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GENERALIZATION OF AN IDENTITY OF RAMANUJAN

S.N. Singh, Satya Prakash Singh and Vijay Yadav*

Department of Mathematics,
Tilak Dhari PG College, Jaunpur (UP) 222002, INDIA

E-mail: snsp39@gmail.com, snsp39@yahoo.com

*Department of Mathematics,
S.P.D.T. College, Andheri (E), Mumbai-59, INDIA

E-mail: vijaychottu@yahoo.com

Dedicated to Prof. A.K. Agarwal on his 70^{th} Birth Anniversary

Abstract: In this paper, an identity of Ramanujan has been generalized. Particular cases of this generalized identity have been discussed.

Keyword and Phrases: Identity, Basic hypergeometric, Basic hypergeometric series of two variables, Basic hypergeometric series of several variables, q-binomial theorem.

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1. Introduction, Notations and Definitions

For complex variables a and q, $|q| < 1$, the $q-$ shifted factorials are given as,

$$(a;q^k)_0 = 1, \quad (a;q^k)_n = (1-a)(1-aq^k)...(1-aq^{(n-1)k}),$$

where n and k are positive integers. For brevity, let

$$(a_1, a_2, ..., a_r; q^k)_n = (a_1;q^k)_n(a_2;q^k)_n...(a_r;q^k)_n.$$

Following [4; (1.2.22), p.4], the generalized basic hypergeometric series is defined by

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q^k; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q^k)_n z^n}{(q, b_1, b_2, ..., b_s; q^k)_n}\{(-1)^n q^{kn(n-1)/2}\}^{1+s-r}. \quad (1.1)$$

The infinite series in (1.1) is absolutely convergent for $|z| < \infty$ if $r \leq s$ and for $r = s+1$, it converges in the region $|z| < 1$. The four basic Appell series are defined as,

$$\Phi^{(1)}[a; b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m(b';q)_n x^m y^n}{(q;q)_m(q;q)_n(c;q)_{m+n}}, \tag{1.2}$$

$$\Phi^{(2)}[a; b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_n(b';q)_n x^m y^n}{(q;q)_m(q;q)_n(c;q)_m(c';q)_n}, \tag{1.3}$$

$$\Phi^{(3)}[a, a'; b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a;q)_m(a';q)_n(b;q)_m(b';q)_n x^m y^n}{(q;q)_m(q;q)_n(c;q)_{m+n}}, \tag{1.4}$$

and

$$\Phi^{(4)}[a; b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_{m+n} x^m y^n}{(q;q)_m(q;q)_n(c;q)_m(c';q)_n}. \tag{1.5}$$

[4; (9.1.1)-(9.1.4), p. 232]

Basic Lauricella function is defined as,

$$\Phi_D[a; b_1, b_2, ..., b_n; c; x_1, x_2, ..., x_n]$$

$$= \sum_{m_1,m_2,...,m_n=0}^{\infty} \frac{(a;q)_{m_1+m_2+...+m_n}(b_1;q)_{m_1}(b_2;q)_{m_2}...(b_n;q)_{m_n}}{(q;q)_{m_1}(q;q)_{m_2}...(q;q)_{m_n}(c;q)_{m_1+m_2+...+m_n}} x_1^{m_1} x_2^{m_2} ... x_n^{m_n}, \tag{1.6}$$

[1; Theorem 4, p. 207]

where max. $(q, |x|_1, |x_2|, ..., |x_n|) < 1$.
We shall use the following $q-$binomial theorem in our analysis,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{1.7}$$

[3; App. II (II.3), p. 354]

2. Main Results

In this section we establish our main results.
Following identity is due to Ramanujan,

$$\sum_{m=0}^{\infty} \frac{(a;q^h)_m(b;q)_{mh} z^m}{(q^h;q^h)_m(c;q)_{mh}} = \frac{(b;q)_\infty(az;q^h)_\infty}{(c;q)_\infty(z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r(z;q^h)_r b^r}{(q;q)_r(az;q^h)_r}, \tag{2.1}$$

[2; Theorem (1.2.1), p. 6]

where $|z| < 1, |b| < 1$.

Proof of (2.1)

Left hand side of (2.1)

$$\sum_{m=0}^{\infty} \frac{(a;q^h)_m (b;q)_{mh} z^m}{(q^h;q^h)_m (c;q)_{mh}} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(a;q^h)_m (cq^{mh};q)_\infty z^m}{(q^h;q^h)_m (bq^{mh};q)_\infty}. \tag{2.2}$$

Using binomial theorem (1.7) we get

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(a;q^h)_m z^m}{(q^h;q^h)_m} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r q^{rmh}}{(q;q)_r}$$

For $|z| < 1$, $|b| < 1$, we have

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r}{(q;q)_r} \sum_{m=0}^{\infty} \frac{(a;q^h)_m}{(q^h;q^h)_m} (zq^{rh})^m$$

Again, applying binomial theorem (1.7) we have,

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r b^r (azq^{rh};q^h)_\infty}{(q;q)_r (zq^{rh};q^h)_\infty},$$

$$= \frac{(b;q)_\infty (az;q^h)_\infty}{(c;q)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q)_r (z;q^h)_r b^r}{(q;q)_r (az;q^h)_r}, \tag{2.3}$$

which is precisely the right hand side of (2.1).
We now establish following transformation

$$\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q^k)_{nh} z^n}{(q^h;q^h)_n (c;q^k)_{nh}} = \frac{(b;q^k)_\infty (az;q^h)_\infty}{(c;q^k)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r (z;q^h)_{kr} b^r}{(q^k;q^k)_r (az;q^h)_{kr}}, \tag{2.4}$$

provided $|z| < 1$, $|b| < 1$.

Proof of (2.4)

Left hand side of (2.4) is

$$= \sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q^k)_{nh} z^n}{(q^h;q^h)_n (c;q^k)_{nh}} = \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n z^n (cq^{knh};q^k)_\infty}{(q^h;q^h)_n (bq^{knh};q^k)_\infty}, \tag{2.5}$$

Applying the $q-$ binomial theorem (1.7)

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n z^n}{(q^h;q^h)_n} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r q^{krnh}}{(q^k;q^k)_r},$$

For $|z| < 1$, $|b| < 1$, it yields

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r}{(q^k;q^k)_r} \sum_{n=0}^{\infty} \frac{(a;q^h)_n b^r}{(q^h;q^h)_n} (zq^{krh})^n$$

Again, applying binomial theorem (1.7) we have,

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r (azq^{krh};q^h)_\infty}{(q^k;q^k)_r (zq^{krh};q^h)_\infty},$$

$$= \frac{(b;q^k)_\infty (az;q^h)_\infty}{(c;q^k)_\infty (z;q^h)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r (z;q^h)_{kr} b^r}{(q^k;q^k)_r (az;q^h)_{kr}}, \tag{2.6}$$

which is precisely the right hand side of (2.4) if we take $k = 1$ in (2.4) it reduces into (2.1).

Two variables generalization of (2.4)

Following is the two variables generalization of the identity (2.4)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1;q^{h_1})_m (a_2;q^{h_2})_n (b;q^k)_{mh_1+nh_2} z_1^m z_2^n}{(q^{h_1};q^{h_1})_m (q^{h_2};q^{h_2})_n (c;q^k)_{mh_1+nh_2}}$$

$$= \frac{(b;q^k)_\infty (a_1 z_1;q^{h_1})_\infty (a_2 z_2;q^{h_2})_\infty}{(c;q^k)_\infty (z_1;q^{h_1})_\infty (z_2;q^{h_2})_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r (z_1;q^{h_1})_{kr} (z_2;q^{h_2})_{kr} b^r}{(q^k;q^k)_r (a_1 z_1;q^{h_1})_{kr} (a_2 z_2;q^{h_2})_{kr}}, \tag{2.7}$$

where max. $(|z_1|, |z_2|, |b|) < 1$.

Proof of (2.7)

Left hand side of (2.7) is

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1;q^{h_1})_m (a_2;q^{h_2})_n (b;q^k)_{mh_1+nh_2} z_1^m z_2^n}{(q^{h_1};q^{h_1})_m (q^{h_2};q^{h_2})_n (c;q^k)_{mh_1+nh_2}}$$

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1;q^{h_1})_m (a_2;q^{h_2})_n (cq^{k(mh_1+nh_2)};q^k)_\infty}{(q^{h_1};q^{h_1})_m (q^{h_2};q^{h_2})_n (bq^{k(mh_1+nh_2)};q^k)_\infty} z_1^m z_2^m$$

Using binomial theorem (1.7) we get

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{m=0}^{\infty} \frac{(a_1;q^{h_1})_m (a_2;q^{h_2})_n z_1^m z_2^n}{(q^{h_1};q^{h_1})_m (q^{h_2};q^{h_2})_n} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r q^{kr(mh_1+nh_2)}}{(q^k;q^k)_r},$$

For max. $(|z_1|,|z_2|,|b|) < 1$, it becomes

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r}{(q^k;q^k)_r} \sum_{m=0}^{\infty} \frac{(a_1;q^{h_1})_m (q^{krh_1} z_1)^m}{(q^{h_1};q^{h_1})_m} \sum_{n=0}^{\infty} \frac{(a_2;q^{h_2})_n (q^{krh_2} z_2)^n}{(q^{h_2};q^{h_2})_n}$$

Again, applying $q-$binomial theorem (1.7), we get

$$= \frac{(b;q^k)_\infty}{(c;q^k)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r (a_1 z_1 q^{krh_1};q^{h_1})_\infty (a_2 z_2 q^{krh_2};q^{h_2})_\infty}{(q^k;q^k)_r (z_1 q^{krh_1};q^{h_1})_\infty (z_2 q^{krh_2};q^{h_2})_\infty}$$

$$= \frac{(b;q^k)_\infty (a_1 z_1;q^{h_1})_\infty (a_2 z_2;q^{h_2})_\infty}{(c;q^k)_\infty (z_1;q^{h_1})_\infty (z_2;q^{h_2})_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r (z_1;q^{h_1})_{kr} (z_2;q^{h_2})_{kr} b^r}{(q^k;q^k)_r (a_1 z_1;q^{h_1})_{kr} (a_2 z_2;q^{h_2})_{kr}}, \tag{2.8}$$

which is right hand side of (2.7).
Proceeding as above, m variables generalization of (2.4) is

$$\sum_{n_1,n_2,...,n_m=0}^{\infty} \frac{(a_1;q^{h_1})_{n_1} (a_2;q^{h_2})_{n_2} ... (a_m;q^{h_m})_{n_m} (b;q^k)_{n_1h_1+n_2h_2+...+n_mh_m} z_1^{n_1} z_2^{n_2} ... z_m^{h_m}}{(q^{h_1};q^{h_1})_{n_1} (q^{h_2};q^{h_2})_{n_2} ... (q^{h_m};q^{h_m})_{n_m} (c;q^k)_{n_1h_1+n_2h_2+...+n_mh_m}}$$

$$= \frac{(b;q^k)_\infty (a_1 z_1;q^{h_1})_\infty ... (a_m z_m;q^{hm})_\infty}{(c;q^k)_\infty (z_1;q^{h_1})_\infty ... (z_m;q^{h_m})_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r (z_1;q^{h_1})_{kr} ... (z_m;q^{h_m})_{kr} b^r}{(q^k;q^k)_r (a_1 z_1;q^{h_1})_{kr} ... (a_m z_m;q^{h_m})_{kr}}, \tag{2.9}$$

provided max. $(|z_1|,|z_2|,...,|z_m|,|b|) < 1$.

3. Special Cases

In this section we discuss special cases of the results established in previous section.

(i) Taking $k = h_1 = h_2 = ... = h_m = 1$ in (2.9) we get,

$$\Phi_D[b;a_1,a_2,...,a_m;c;q;z_1,z_2,...,z_m] = \frac{(b,a_1z_1,a_2z_2,...,a_mz_m;q)_\infty}{(c,z_1,z_2,...,z_m;q)_\infty}$$

$$\times \,_{m+1}\Phi_m \left[\begin{matrix} c/b, z_1, z_2, ..., z_m; q; b \\ a_1z_1, a_2z_2, ..., a_mz_m \end{matrix} \right], \tag{3.1}$$

where max. $(|z_1|, |z_2|, ..., |z_m|, |b|) < 1$ and Φ_D is the basic analogue of Lauricella function F_D defined in [4; (8.6.4), p. 228] [1; Theorem 5, p. 207].

(ii) Putting $z_1/a_1, z_2/a_2, ..., z_m/a_m$ for $z_1, z_2, ..., z_m$ respectively in (2.9) and then taking $a_1, a_2, ..., a_m \to \infty$ in (2.9) we get,

$$\sum_{n_1,n_2,...,n_m=0}^{\infty} \frac{(b;q^k)_{n_1h_1+n_2h_2+...+n_mh_m}}{(c;q^k)_{n_1h_1+n_2h_2+...+n_mh_m}} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$$

$$\times \frac{(-)^{n_1+n_2+...+n_m} q^{h_1n_1(n_1-1)/2+h_2n_2(n_2-1)/2+...+h_mn_m(n_m-1)/2}}{(q^{h_1};q^{h_1})_{n_1}(q^{h_2};q^{h_2})_{n_2}...(q^{h_m};q^{h_m})_{n_m}}$$

$$= \frac{(b;q^k)_\infty (z_1;q^{h_1})_\infty ... (z_m;q^{h_m})_\infty}{(c;q^k)_\infty} \sum_{r=0}^{\infty} \frac{(c/b;q^k)_r b^r}{(q^k;q^k)_r (z_1;q^{h_1})_{kr} (z_2;q^{h_2})_{kr} ... (z_m;q^{h_m})_{kr}}. \tag{3.2}$$

(iii) For $h = k = 1$, (2.4) yields

$${}_2\Phi_1 \left[\begin{array}{l} a, b; q; z \\ c \end{array} \right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} \, {}_2\Phi_1 \left[\begin{array}{l} c/b, z; q; b \\ az \end{array} \right], \tag{3.3}$$

which is Heine's fundamental transformation.

(iv) For $h = k = 2$, (2.4) gives

$${}_3\Phi_2 \left[\begin{array}{l} a, \sqrt{b}, -\sqrt{b}; q; z \\ \sqrt{c}, -\sqrt{c} \end{array} \right] = \frac{(b;q^2)_\infty (az;q)_\infty}{(c;q^2)_\infty (z;q)_\infty} \, {}_3\Phi_2 \left[\begin{array}{l} c/b, z, zq; q^2; b \\ az, azq \end{array} \right], \tag{3.4}$$

where $|z|, |b| < 1$.

(v) For $h = 2$ and $k = 1$, (2.4) gives

$${}_3\Phi_2 \left[\begin{array}{l} a, b, bq; q^2; z \\ c, cq \end{array} \right] = \frac{(b;q)_\infty (az;q^2)_\infty}{(c;q)_\infty (z;q^2)_\infty} \, {}_3\Phi_2 \left[\begin{array}{l} c/b, \sqrt{z}, -\sqrt{z}; q; b \\ \sqrt{az}, -\sqrt{az} \end{array} \right], \tag{3.5}$$

where $|z|, |b| < 1$.

(vi) Taking $h = 2$, $k = 2$ in (2.4) we find,

$${}_5\Phi_4 \left[\begin{array}{l} a, \sqrt{b}, -\sqrt{b}, q\sqrt{b}, -q\sqrt{b}; q^2; z \\ \sqrt{c}, -\sqrt{c}, q\sqrt{c}, -q\sqrt{c} \end{array} \right]$$

$$= \frac{(b, az; q^2)_\infty}{(c, z; q^2)_\infty} \, {}_5\Phi_4 \left[\begin{array}{l} c/b, \sqrt{z}, -\sqrt{z}, q\sqrt{z}, -q\sqrt{z}; q^2; b \\ \sqrt{az}, -\sqrt{az}, q\sqrt{az}, -q\sqrt{az} \end{array} \right], \tag{3.6}$$

where $|z| < 1$, $|b| < 1$.
(vii) Taking $h_1 = h_2 = k = 1$ in (2.7) we get,

$$\Phi^{(1)}[b; a_1, a_2; c; q; z_1, z_2] = \frac{(b, a_1 z_1, a_2 z_2; q)_\infty}{(c, z_1, z_2; q)_\infty} {}_3\Phi_2[c/b, z_1, z_2; a_1 z_1, a_2 z_2; q; b], \quad (3.7)$$

where $|z_1| < 1, |z_2| < 1, |b| < 1$ and $\Phi^{(1)}$ is the basic Appell function [4; (9.1.1), p. 232].
Taking different values of h, k one can have large number of transformations from (2.4), (2.7) and also from (2.9).

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