1, g=1.\end{cases}$$

Theorem 6.6. *For* $p\equiv 1(mod 8)$*, the expressions for the primitive idempotents corresponding to* P_{p^i}*,* $P_{\lambda p^i}$*,* $P_{\mu p^i}$*,* $P_{\nu p^i}$*,* $P_{\eta p^i}$*,* $P_{\xi p^i}$*,* $P_{\rho p^i}$*,* $P_{\chi p^i}$*,* P_{gp^i}*,* $P_{\lambda gp^i}$*,* $P_{\mu gp^i}$*,* $P_{\nu gp^i}$*,* $P_{\eta gp^i}$*,* $P_{\xi gp^i}$*,* $P_{\rho gp^i}$ *and* $P_{\chi gp^i}$ *are given by* $P_{p^j}(x)=\frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{t\nu p^{n+j}}\overline{C}_{tp^n}\}$ $+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{a\nu p^{i+j}}\overline{C}_{ap^i}+\sum_{a\in A}(-1)^a\alpha^{a\nu gp^{i+j}}\overline{C}_{agp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-M_{i+j}\overline{C}_{p^i}-K_{i+j}\overline{C}_{2p^i}-P_{i+j}\overline{C}_{4p^i}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}-L_{i+j}\overline{C}_{\lambda p^i}-O_{i+j}\overline{C}_{2\lambda p^i}+P_{i+j}\overline{C}_{4\lambda p^i}-J_{i+j}\overline{C}_{\mu p^i}+K_{i+j}\overline{C}_{2\mu p^i}-N_{i+j}\overline{C}_{\nu p^i}+O_{i+j}\overline{C}_{2\nu p^i}+M_{i+j}\overline{C}_{\eta p^i}+L_{i+j}\overline{C}_{\xi p^i}+J_{i+j}\overline{C}_{\rho p^i}+N_{i+j}\overline{C}_{\chi p^i}-E_{i+j}\overline{C}_{gp^i}-C_{i+j}\overline{C}_{2gp^i}-G_{i+j}\overline{C}_{4gp^i}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}-D_{i+j}\overline{C}_{\lambda gp^i}-F_{i+j}\overline{C}_{2\lambda gp^i}+G_{i+j}\overline{C}_{4\lambda gp^i}-B_{i+j}\overline{C}_{\mu gp^i}+C_{i+j}\overline{C}_{2\mu gp^i}-A_{i+j}\overline{C}_{\nu gp^i}+F_{i+j}\overline{C}_{2\nu gp^i}+E_{i+j}\overline{C}_{\eta gp^i}+D_{i+j}\overline{C}_{\xi gp^i}+B_{i+j}\overline{C}_{\rho gp^i}+A_{i+j}\overline{C}_{\chi gp^i}\}]$

$P_{\lambda p^j}(x)=\frac{1}{16p^n}[\frac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{t\mu p^{n+j}}\overline{C}_{tp^n}\}+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{a\mu p^{i+j}}\overline{C}_{ap^i}+\sum_{a\in A}(-1)^a\alpha^{a\mu gp^{i+j}}\overline{C}_{agp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-L_{i+j}\overline{C}_{p^i}-O_{i+j}\overline{C}_{2p^i}+P_{i+j}\overline{C}_{4p^i}+Q_{i+j}\overline{C}_{8p^i}+R_{i+j}\overline{C}_{16p^i}+M_{i+j}\overline{C}_{\lambda p^i}-K_{i+j}\overline{C}_{2\lambda p^i}-P_{i+j}\overline{C}_{4\lambda p^i}+N_{i+j}\overline{C}_{\mu p^i}+O_{i+j}\overline{C}_{2\mu p^i}-J_{i+j}\overline{C}_{\nu p^i}+K_{i+j}\overline{C}_{2\nu p^i}+L_{i+j}\overline{C}_{\eta p^i}-M_{i+j}\overline{C}_{\xi p^i}-N_{i+j}\overline{C}_{\rho p^i}+J_{i+j}\overline{C}_{\chi p^i}-D_{i+j}\overline{C}_{gp^i}-F_{i+j}\overline{C}_{2gp^i}+G_{i+j}\overline{C}_{4gp^i}+H_{i+j}\overline{C}_{8gp^i}+I_{i+j}\overline{C}_{16gp^i}+E_{i+j}\overline{C}_{\lambda gp^i}-C_{i+j}\overline{C}_{2\lambda gp^i}-G_{i+j}\overline{C}_{4\lambda gp^i}+A_{i+j}\overline{C}_{\mu gp^i}+F_{i+j}\overline{C}_{2\mu gp^i}-B_{i+j}\overline{C}_{\nu gp^i}+C_{i+j}\overline{C}_{2\nu gp^i}+D_{i+j}\overline{C}_{\eta gp^i}-E_{i+j}\overline{C}_{\xi gp^i}-A_{i+j}\overline{C}_{\rho gp^i}+B_{i+j}\overline{C}_{\chi gp^i}\}]$

$$P_{\mu p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{\lambda tp^{n+j}}\overline{C}_{tp^n}\}+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{a\lambda p^{i+j}}\overline{C}_{ap^i}+ \sum_{a\in A}(-1)^a\alpha^{a\lambda gp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-J_{i+j}\overline{C}_{p^i} + K_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + N_{i+j}\overline{C}_{\lambda p^i} + O_{i+j}\overline{C}_{2\lambda p^i} + P_{i+j}\overline{C}_{4\lambda p^i} + M_{i+j}\overline{C}_{\mu p^i} - K_{i+j}\overline{C}_{2\mu p^i} - L_{i+j}\overline{C}_{\nu p^i} - O_{i+j}\overline{C}_{2\nu p^i} + J_{i+j}\overline{C}_{\eta p^i} - N_{i+j}\overline{C}_{\xi p^i} - M_{i+j}\overline{C}_{\rho p^i} + L_{i+j}\overline{C}_{\chi p^i} - B_{i+j}\overline{C}_{gp^i} + C_{i+j}\overline{C}_{2gp^i} - G_{i+j}\overline{C}_{4gp^i} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + A_{i+j}\overline{C}_{\lambda gp^i} + F_{i+j}\overline{C}_{2\lambda gp^i} + G_{i+j}\overline{C}_{4\lambda gp^i} + E_{i+j}\overline{C}_{\mu gp^i} - C_{i+j}\overline{C}_{2\mu gp^i} - D_{i+j}\overline{C}_{\nu gp^i} - F_{i+j}\overline{C}_{2\nu gp^i} + B_{i+j}\overline{C}_{\eta gp^i} - A_{i+j}\overline{C}_{\xi gp^i} - E_{i+j}\overline{C}_{\rho gp^i} + D_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{\nu p^j}(x) = \tfrac{1}{16p^n}[\tfrac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}(-1)^t\alpha^{tp^{n+j}}\overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}(-1)^a\alpha^{ap^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}(-1)^a\alpha^{agp^{i+j}}\overline{C}_{agp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-N_{i+j}\overline{C}_{p^i} + O_{i+j}\overline{C}_{2p^i} + P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} - J_{i+j}\overline{C}_{\lambda p^i} + K_{i+j}\overline{C}_{2\lambda p^i} - P_{i+j}\overline{C}_{4\lambda p^i} - L_{i+j}\overline{C}_{\mu p^i} - O_{i+j}\overline{C}_{2\mu p^i} - M_{i+j}\overline{C}_{\nu p^i} - K_{i+j}\overline{C}_{2\nu p^i} + N_{i+j}\overline{C}_{\eta p^i} + J_{i+j}\overline{C}_{\xi p^i} + L_{i+j}\overline{C}_{\rho p^i} + M_{i+j}\overline{C}_{\chi p^i} - A_{i+j}\overline{C}_{gp^i} + F_{i+j}\overline{C}_{2gp^i} + G_{i+j}\overline{C}_{4gp^i} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - B_{i+j}\overline{C}_{\lambda gp^i} + C_{i+j}\overline{C}_{2\lambda gp^i} - G_{i+j}\overline{C}_{4\lambda gp^i} - D_{i+j}\overline{C}_{\mu gp^i} - F_{i+j}\overline{C}_{2\mu gp^i} - E_{i+j}\overline{C}_{\nu gp^i} - C_{i+j}\overline{C}_{2\nu gp^i} + A_{i+j}\overline{C}_{\eta gp^i} + B_{i+j}\overline{C}_{\xi gp^i} + D_{i+j}\overline{C}_{\rho gp^i} + E_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{\eta p^j}(x) = \tfrac{1}{16p^n}{}_{i+j}\ [\tfrac{\phi(p^{n-j})}{2}\{\sum_{t=0}^{15}\alpha^{t\nu p^{n+j}}\overline{C}_{tp^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}\alpha^{a\nu p^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}\alpha^{a\nu gp^{i+j}}\overline{C}_{agp^i}\} + \tfrac{1}{p^j}\sum_{i=0}^{n-j-1}\{M_{i+j}\overline{C}_{p^i} - K_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} + L_{i+j}\overline{C}_{\lambda p^i} - O_{i+j}\overline{C}_{2\lambda p^i} + P_{i+j}\overline{C}_{4\lambda p^i} + J_{i+j}\overline{C}_{\mu p^i} + K_{i+j}\overline{C}_{2\mu p^i} + N_{i+j}\overline{C}_{\nu p^i} + O_{i+j}\overline{C}_{2\nu p^i} - M_{i+j}\overline{C}_{\eta p^i} - L_{i+j}\overline{C}_{\xi p^i} - J_{i+j}\overline{C}_{\rho p^i} - N_{i+j}\overline{C}_{\chi p^i} + E_{i+j}\overline{C}_{gp^i} - C_{i+j}\overline{C}_{2gp^i} - G_{i+j}\overline{C}_{4gp^i} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + D_{i+j}\overline{C}_{\lambda gp^i} - F_{i+j}\overline{C}_{2\lambda gp^i} + G_{i+j}\overline{C}_{4\lambda gp^i} + B_{i+j}\overline{C}_{\mu gp^i} + C_{i+j}\overline{C}_{2\mu gp^i} + A_{i+j}\overline{C}_{\nu gp^i} + F_{i+j}\overline{C}_{2\nu gp^i} - E_{i+j}\overline{C}_{\eta gp^i} - D_{i+j}\overline{C}_{\xi gp^i} - B_{i+j}\overline{C}_{\rho gp^i} - A_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{\xi p^j}(x) = \tfrac{1}{16p^n}\ [\tfrac{\phi(p^{n-j})}{2}\ \{\sum_{t=0}^{15}\alpha^{t\mu p^{n+j}}\overline{C}_{tp^n}\} + \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\sum_{a\in A}\alpha^{a\mu p^{i+j}}\overline{C}_{ap^i} + \sum_{a\in A}\alpha^{a\mu gp^{i+j}}\overline{C}_{agp^i}\} + \tfrac{1}{p^j}\sum_{i=0}^{n-j-1}\{L_{i+j}\overline{C}_{p^i} - O_{i+j}\overline{C}_{2p^i} + P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} - M_{i+j}\overline{C}_{\lambda p^i} - K_{i+j}\overline{C}_{2\lambda p^i} - P_{i+j}\overline{C}_{4\lambda p^i} - N_{i+j}\overline{C}_{\mu p^i} + O_{i+j}\overline{C}_{2\mu p^i} + J_{i+j}\overline{C}_{\nu p^i} + K_{i+j}\overline{C}_{2\nu p^i} - L_{i+j}\overline{C}_{\eta p^i} + M_{i+j}\overline{C}_{\xi p^i} + N_{i+j}\overline{C}_{\rho p^i} - J_{i+j}\overline{C}_{\chi p^i} + D_{i+j}\overline{C}_{gp^i} - F_{i+j}\overline{C}_{2gp^i} + G_{i+j}\overline{C}_{4gp^i} + H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - E_{i+j}\overline{C}_{\lambda gp^i} - C_{i+j}\overline{C}_{2\lambda gp^i} - G_{i+j}\overline{C}_{4\lambda gp^i} - A_{i+j}\overline{C}_{\mu gp^i} + F_{i+j}\overline{C}_{2\mu gp^i} + B_{i+j}\overline{C}_{\nu gp^i} + C_{i+j}\overline{C}_{2\nu gp^i} - D_{i+j}\overline{C}_{\eta gp^i} + E_{i+j}\overline{C}_{\xi gp^i} + A_{i+j}\overline{C}_{\rho gp^i} - B_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{\rho p^j}(x) = \frac{1}{16p^n} \ [\frac{\phi(p^{n-j})}{2} \ \{\sum_{t=0}^{15} \alpha^{t\lambda p^{n+j}} \overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2} \sum_{i=n-j}^{n-1} \{\sum_{a\in A} \alpha^{a\lambda p^{i+j}} \overline{C}_{ap^i}$$

$$+\sum_{a\in A} \alpha^{a\lambda gp^{i+j}} \overline{C}_{agp^i}\} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{J_{i+j}\overline{C}_{p^i} + K_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} -$$
$$N_{i+j}\overline{C}_{\lambda p^i} + O_{i+j}\overline{C}_{2\lambda p^i} + P_{i+j}\overline{C}_{4\lambda p^i} - M_{i+j}\overline{C}_{\mu p^i} - K_{i+j}\overline{C}_{2\mu p^i} + L_{i+j}\overline{C}_{\nu p^i} - O_{i+j}\overline{C}_{2\nu p^i} -$$
$$J_{i+j}\overline{C}_{\eta p^i} + N_{i+j}\overline{C}_{\xi p^i} + M_{i+j}\overline{C}_{\rho p^i} - L_{i+j}\overline{C}_{\chi p^i} + B_{i+j}\overline{C}_{gp^i} + C_{i+j}\overline{C}_{2gp^i} - G_{i+j}\overline{C}_{4gp^i} +$$
$$H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} - A_{i+j}\overline{C}_{\lambda gp^i} + F_{i+j}\overline{C}_{2\lambda gp^i} + G_{i+j}\overline{C}_{4\lambda gp^i} - E_{i+j}\overline{C}_{\mu gp^i} - C_{i+j}\overline{C}_{2\mu gp^i} +$$
$$D_{i+j}\overline{C}_{\nu gp^i} - F_{i+j}\overline{C}_{2\nu gp^i} - B_{i+j}\overline{C}_{\eta gp^i} + A_{i+j}\overline{C}_{\xi gp^i} + E_{i+j}\overline{C}_{\rho gp^i} - D_{i+j}\overline{C}_{\chi gp^i}\}]$$

$$P_{\chi p^j}(x) = \frac{1}{16p^n} \ [\frac{\phi(p^{n-j})}{2} \{\sum_{t=0}^{15} \alpha^{tp^{n+j}} \overline{C}_{tp^n}\} + \frac{\phi(p^{n-j})}{2} \sum_{i=n-j}^{n-1} \{\sum_{a\in A} \alpha^{ap^{i+j}} \overline{C}_{ap^i}$$

$$+\sum_{a\in A} \alpha^{agp^{i+j}} \overline{C}_{agp^i}\} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{N_{i+j}\overline{C}_{p^i} + O_{i+j}\overline{C}_{2p^i} + P_{i+j}\overline{C}_{4p^i} + Q_{i+j}\overline{C}_{8p^i} + R_{i+j}\overline{C}_{16p^i} +$$
$$J_{i+j}\overline{C}_{\lambda p^i} + K_{i+j}\overline{C}_{2\lambda p^i} - P_{i+j}\overline{C}_{4\lambda p^i} + L_{i+j}\overline{C}_{\mu p^i} - O_{i+j}\overline{C}_{2\mu p^i} + M_{i+j}\overline{C}_{\nu p^i} - K_{i+j}\overline{C}_{2\nu p^i} -$$
$$N_{i+j}\overline{C}_{\eta p^i} - J_{i+j}\overline{C}_{\xi p^i} - L_{i+j}\overline{C}_{\rho p^i} - M_{i+j}\overline{C}_{\chi p^i} + A_{i+j}\overline{C}_{gp^i} + F_{i+j}\overline{C}_{2gp^i} + G_{i+j}\overline{C}_{4gp^i} +$$
$$H_{i+j}\overline{C}_{8gp^i} + I_{i+j}\overline{C}_{16gp^i} + B_{i+j}\overline{C}_{\lambda gp^i} + C_{i+j}\overline{C}_{2\lambda gp^i} - G_{i+j}\overline{C}_{4\lambda gp^i} + D_{i+j}\overline{C}_{\mu gp^i} - F_{i+j}\overline{C}_{2\mu gp^i} +$$
$$E_{i+j}\overline{C}_{\nu gp^i} - C_{i+j}\overline{C}_{2\nu gp^i} - A_{i+j}\overline{C}_{\eta gp^i} - B_{i+j}\overline{C}_{\xi gp^i} - D_{i+j}\overline{C}_{\rho gp^i} - E_{i+j}\overline{C}_{\chi gp^i}\}].$$

The expressions for P_{gp^j}, $P_{\lambda gp^j}$, $P_{\mu gp^j}$, $P_{\nu gp^j}$, $P_{\eta gp^j}$, $P_{\xi gp^j}$, $P_{\rho gp^j}$ and $P_{2\chi gp^j}$ can be obtained by replacing P, Q, R, K, O, L, M, N, J by G, H, I, C, F, D, E, A, B and $\alpha^{p^{i+j}}$ by $\alpha^{gp^{i+j}}$ respectively, where A_{i+j}, B_{i+j}, D_{i+j},E_{i+j}, J_{i+j}, L_{i+j}, M_{i+j}, N_{i+j} can be obtained, using Lemma 6.1 *and the following relations,*

$$\frac{1}{p^j} \sum_{i=0}^{n-j-1} \{\frac{M_{i+j}N_{i+j} + A_{i+j}E_{i+j}}{p^i}\} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{\frac{J_{i+j}L_{i+j} + B_{i+j}D_{i+j}}{p^i}\} = -p^{n-1}(p+1)$$

$$\frac{4}{p^j} \sum_{i=0}^{n-j-1} \{\frac{M_{i+j}B_{i+j}}{p^i} + \frac{K_{i+j}C_{i+j}}{p^i} + \frac{O_{i+j}F_{i+j}}{p^i} + \frac{N_{i+j}D_{i+j}}{p^i}\} - \frac{4}{p^j} \sum_{i=0}^{n-j-1} \{\frac{P_{i+j}G_{i+j} + L_{i+j}A_{i+j} + J_{i+j}E_{i+j}}{p^i}\}$$

$$= \frac{2}{p^j} \sum_{i=0}^{n-j-1} \{\frac{Q_{i+j}H_{i+j} + R_{i+j}I_{i+j}}{p^i}\} + \phi(p^{n-j}) \ \{1 - \alpha^{(\nu+\lambda g)p^{n+j}} - \alpha^{2\lambda(1+g)p^{n+j}} + \alpha^{3(\chi+\lambda g)p^{n+j}} +$$
$$\alpha^{4(1+g)p^{n+j}} + \alpha^{5(\chi+\lambda g)p^{n+j}} + \alpha^{6(\chi+\lambda g)p^{n+j}} + \alpha^{7(\chi+\lambda g)p^{n+j}}\} + \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \phi(p^{n-i})\{1 - \alpha^{(\nu+\lambda g)p^{i+j}} -$$
$$\alpha^{2\lambda(1+g)p^{i+j}} + \alpha^{4(1+g)p^{i+j}} + \alpha^{(\mu+g)p^{i+j}} - \alpha^{2(1+g)p^{i+j}} + \alpha^{(\nu g+\lambda)p^{i+j}} - \alpha^{(1+\mu g)p^{i+j}}\}$$

$$\frac{1}{p^j} \sum_{i=0}^{n-j-1} \{\frac{M^2{}_{i+j} + E^2{}_{i+j}}{p^i} + \frac{L^2{}_{i+j} + D^2{}_{i+j}}{p^i} + \frac{J^2{}_{i+j} + B^2{}_{i+j}}{p^i} + \frac{N^2{}_{i+j} + A^2{}_{i+j}}{p^i}\} = 0.$$

$\frac{4}{p^j}\sum_{i=0}^{n-j-1}\{\frac{E_{i+j}L_{i+j}}{p^i}+\frac{J_{i+j}A_{i+j}}{p^i}+\frac{P_{i+j}G_{i+j}}{p^i}-\frac{O_{i+j}C_{i+j}}{p^i}-\frac{M_{i+j}D_{i+j}}{p^i}-\frac{K_{i+j}F_{i+j}}{p^i}-\frac{N_{i+j}B_{i+j}}{p^i}\}=\frac{2}{p^j}\sum_{i=0}^{n-j-1}\{\frac{Q_{i+j}H_{i+j}}{p^i}+\frac{R_{i+j}I_{i+j}}{p^i}\}+\phi(p^{n-j})\sum_{i=n-j}^{n-1}\phi(p^{n-j})\{1-\alpha^{(\mu+\nu g)p^{i+j}}+\alpha^{2(1+\lambda g)p^{i+j}}-\alpha^{4(1+g)p^{i+j}}+\alpha^{(\nu+\mu g)p^{i+j}}+\alpha^{2(\lambda+g)p^{i+j}}+\alpha^{(1+\lambda g)p^{i+j}}-\alpha^{(\lambda+g)p^{i+j}}\}$.

Proof. These expressions can be obtained using Lemmas $4.3-4.6$, $5.2-5.4$ and $6.2-6.5$ and similar procedure as in Theorem 4.7. Also the relations can be derived using $P_{p^j}(\alpha^{p^i})=1$, $P_{p^j}(\alpha^{\lambda gp^i})=0$, $P_{p^j}(\alpha^{\nu p^j})=0$ and $P_{\lambda p^j}(\alpha^{\nu gp^j})=0$.

7. Dimension and Generating Polynomials

The polynomial $m_s(x)=\prod_{s\epsilon\Omega_s}(x-\alpha^s)$ denote the minimal polynomial for α^s and so the generating polynomial for cyclic code M_s of length $16p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{16p^n}-1}{m_s(x)}$ and the dimension of M_s is equal to the cardinality of the class Ω_s [5].

Theorem 7.1. (i) *The generating polynomial for the codes* M_{tp^n}, *for* $0\le t\le 15$ *are* $(1+x+x^2+...+x^{(16p^n-1)}$, $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x+\beta)(1+x^{16}+...+x^{16(p^n-1)})$, $(x+\beta^2)(x^2+\beta^4)(x^4-1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x+\beta^3)(x^2+\beta^6)(x^4-\beta^4)(x^8-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^2-1)(x+\beta^4)(x^4+1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x+\beta^5)(x^2-\beta^2)(x^4+\beta^4)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^4-1)(x+\beta^6)(x^2-\beta^4)(x^8-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-\beta^4)(x^2-\beta^6)(x+\beta^7)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8+1)(x^4+1)(x^2+1)(x-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x-\beta)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-1)(x^2+\beta^4)(x-\beta^2)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-\beta^4)(x^2+\beta^6)(x-\beta^3)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8+1)(x^4+1)(x^2-1)(x-\beta^4)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4+\beta^4)(x^2-\beta^2)(x-\beta^5)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^2-\beta^4)(x^4-1)(x-\beta^6)(1+x^{16}+...+x^{16(p^n-1)})$ *and* $(x^8-1)(x^4-\beta^4)(x^2-\beta^6)(x-\beta^7)(1+x^{16}+...+x^{16(p^n-1)})$ *respectively, where* β *is* $16th$ *root of unity.*

(ii) *The generating polynomial for* $M_{8p^i}\oplus M_{8gp^i}$, $M_{16p^i}\oplus M_{16gp^i}$ *and* $M_{p^i}\oplus M_{2p^i}\oplus M_{4p^i}\oplus M_{\lambda p^i}\oplus M_{2\lambda p^i}\oplus M_{4\lambda p^i}\oplus M_{\mu p^i}\oplus M_{2\mu p^i}\oplus M_{\nu p^i}\oplus M_{2\nu p^i}\oplus M_{\eta p^i}\oplus M_{\xi p^i}\oplus M_{\rho p^i}\oplus M_{\chi p^i}\oplus M_{gp^i}\oplus M_{2gp^i}\oplus M_{4gp^i}\oplus M_{\lambda gp^i}\oplus M_{2\lambda gp^i}\oplus M_{4\lambda gp^i}\oplus M_{\mu gp^i}\oplus M_{2\mu gp^i}\oplus M_{\nu gp^i}\oplus M_{2\nu gp^i}\oplus M_{\eta gp^i}\oplus M_{\xi gp^i}\oplus M_{\rho gp^i}\oplus M_{\chi gp^i}$ *are* $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$, $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$ *and* $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$ *respectively.*

Proof. (i) The minimal polynomial for α^{tp^n}, for $0\le t\le 15$ are $(x-1)$, $(x-\beta)$, $(x-\beta^2)$, $(x-\beta^3)$, $(x-\beta^4)$, $(x-\beta^5)$, $(x-\beta^6)$, $(x-\beta^7)$, $(x+1)$, $(x+\beta)$, $(x+\beta^2)$,$(x+\beta^4)$,

$(x+\beta^5)$,$(x+\beta^6)$ and $(x+\beta^7)$ respectively. The corresponding generating polynomials are $(1+x+x^2+...+x^(16p^n-1))$, $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x+\beta)(1+x^{16}+...+x^{16(p^n-1)})$, $(x+\beta^2)(x^2+\beta^4)(x^4-1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x+\beta^3)(x^2+\beta^6)(x^4-\beta^4)(x^8-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^2-1)(x+\beta^4)(x^4+1)(x^8+1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x+\beta^5)(x^2-\beta^2)(x^4+\beta^4)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^4-1)(x+\beta^6)(x^2-\beta^4)(x^8-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-\beta^4)(x^2-\beta^6)(x+\beta^7)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8+1)(x^4+1)(x^2+1)(x-1)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x-\beta)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-1)(x^2+\beta^4)(x-\beta^2)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4-\beta^4)(x^2+\beta^6)(x-\beta^3)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8+1)(x^4+1)(x^2-1)(x-\beta^4)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^4+\beta^4)(x^2-\beta^2)(x-\beta^5)(1+x^{16}+...+x^{16(p^n-1)})$, $(x^8-1)(x^2-\beta^4)(x^4-1)(x-\beta^6)(1+x^{16}+...+x^{16(p^n-1)})$ and $(x^8-1)(x^4-\beta^4)(x^2-\beta^6)(x-\beta^7)(1+x^{16}+...+x^{16(p^n-1)})$.

(ii) The product of minimal polynomial satisfied by α^{8p^i} and α^{8gp^i} is $(\frac{x^{p^{n-i}}+1}{x^{p^{n-i-1}}+1})$. Therefore, the generating polynomial for $M_{8p^i}\oplus M_{8gp^i}$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$. The product of minimal polynomial satisfied by α^{16p^i} and α^{16gp^i} is $(\frac{x^{p^{n-i}}-1}{x^{p^{n-i-1}}-1})$. Therefore, the generating polynomial for $M_{16p^i}\oplus M_{16gp^i}$ is $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$. Also the product of minimal polynomial satisfied by $\alpha^{p^i},\alpha^{2p^i},\alpha^{4p^i},...\alpha^{\rho gp^i},\alpha^{\chi gp^i}$ is $\frac{(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)}{(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)}$. Therefore, the generating polynomial for $M_{p^i}\oplus M_{2p^i}\oplus M_{4p^i}\oplus M_{\lambda p^i}\oplus M_{2\lambda p^i}\oplus M_{4\lambda p^i}\oplus M_{\mu p^i}\oplus M_{2\mu p^i}\oplus M_{\nu p^i}\oplus M_{2\nu p^i}\oplus M_{\eta p^i}\oplus M_{\xi p^i}\oplus M_{\rho p^i}\oplus M_{\chi p^i}\oplus M_{gp^i}\oplus M_{2gp^i}\oplus M_{4gp^i}\oplus M_{\lambda gp^i}\oplus M_{2\lambda gp^i}\oplus M_{4\lambda gp^i}\oplus M_{\mu gp^i}\oplus M_{2\mu gp^i}\oplus M_{\nu gp^i}\oplus M_{2\nu gp^i}\oplus M_{\eta gp^i}\oplus M_{\xi gp^i}\oplus M_{\rho gp^i}\oplus M_{\chi gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$.

8. Minimum Distance

If C is a cyclic code of length m generated by $g(x)$ and its minimum distance is d, then the code $\bar{C}$ of length mk generated by $g(x)(1+x^m+x^{2m}+...+x^{(k-1)m})$ is a repetition code of C repeated k times and its minimum distance is dk[3]. Here, we find the minimum distance of the minimal cyclic code M_s of length $16p^n$, generated by the primitive idempotent P_s.

Theorem 8.1. *Each of the codes M_{tp^n}, for $0\leq t\leq 15$ are of minimum distance $16p^n$. For $0\leq i\leq n-1$, the minimum distance of the cyclic codes M_{cp^i}, M_{cgp^i}, $c=\{8,16\}$ are greater than or equal $32p^i$ and minimum distance for the codes M_{ap^i}, M_{agp^i}, $a\in A-\{8,16\}$ are greater than or equal to $16p^i$.*

Proof. As the generating polynomial for the code M_0 is a polynomial of length $16p^n$, hence its minimum distance is $16p^n$. Also, the generating polynomial for the cyclic code M_{p^n} is $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x+\beta)(1+x^{16}+...+x^{16(p^n-1)})$. Which

is a repetition code of the cyclic code of length 16 with generating polynomial $(x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x+\beta)$, repeated p^n times. Therefore its minimum distance is $16p^n$. Similarly, the minimum distance of each of the cyclic codes M_{tp^n}, where $2 \leq t \leq 15$ is $16p^n$.
The cyclic codes M_{8p^i} and M_{8gp^i}, with genrating polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$ is a repetition code of the code generated by $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(x^{8p^{n-i}}+1)$ of length $16p^{n-i}$ and minimum distance 32 repeated p^i times. Therefore its minimum distance is $32p^i$. The codes corresponding to M_{8p^i} and M_{8gp^i} are the sub codes of the above code, so their minimum distances are greater than or equal to $32p^i$.
Similarly, the minimum distance of the cyclic code M_{16p^i} and M_{16gp^i} of length $16p^n$ are also greater than or equal to $32p^i$.
The product of generating polynomial for the cyclic codes M_{ap^i}, M_{agp^i}, $a \in A - \{8, 16\}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}+1)(1+x^{16p^{n-i}}+...+x^{16p^{n-i}(p^i-1)})$. If we consider a code C of length $16p^{n-i}$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{8p^{n-i-1}}+1)(x^{2p^{n-i}}+1)$, then the minimum distance of this code is 16. Since the cyclic code of length $16p^n$ generated by the said polynomial is a repetition of the code C repeated p^i times. Hence its minimum distance is $16p^i$.
Since the codes corresponding to Ω_{ap^i}, Ω_{agp^i}, $a \in A$ are the sub codes of the above code, so their minimum distance is greater than or equal to $16p^i$.

Example 9.1. Cyclic Codes of length 48.
Take $p = 3$, $n = 1$, $q = 193$. Then the q-cyclotomic cosets are $\Omega_t = \{t\}$, $0 \leq t \leq 47$ and the expressions for primitive idempotents in $\frac{GF(193)[x]}{<x^{48}-1>}$ are $P_b(x) = \frac{1}{48}[\sum_{t=0}^{47}\{(-48)^{bt}(mod 193)\}\overline{C}_t]$ where $b = 0$ to 47.
Minimal polynomials for α^t, $0 \leq t \leq 47$ are $(x-r)$ where $\alpha^t \equiv r(mod 193)$. The minimal codes M_t, $0 \leq t \leq 47$ of length 48 are as follows:

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_{ap^n}	1	48	$\sum_{t=0}^{47}\{4^{ap^n(47-t)}x^t\}(mod 193)$ where $0 \leq a \leq 15$
M_c	1	$16 \leq d \leq 48$	$\sum_{t=0}^{47}\{4^{c(47-t)}x^t\}(mod 193)$ where $c = \{3t+1, 3t+2; 0 \leq t \leq 15\} - \{8, 16, 32, 40\}$
M_d	1	$32 \leq d \leq 48$	$\sum_{t=0}^{47}\{4^{d(47-t)}x^t\}(mod 193)$ where $d = 8, 16, 32, 40$

References

[1] A. Sahni, P.T. Sehgal, Minimal Cyclic Codes of Length $p^n q$, Finite Fields Appl, 18(2012), 1017-1036.

[2] F. Li, Q. Yue, C. Li, Irreducible Cyclic codes of length $4p^n$ and $8p^n$, Finite Fields Appl., 34(2015), 208-234.

[3] G.K. Bakshi, M. Raka, Minimal Cyclic Codes of Length $p^n q$, Finite Fields Appl., 9(2003), 432-448.

[4] J. Singh, S.K. Arora, Minimal Cyclic Codes of Length $8p^n$ over $GF(q)$, where q is prime power of the form $8k+5$, J.Appl. Math. Comput, 48(2015) , 55-69.

[5] J. Singh, S.K. Arora, S. Chawla, Some Cyclic Codes of Length $8p^n$, International Journal of Pure and Applied Mathematics, 116(1)(2017), 217-241.

[6] M. Pruthi, S.K. Arora, Minimal codes of prime power length, Finite Field Appl, 3(1997), 99-113.

[7] S. Batra, S.K. Arora, Minimal Quadratic Cyclic Codes of Length p^n (p odd prime), Korean J. Comput. Appli. Math., 8(2001), 531-547.

[8] S. Batra, S.K. Arora, Some Cyclic Codes of Length $2p^n$, Des. Codes Cryptogr., 61(2010), 41-69.

[9] S.K. Arora, M. Pruthi, Minimal cyclic codes of length $2p^n$, Finite Field Appl, 5(1999), 177-187.

[10] S.K. Arora, S. Batra, S.D. Cohen, M. Pruthi, The Primitive Idempotents of a Cyclic Group Algebra, Southest Asian Bulletin of Mathematics, 26(2002), 549-557.

[11] V. Pless, Introduction of the theory of Error Correcting Codes, Wiley, New York, 1981.