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On Transformation Formulae for theta hypergeometric functions

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Abstract: In this paper, making use of an identity and certain summation formulae for truncated theta hypergeometric series, we have established transformation formulae for finite-bilateral theta hypergeometric series.
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1. Introduction, Notations and Definitions

Elliptic hypergeometric series and their extensions to theta hypergeometric series has become an increasingly active area of research these days. In the present paper, we have established transformation formulae for bilateral theta hypergeometric series. Special cases of the results established in this paper have also been deduced.

A modified Jacobi's theta function with argument x and nome p is defined by

$$\theta(x;p) = [x;p]_\infty [p/x;p]_\infty \equiv [x, p/x;p]_\infty \tag{1.1}$$

$$\theta(x_1, x_2, ..., x_r; p) = \theta(x_1;p)\theta(x_2;p), ..., \theta(x_r;p)$$

and

$$[x;p]_\infty = \prod_{r=0}^{\infty}(1 - xp^r).$$

Following Gasper and Rahman [1; Chapter 11 (11.2.5) and (11.2.53)] theta shifted factorial is defined by,

$$[a;p,q]_n = \begin{cases} \theta(a;p)\theta(aq;p)...\theta(aq^{n-1};p); & \text{for } n > 0 \\ 1, & \text{n=0,} \end{cases}$$

Also,

$$[a;q,p]_{-n} = \frac{q^{n(n+1)/2}}{(-a)^n[q/a;q,p]_n}, \qquad n \geq 1 \tag{1.2}$$

and,

$$[a_1,a_2,...,a_r;q,p]_n = [a_1;q,p]_n[a_2;q,p]_n...[a_r;q,p]_n. \tag{1.3}$$

Corresponding to Spiridonov [2], theta hypergeometric series is defined by

$${}_{r+1}E_r\left[\begin{array}{l} a_1,a_2,...,a_{r+1};q,p;z \\ b_1,b_2,...,b_r \end{array}\right] = \sum_{n=0}^{\infty}\frac{[a_1,a_2,...,a_{r+1};q,p]_n z^n}{[q,b_1,b_2,...,b_r;q,p]_n}, \tag{1.4}$$

where max. $\{|z|,|q|,|p|\} < 1$.
Corresponding to Spiridonov [3] a very well-poised theta hypergeometric series is defined by,

$$\begin{aligned} &{}_{r+1}V_r[a_1;a_6,...,a_{r+1};q,p;z] \\ &= \sum_{n=0}^{\infty}\frac{\theta(a_1q^{2n};p)[a_1,a_6,...,a_{r+1};q,p]_n}{\theta(a_1;p)[a_1q/q_6,...,a_1q/a_{r+1};q,p]_n}(zq)^n \\ &= {}_{r+1}E_r\left[\begin{array}{l} a_1,q\sqrt{a_1},-q\sqrt{a_1},q\sqrt{a_1/p},-q\sqrt{a_1p},a_6,...,a_{r+1};q,p;-z \\ \sqrt{a_1},-\sqrt{a_1},\sqrt{a_1p},-\sqrt{a_1/p},a_1q/a_6,...,a_1q/a_{r+1} \end{array}\right] \end{aligned} \tag{1.5}$$

A truncated very well poised theta hypergeometric series is defined by,

$${}_{r+1}V_r[a_1;a_6,...,a_{r+1};q,p;z]_N = \sum_{n=0}^{N}\frac{\theta(a_1q^{2n};p)[a_1,a_6,...,a_{r+1};q,p]_n}{\theta(a_1;p)[a_1q/a_6,...,a_1q/a_{r+1};q,p]_n}(zq)^n \tag{1.6}$$

We call a series of the form,

$$\sum_{k=-m}^{n}\frac{[a_1,a_2,...,a_{r+1};q,p]_k z^k}{[q,b_1,b_2,...,b_r;q,p]_k}$$

a finite bilateral theta hypergeometric series.
We shall make use of the following identity,

$$\sum_{k=-m}^{n}\lambda_{k+m}\sum_{j=0}^{n-k}A_j = \sum_{k=-m}^{n}A_{k+m}\sum_{j=0}^{n-k}\lambda_j \tag{1.7}$$

Proof of (1.7) In order to prove (1.7) let us consider the following well known identity,

$$\sum_{k=0}^{n}\lambda_k\sum_{j=0}^{n-k}A_j = \sum_{k=0}^{n}A_k\sum_{j=0}^{n-k}\lambda_j \tag{1.8}$$

[Gasper & Rahman 1; (11.6.18) p.321]

Taking n+m for n and replacing k by k+m in (1.8), we get (1.7) after some simplification.
Following summations are also needed in our analysis,

$$ {}_{10}V_9\left[a; b, c, d, e, q^{-n}; q, p\right] = \frac{[aq, aq/bc, aq/bd, aq/cd; q, p]_n}{[aq/b, aq/c, aq/d, aq/bcd; q, p]_n} \tag{1.9} $$

[Gasper & Rahman 1; (11.4.1) p.321]

where $bcdeq^{-n} = a^2q$
Now, setting $e = aq^{n+1}$ in (1.9) we get,

$$ {}_8V_7[a; b, c, a/bc; q, p]_n = \frac{[aq, aq/bc, bq, cq; q, p]_n}{[q, aq/b, aq/c, bcq; q, p]_n} \tag{1.10} $$

[Gasper & Rahman 1; (11.4.10) p.322]

Again, we have

$$ {}_{10}V_9[dp; a, b, dpq/c, cdpq^n/ab, q^{-n}; q, p^2] = \frac{[dpq, c/a, c/b, dpq/ab; q, p^2]_n}{[c/ab, dpq/a, c, dpq/b; q, p^2]_n} \tag{1.11} $$

[Gasper & Rahman 1; (11.4.11) p.323]

Now, Taking c=abq in (1.11) we get

$$ {}_8V_7[dp; a, b, dp/ab; q, p^2]_n = \frac{[dpq, aq, bq, dpq/ab; q, p^2]_n}{[q, abq, dpq/a, dpq/b; q, p^2]_n} \tag{1.12} $$

We also have

$$ \begin{aligned} &\sum_{k=0}^{n} \frac{\theta\{ad(rst/q)^k, (b/d)(r/q)^k, (c/d)(s/q)^k, (ad/bc)(t/q)^k; p\}}{\theta(ad, b/d, c/d, ad/bc; p)} \times \\ &\times \frac{[a; rst/q^2, p]_k [b; r, p]_k [c; s, p]_k [ad^2/bc; t, p]_k q^k}{[dq; q, p]_k [adst/bq; st/q, p]_k [adrt/cq; rt/q, p]_k [bcrs/dq; rs/q, p]_k} \\ &= \frac{\theta(a, b, c, ad^2/bc; p)[arst/q^2; srt/q^2, p]_n}{d\theta(ad, b/d, c/d, ad/bc; p)[dq; q, p]_n [adst/bq; st/q, p]_n} \times \\ &\times \frac{[br; r, p]_n [cs; s, p]_n [ad^2t/bc; t, p]_n}{[adrt/cq; rt/q, p]_n [bcrs/dq; rs/q, p]_n} - \frac{\theta(d, ad/b, ad/c, bc/d; p)}{d\theta(ad, b/d, c/d, ad/bc; p)} \end{aligned} \tag{1.13} $$

[Gasper & Rahman 1; (11.6.9) p.327]

Taking d=1 in the above we get

$$\sum_{k=0}^{n} \frac{\theta\{a(rst/q)^k, b(r/q)^k, c(s/q)^k, (a/bc)(t/q)^k; p\}}{\theta(a, b, c, a/bc; p)} \times$$

$$\times \frac{[a; rst/q^2, p]_k [b; r, p]_k [c; s, p]_k [a/bc; t, p]_k q^k}{[q; q, p]_k [ast/bq; st/q, p]_k [art/cq; rt/q, p]_k [bcrs/q; rs/q, p]_k}$$

$$= \frac{[arst/q^2; srt/q^2, p]_n [br; r, p]_n [cs; s, p]_n [at/bc; t, p]_n}{[q; q, p]_n [ast/bq; st/q, p]_n [art/cq; rt/q, p]_n [bcrs/q; rs/q, p]_n} \tag{1.14}$$

2. Main Results

In this section we shall establish our main transformations. We start by setting

$$\lambda_k = \frac{\theta(aq^{2k}; p)[a, b, c, a/bc; q, p]_k q^k}{\theta(a; p)[q, aq/b, aq/c, bcq; q, p]_k}$$

and

$$A_k = \frac{\theta(\alpha p_1 q_1^{2k}; p_1^2)[\alpha p_1, \beta, \gamma, \alpha p_1/\beta\gamma; q_1, p_1^2]_k q_1^k}{\theta(\alpha p_1; p_1^2)[q_1, \alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; q_1, p_1^2]_k}$$

in (1.7) and using (1.10) and (1.12) we get

$$\frac{\theta(aq^{2m}; p)[a, b, c, a/bc; q, p]_m [\alpha p_1 q_1, \alpha p_1 q_1/\beta\gamma, \beta q_1, \gamma q_1; q_1, p_1^2]_n q^m}{\theta(a; p)[q, aq/b, aq/c, bcq; q, p]_m [\alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; q_1, p_1^2]_n} \times$$

$$\sum_{k=-m}^{n} \frac{\theta(aq^{2m+2k}; p)[aq^m, bq^m, cq^m, aq^m/bc; q, p]_k q^k}{\theta(aq^{2m}; p)[q^{1+m}, aq^{1+m}/b, aq^{1+m}/c, bcq^{1+m}; q, p]_k} \times$$

$$\frac{[q_1^{-n}, \beta q_1^{-n}/\alpha p_1, \gamma q_1^{-n}/\alpha p_1, q_1^{-n}/\beta\gamma; q_1, p_1^2]_k}{[q_1^{-n}/\alpha p_1, q_1^{-n}/\beta, q_1^{-n}/\gamma, \beta\gamma q_1^{-n}/\alpha p_1; q_1, p_1^2]_k}$$

$$= \frac{\theta(\alpha p_1 q_1^{2m}; p_1^2)[\alpha p_1, \beta, \gamma, \alpha p_1/\beta\gamma; q_1, p_1^2]_m [aq, bq, cq, aq/bc; q, p]_n q_1^m}{\theta(\alpha p_1; p_1^2)[q_1, \alpha p_1 q_1/\beta, \alpha p_1 q_1/\gamma, \beta\gamma q_1; q_1, p_1^2]_m [q, aq/b, aq/c, bcq; q, p]_n} \times$$

$$\sum_{k=-m}^{n} \frac{\theta(\alpha p_1 q_1^{2m+2k}; p_1^2)[\alpha p_1 q_1^m, \beta q_1^m, \gamma q_1^m, \alpha p_1 q_1^m/\beta\gamma; q_1, p_1^2]_k q_1^k}{\theta(\alpha p_1 q_1^{2m}; p_1^2)[q_1^{1+m}, \alpha p_1 q_1^{1+m}/\beta, \alpha p_1 q_1^{1+m}/\gamma, \beta\gamma q_1^{1+m}; q_1, p_1^2]_k} \times$$

$$\frac{[q^{-n}, bq^{-n}/a, cq^{-n}/a, q^{-n}/bc; q, p]_k}{[q^{-n}/a, q^{-n}/b, q^{-n}/c, bcq^{-n}/a; q, p]_k} \tag{2.1}$$

Next, putting

$$\lambda_k = \frac{\theta(aq^{2k};p)[a,b,c,a/bc;q,p]_k q^k}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_k}$$

and

$$A_k = \frac{\theta\{\alpha(rst/q_1)^k,\beta(r/q_1)^k,\gamma(s/q_1)^k,(\alpha/\beta\gamma)(t/q_1)^k;p_1\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p_1)}\times$$

$$\times\frac{[\alpha;rst/q_1^2,p_1]_k[\beta;r,p_1]_k[\gamma;s,p_1]_k[\alpha/\beta\gamma;t,p_1]_k q_1^k}{[q_1;q_1,p_1]_k[\alpha st/\beta q_1;st/q_1,p_1]_k[\alpha rt/\gamma q_1;rt/q_1,p_1]_k[\beta\gamma rs/q_1;rs/q_1,p_1]_k}$$

in (1.7) and using (1.10) and (1.14) we get

$$\frac{\theta(aq^{2m};p)[a,b,c,a/bc;q,p]_m[\alpha rst/q_1^2;rst/q_1^2,p_1]_n[\beta r;r,p_1]_n}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_m[q_1;q_1,p_1]_n[\alpha st/\beta q_1;st/q_1,p_1]_n}\times$$

$$\times\frac{[\gamma s;s,p_1]_n[\alpha t/\beta\gamma;t,p_1]_n q^m}{[\alpha rt/\gamma q_1;rt/q_1,p_1]_n[\beta\gamma rs/q_1;rs/q_1,p_1]_n}$$

$$\times\sum_{k=-m}^{n}\frac{\theta(aq^{2m+2k};p)[aq^m,bq^m,cq^m,aq^m/bc;q,p]_k q^k}{\theta(aq^{2m};p)[q^{1+m},aq^{1+m}/b,aq^{1+m}/c,bcq^{1+m};q,p]_k}\times$$

$$\frac{[q_1^{-n};q_1,p_1]_k[\beta(st/q_1)^{-n}/\alpha;st/q_1,p_1]_k}{[(rst/q_1^2)^{-n}/\alpha;rst/q_1^2,p_1]_k[r^{-n}/\beta;r,p_1]_k}\times$$

$$\frac{[\gamma(rt/q_1)^{-n}/\alpha;rt/q_1,p_1]_k[(rs/q_1)^{-n}/\beta\gamma;rs/q_1,p_1]_k}{[s^{-n}/\gamma;s,p_1]_k[\beta\gamma t^{-n}/\alpha;t,p_1]_k}$$

$$=\frac{[\alpha;rst/q_1^2,p_1]_m[\beta;r,p_1]_m[\gamma;s,p_1]_m[\alpha/\beta\gamma;t,p_1]_m q_1^m}{[q_1;q_1,p_1]_m[\alpha st/\beta q_1;st/q_1,p_1]_m[\alpha rt/\gamma q_1;rt/q_1,p_1]_m[\beta\gamma rs/q_1;rs/q_1,p_1]_m}\times$$

$$\frac{[aq,bq,cq,aq/bc;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n}\times$$

$$\sum_{k=-m}^{n}\frac{\theta\{\alpha(rst/q_1)^{k+m},\beta(r/q_1)^{k+m},\gamma(s/q_1)^{k+m},(\alpha/\beta\gamma)(t/q_1)^{k+m};p_1\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p_1)}\times$$

$$\frac{[\alpha(rst/q_1^2)^m;rst/q_1^2,p_1]_k[\beta r^m;r,p_1]_k[\gamma s^m;s,p_1]_k}{[q_1^{1+m};q_1,p_1]_k[\alpha(st/q_1)^{1+m}/\beta;st/q_1,p_1]_k[\alpha(rt/q_1)^{m+1}/\gamma;rt/q_1,p_1]_k}\times$$

$$\frac{[\alpha t^m/\beta\gamma;t,p_1]_k[q^{-n},bq^{-n}/a,cq^{-n}/a,q^{-n}/bc;q,p]_k}{[\beta\gamma(rs/q_1)^{m+1};rs/q_1,p_1]_k[q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a;q,p]_k}. \tag{2.2}$$

Next, if we put

$$\lambda_k = \frac{\theta(apq^{2k};p^2)[ap,b,c,ap/bc;q,p^2]_k q^k}{\theta(ap;p^2)[q,apq/b,apq/c,bcq;q,p^2]_k}$$

and

$$A_k = \frac{\theta\{\alpha(rst/q_1)^k, \beta(r/q_1)^k, \gamma(s/q_1)^k, (\alpha/\beta\gamma)(t/q_1)^k;p_1\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p_1)}\times$$

$$\times\frac{[\alpha;rst/q_1^2,p_1]_k[\beta;r,p_1]_k[\gamma;s,p_1]_k[\alpha/\beta\gamma;t,p_1]_k q_1^k}{[q_1;q_1,p_1]_k[\alpha st/\beta q_1;st/q_1,p_1]_k[\alpha rt/\gamma q_1;rt/q_1,p_1]_k[\beta\gamma rs/q_1;rs/q_1,p_1]_k}$$

in (1.7) and using (1.12) and (1.14) we get

$$\frac{\theta(apq^{2m};p^2)[ap,b,c,ap/bc;q,p^2]_m q^m}{\theta(ap;p^2)[q,apq/b,apq/c,bcq;q,p^2]_m}\times$$

$$\times\frac{[\alpha rst/q_1^2;rst/q_1^2,p_1]_n[\beta r;r,p_1]_n[\gamma s;s,p_1]_n[\alpha t/\beta\gamma;t,p_1]_n}{[q_1;q_1,p_1]_n[\alpha st/\beta q_1;st/q_1,p_1]_n[\alpha rt/\gamma q_1;rt/q_1,p_1]_n[\beta\gamma rs/q_1;rs/q_1,p_1]_n}$$

$$\times\sum_{k=-m}^{n}\frac{\theta(apq^{2m+2k};p^2)[apq^m,bq^m,cq^m,apq^m/bc;q,p]_k q^k}{\theta(apq^{2m};p^2)[q^{1+m},apq^{1+m}/b,apq^{1+m}/c,bcq^{1+m};q,p]_k}\times$$

$$\frac{[q_1^{-n};q_1,p_1]_k[\beta(st/q_1)^{-n}/\alpha;st/q_1,p_1]_k}{[(rst/q_1^2)^{-n}/\alpha;rst/q_1^2,p_1]_k[r^{-n}/\beta;r,p_1]_k}\times$$

$$\frac{[\gamma(rt/q_1)^{-n}/\alpha;rt/q_1,p_1]_k[(rs/q_1)^{-n}/\beta\gamma;rs/q_1,p_1]_k}{[s^{-n}/\gamma;s,p_1]_k[\beta\gamma t^{-n}/\alpha;t,p_1]_k}$$

$$=\frac{[\alpha;rst/q_1^2,p_1]_m[\beta;r,p_1]_m[\gamma;s,p_1]_m[\alpha/\beta\gamma;t,p_1]_m q_1^m}{[q_1;q_1,p_1]_m[\alpha st/\beta q_1;st/q_1,p_1]_m[\alpha rt/\gamma q_1;rt/q_1,p_1]_m[\beta\gamma rs/q_1;rs/q_1,p_1]_m}\times$$

$$\frac{[aq,bq,cq,aq/bc;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n}\times$$

$$\sum_{k=-m}^{n}\frac{\theta\{\alpha(rst/q_1)^{k+m},\beta(r/q_1)^{k+m},\gamma(s/q_1)^{k+m},(\alpha/\beta\gamma)(t/q_1)^{k+m};p_1\}}{\theta(\alpha,\beta,\gamma,\alpha/\beta\gamma;p_1)}\times$$

$$\frac{[\alpha(rst/q_1^2)^m;rst/q_1^2,p_1]_k[\beta r^m;r,p_1]_k[\gamma s^m;s,p_1]_k}{[q_1^{1+m};q_1,p_1]_k[\alpha(st/q_1)^{1+m}/\beta;st/q_1,p_1]_k[\alpha(rt/q_1)^{m+1}/\gamma;rt/q_1,p_1]_k}\times$$

$$\frac{[\alpha t^m/\beta\gamma;t,p_1]_k[q^{-n},bq^{-n}/ap,cq^{-n}/ap,q^{-n}/bc;q,p^2]_k}{[\beta\gamma(rs/q_1)^{m+1};rs/q_1,p_1]_k[q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a;q,p]_k}. \tag{2.3}$$

3. Special Cases

In this section we shall deduce certain interesting special cases of our results. If we set $r = s = t = q_1$ in (2.2), we get after some simplification

$$\frac{\theta(aq^{2m};p)[a,b,c,a/bc;q,p]_m[\alpha q_1/\beta\gamma,\beta q_1,\gamma q_1;q_1,p_1]_n q^m}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_m[\alpha q_1/\beta,\alpha q_1/\gamma,\beta\gamma q_1;q_1,p_1]_n}\times$$
$$\sum_{k=-m}^{n}\frac{\theta(aq^{2m+2k};p)[aq^m,bq^m,cq^m,aq^m/bc;q,p]_k q^k}{\theta(aq^{2m};p)[q^{1+m},aq^{1+m}/b,aq^{1+m}/c,bcq^{1+m};q,p]_k}\times$$
$$\frac{[q_1^{-n},\beta q_1^{-n}/\alpha,\gamma q_1^{-n}/\alpha,q_1^{-n}/\beta\gamma;q_1,p_1]_k}{[q_1^{-n}/\alpha,q_1^{-n}/\beta,q_1^{-n}/\gamma,\beta\gamma q_1^{-n}/\alpha;q_1,p_1]_k}$$
$$=\frac{[\alpha,\beta,\gamma,\alpha/\beta\gamma,q_1,p_1]_m[aq,bq,cq,aq/bc;q,p]_n q_1^m}{[q_1,\alpha q_1/\beta,\alpha q_1/\gamma,\beta\gamma q_1;q_1,p_1]_m[q,aq/b,aq/c,bcq;q,p]_n}\times$$
$$\sum_{k=-m}^{n}\frac{\theta(\alpha q_1^{2m+2k};p_1)[\alpha q_1^m,\beta q_1^m,\gamma q_1^m,\alpha q_1^m/\beta\gamma;q_1,p_1]_k q_1^k}{\theta(\alpha;p_1)[q_1^{1+m},\alpha q_1^{1+m}/\beta,\alpha q_1^{1+m}/\gamma,\beta\gamma q_1^{1+m};q_1,p_1]_k}\times$$
$$\frac{[q^{-n},bq^{-n}/a,cq^{-n}/a,q^{-n}/bc;q,p]_k}{[q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a;q,p]_k} \tag{3.1}$$

Now, setting $\alpha = \beta\gamma$ in (3.1), we get

$$\sum_{k=0}^{n}\frac{\theta(aq^{2k};p)[a,b,c,a/bc;q,p]_k}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_k}=\frac{[aq,aq/bc,bq,cq;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n} \tag{3.2}$$

which is (1.9)

Again, if we take $\alpha = \beta\gamma q_1$ in (3.1) we get

$$\frac{\theta(aq^{2m};p)[a,b,c,a/bc;q,p]_m[\beta q_1,\gamma q_1,\beta\gamma q_1^{1+n},q_1^{1+n};p_1]_m q^m}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_m[q_1,\beta\gamma q_1,\beta q_1^{1+n},\gamma q_1^{1+n};p_1]_n}\times$$
$$\sum_{k=-m}^{n}\frac{\theta(aq^{2m+2k};p)[aq^m,bq^m,cq^m,aq^m/bc;q,p]_k q^k}{\theta(aq^{2m};p)[q^{1+m},aq^{1+m}/b,aq^{1+m}/c,bcq^{1+m};q,p]_k}\times$$
$$\frac{[q_1^{-n},q_1^{-n-1}/\beta,q_1^{-n-1}/\gamma,q_1^{-n}/\beta\gamma;q_1,p_1]_k}{[q_1^{-n-1}/\beta\gamma,q_1^{-n}/\beta,q_1^{-n}/\gamma,q_1^{-n-1};q_1,p_1]_k}$$
$$=\frac{[\beta,\gamma;q_1,p_1]_m[aq,bq,cq,aq/bc;q,p]_n q_1^m}{[\gamma q_1^2,\beta q_1^2;q_1,p_1]_m[q,aq/b,aq/c,bcq;q,p]_n}\times$$

$$\sum_{k=-m}^{n} \frac{\theta(\beta\gamma q_1^{2m+2k+1};p_1)[\beta q_1^m,\gamma q_1^m;q_1,p_1]_k q_1^k}{\theta(\beta\gamma;p_1)[\beta q_1^{2+m},\gamma q_1^{2+m};q_1,p_1]_k}\times$$
$$\frac{[q^{-n},bq^{-n}/a,cq^{-n}/a,q^{-n}/bc;q,p]_k}{[q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a;q,p]_k} \tag{3.3}$$

Next, if we put m=0 in (3.1) we get

$$\frac{[\alpha q_1,\beta q_1,\gamma q_1,\alpha q_1/\beta\gamma;q_1,p_1]_n}{[q_1,\alpha q_1/\beta,\alpha q_1/\gamma,\beta\gamma q_1;q_1,p_1]_n}$$

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)[a,b,c,a/bc;q,p]_k[q_1^{-n},\beta q_1^{-n}/\alpha,\gamma q_1^{-n}/\alpha,q_1^{-n}/\beta\gamma;q_1,p]_k q^k}{\theta(a;p)[q,aq/b,aq/c,bcq;q,p]_k[q_1^{-n}/\alpha,q_1^{-n}/\beta,q_1^{-n}/\gamma,\beta\gamma q_1^{-n}/\alpha;q_1,p_1]_k}$$

$$=\frac{[aq,bq,cq,aq/bc;q,p]_n}{[q,aq/b,aq/c,bcq;q,p]_n}\times$$

$$\sum_{k=0}^{n} \frac{\theta(aq_1^{2k};p_1)[\alpha,\beta,\gamma,\alpha/\beta\gamma;q_1,p_1]_k[q^{-n},bq^{-n}/a,cq^{-n}/a,q^{-n}/bc;q,p]_k q_1^k}{\theta(a;p_1)[q_1,\alpha q_1/\beta,\alpha q_1/\gamma,\beta\gamma q_1;q_1,p_1]_k[q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a;q,p]_k} \tag{3.4}$$

If we set $p=p_1=0$ and $q_1=q$ in (3.4), we get the following interesting transformation

$${}_{10}\Phi_9\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,c,a/bc,\beta q^{-n}/\alpha,\gamma q^{-n}/\alpha,q^{-n}/\beta\gamma,q^{-n};q;q \\ \sqrt{a},-\sqrt{a},aq/b,aq/c,bcq,q^{-n}/\alpha,q^{-n}/\beta,q^{-n}/\gamma,\beta\gamma q^{-n}/\alpha \end{matrix}\right]$$

$$=\frac{[aq,bq,aq/bc,cq,\alpha q/\beta,\alpha q/\gamma,\beta\gamma q;q]_n}{[aq/b,aq/c,bcq,\alpha q,\beta q,\gamma q,\alpha q/\beta\gamma;q]_n}\times$$

$${}_{10}\Phi_9\left[\begin{matrix} \alpha,q\sqrt{\alpha},-q\sqrt{\alpha},\beta,\gamma,\alpha/\beta\gamma,bq^{-n}/a,cq^{-n}/a,q^{-n}/bc,q^{-n};q;q \\ \sqrt{\alpha},-\sqrt{\alpha},\alpha q/\beta,\alpha q/\gamma,\beta\gamma q,q^{-n}/a,q^{-n}/b,q^{-n}/c,bcq^{-n}/a \end{matrix}\right] \tag{3.5}$$

Now, letting $\beta\to 1$ in (3.5) we get the following summation of a truncated very well poised ${}_6\Phi_5$,

$${}_6\Phi_5\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,c,a/bc;q;q \\ \sqrt{a},-\sqrt{a},aq/b,aq/c,bcq \end{matrix}\right]_n=\frac{[aq,bq,aq/bc,cq;q]_n}{[q,aq/b,aq/c,bcq;q]_n} \tag{3.6}$$

It is evident that several other interesting results involving theta hypergeometric functions can be established.

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