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POLYNOMIALS YIELDING QUADRUPLES WITH PROPERTY D(k)

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Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: Let k be a natural number. Two integers α and β are said to have the property D(k) (resp. D(-k)) if $\alpha\beta$+k (resp. $\alpha\beta$−k) is a perfect square. The purpose of this paper is identification of polynomials producing quadruples with property D(k) for certain values of k. Incidentally the paper brings out an attribute of Ramanujan number 1729 in contributing two quadruples of polynomials with property D(k).

Keyword and Phrases: Property p_k, extendable set, $P_{r,k}$ sequence, Pell's equation, quadruple with Diophantine property, Ramanujan number.

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1. Introduction

The Greek mathematician Diophantus raised the question as to four numbers such that the product of any two increased by a given number shall be a square. M.Gardner [11] asked for a fifth number that can be added to the set $\{1, 3, 8, 120\}$ without destroying the property that the product of any two integers is one less than a perfect square. For historical details of the problem, one may refer to J.Roberts [24] and the author [19].

It is seen that the polynomials x, x+2, 4x+4 have the property that the product of any two of them increased by 1 is a square. A fourth polynomial that works with these three is $16x^3 + 48x^2 + 44x + 12$. B.W.Jones [12, 13] considered polynomials for this problem. He found all polynomials that work with x and x+2. He defined

$$f_r(x) = \frac{\alpha^r - \beta^r}{\alpha - \beta} \text{ and } c_r(x) \ = 2 f_r(x)\ f_{r+1}(x),$$

where $\alpha(= \alpha(x))$ and $\beta(= \beta(x))$ are the roots of the quadratic equation

$$w^2 - 2(x+1)w + 1 = 0.$$

He then showed that

$$x,\ x+2, c_r(x), c_{r+1}(x)$$

are such that the product of any two of them is one less than a square. Taking $x = r = 1$, we find $\alpha = 2 + \sqrt{3}$, $\beta = 2 - \sqrt{3}$, $f_1(1) = 1$, $f_2(1) = 4$, $f_3(1) = 15$, $c_1(1) = 8$, $c_2(1) = 120$. Thus, in this case, the quadruple $\{x, x+2, c_1(1), c_2(1)\}$ becomes $\{1, 3, 8, 120\}$.

2. Definitions

Let N denote the set of all natural numbers. We recall the definitions furnished in [16].

Definition 2.1 *Let k be a given element of N. Two integers α and β are said to have the property p_k (resp. p_{-k}) if $\alpha\beta$+k (resp. $\alpha\beta$-k) is a perfect square.*

Definition 2.2 *Let k be a given element of N. A set S of elements of N is said to be a P_k set or a set with property p_k if any two elements α and β in S have the property p_k. A set with property p_k is also referred to as a set with property D(k) where D stands for Diophantus from whom the problem originated.*

Definition 2.3 *A P_k set S is called extendable if, for some integer d, $d \notin S$, the set $S \cup \{d\}$ is a P_k set.*

Definition 2.4 *A sequence of integers is said to be a $P_{r,k}$ sequence if every r consecutive terms of the sequence constitute a P_k set. For the construction of a $P_{3,k}$ sequence of integers, one may refer [16].*

Following the concept of J. Arkin, V.E. Hoggatt and E.G. Strauss [1], we have a classification of triples and quadruples with property D(k) as detailed below:

Definition 2.5 *A triple $\{a_1, a_2, a_3\}$ with property D(k), where $a_1 < a_2 < a_3$ is said to be regular if*

$$(a_3 - a_2 - a_1)^2 = 4(a_1 a_2 + k).$$

Definition 2.6 *A quadruple $\{a_1, a_2, a_3, a_4\}$ with property D(k), where $a_1 < a_2 < a_3 < a_4$ is said to be regular if*

$$k(a_4 + a_3 - a_2 - a_1)^2 = 4(a_1 a_2 + k)(a_3 a_4 + k).$$

It is seen that a regular quadruple with property D(k) can exist only if k is a square. A quadruple with property D(k) which is not regular is said to be irregular.

3. Linkage with Pell's equation

Starting from a triple with property D(k), one would like to extend it to a quadruple with property D(k). In this process, there arises a system of Pell's equations as illustrated in several papers (see for e.g., [2, 3, 4, 5, 8, 9, 13, 14, 15, 16, 17, 18, 20, 22, 23]).

4. Overview of previous results

A. Baker and H. Davenport [2] dealt with the quadruple $\{1, 3, 8, 120\}$ with property D(1) in which the first three terms are the Fibonacci numbers F_2, F_4 and F_6. They proved that the set cannot be extended further. Their result was also proved later by P. Kanagasabapathy and T. Ponnudurai [14] employing a different method.

Let us consider the previous works pertaining to polynomials producing P_k sets.

Theorem 4.1. *(S.P. Mohanty and the Author [16]) If* $k \equiv 2 \pmod 4$*, then there is no* $P_{r,k}$ *sequence with* $r \geq 4$.

Theorem 4.2. *(A. Filipin and Y.Fujita [10]) Any polynomial quadruple with property D(4) is regular.*

Several results on polynomials for sets of numbers with Diophantine property have been furnished by A.Dujella in [3, 5, 6, 7].

5. Special kind of polynomials producing quadruples with Diophantine property

Consider the following sequences $\{a_n\}$ with given k indicated against each: (the author [18])

i. $\{2t+1, 11t^2+6t+1, 33t^2+16t+2, 88t^2+42t+5, ...\}$ with $k = 121t^4+66t^3-7t^2-8t-1$, $t \geq 1$

ii. $\{2t+1, 33t^2+28t+6, 99t^2+82t+17, 264t^2+218t+45, ...\}$ with $k = 1089t^4+1650t^3+917t^2+220t+19$, $t \geq 0$

iii. $\{2t+3, 3t^2+4t+2, 9t^2+10t+3, 24t^2+26t+7, ...\}$ with $k = 9t^4+6t^3-19t^2-20t-5$, $t \geq 1$

iv. $\{2t+7, t^2+2t+3, 3t^2+4t+2, 8t^2+10t+3, ...\}$ with $k = t^4-2t^3-19t^2-20t-5$, $t \geq 1$.

One can check that the set consisting of the first three terms of the sequence $\{a_n\}$ has the property D(k). The set consisting of the second, third and fourth terms of the sequence also has the property D(k). The terms of the sequence are defined by the relation

$$a_{n+2} = 3a_{n+1} - a_n$$

for n $\geq$ 1. It is ascertained that each one of these sequences $\{a_n\}$ is a $P_{3,k}$ sequence. A special property of these sequences is that the fourth term also shares the property D(k) with the first term.

6. Theorems on polynomials

In the following theorems we give some polynomials which produce 4 numbers possessing the property D(k) for some integer k.

Theorem 6.1. *Let t be a given integer* ≥ 2. *Let* d_1, d_2 *be two positive divisors of* $t^2 - 1$ *such that* $d_1 < d_2$ *and* $t^2 - 1 = d_1 d_2$. *Then the numbers* $\{d_1, d_2, d_1 + d_2 + 2t, 4t(d_1 + t)(d_2 + t)\}$ *have the property D(1).*

Proof. We have $d_1 d_2 + 1 = t^2, d_1(d_1 + d_2 + 2t) + 1 = (t + d_1)^2$,
$d_1.4t(d_1 + t)(d_2 + t) + 1 = (2t^2 + 2d_1 t - 1)^2$,
$d_2(d_1 + d_2 + 2t) + 1 = (t + d_2)^2, d_2.4t(d_1 + t)(d_2 + t) + 1 = (2t^2 + 2d_2 t - 1)^2$,
$(d_1 + d_2 + 2t).4t(d_1 + t)(d_2 + t) + 1 = (4t^2 + 2(d_1 + d_2)t - 1)^2$.

Corollary 6.2. *The following are sets with property D(1) for all integers* $t \geq 0$:
$\{t + 1, t + 3, 4t + 8, 16t^3 + 96t^2 + 128t + 120\}$,
$\{1, t^2 + 4t + 3, t^2 + 6t + 8, 4t^4 + 40t^3 + 144t^2 + 220t + 120\}$,
$\{t + 2, 4t + 4, 9t + 12, 144t^3 + 624t^2 + 892t + 420\}$.

Theorem 6.3. *Let t be a given integer* ≥ 3. *Let* d_1, d_2 *be two positive divisors of* $t^2 - 4$ *such that* $d_1 < d_2$ *and* $t^2 - 4 = d_1 d_2$. *Then* $d_1, d_2, d_1 + d_2 + 2t, t(d_1 + t)(d_2 + t)$ *have the property D(4).*

Proof. It is seen that
$d_1 d_2 + 4 = t^2, d_1(d_1 + d_2 + 2t) + 4 = (t + d_1)^2$,
$d_1 t(d_1 + t)(d_2 + t) + 4 = (2t^2 + 2d_1 t - 1)^2$,
$d_2(d_1 + d_2 + 2t) + 4 = (t + d_2)^2$,
$d_2 t(d_1 + t)(d_2 + t) + 4 = (2t^2 + 2d_2 t - 1)^2$,
$(d_1 + d_2 + 2t).t(d_1 + t)(d_2 + t) + 4 = (4t^2 + 2(d_1 + d_2)t - 1)^2$.

Corollary 6.4. $\{1, t^2 + 6t + 5, t^2 + 8t + 12, t^4 + 14t^3 + 69t^2 + 140t + 96\}$ *and* $\{2t + 1, 2t + 5, 8t + 12, 32t^3 + 144t^2 + 208t + 96\}$ *are sets with property D(4) where* $t \geq 0$.

This is a generalization of the result contained in [17] regarding the quadruple $\{1, 5, 12, 96\}$ *with property D(4).*

Theorem 6.5. *The set* $\{1, 4t^4 + 24t^3 + 52t^2 + 48t + 16, 4t^6 + 36t^5 + 133t^4 + 258t^3 +$

$273t^2+144t+27, 16t^8+192t^7+1000t^6+2952t^5+5393t^4+6222t^3+4399t^2+1722t+280\}$ *has the property* $D((2t+3)^2)$ *for* $t \geq 0$.

Theorem 6.6. *The numbers* $\{1, 5(t+1)^2, 5t^2+16t+12, 45t^2+132t+96\}$, $t \geq 0$, *have the property* $D(4(t+1)^2)$.
This is yet another generalization of the result in [17] concerning the quadruple $\{1, 5, 12, 96\}$ *with property D(4).*

Theorem 6.7. *The numbers* $\{1, (m^2+2m)t^2, (m^2+2m)t^2+(2m+2)t+1, (4m^4+16m^3+20m^2+8m)t^2+(8m^3+24m^2+20m+4)t+4m^2+8m+4\}$ *have the property* $D(t^2)$ *where* $m, t \geq 1$.

Corollary 6.8. *Given a natural number t, there exist an infinite number of sets of size 4 with property* $D(t^2)$.

Theorem 6.9 *Let t be any given integer* ≥ 0. *Then the numbers* $\{(t+1)^2, 2t^3+7t^2+8t+3, 4t^3+16t^2+20t+8, 12t^3+80t^2+172t+120\}$ *have the property* $D((t+1)^6)$. *This is a generalization of the result contained in [2] regarding the quadruple* $\{1, 3, 8, 120\}$ *with property D(1).*

7. Polynomials producing quadruples with property $D(k)$, $k = 1, 4, 16, 64, 256$

Identification of polynomials that would lead to quadruples with property D(k) is an interesting as well as a challenging task. In this section, we consider the case when k is a square.
Let us recall the following result of Euler: If $a_1a_2+\lambda^2 = b^2$ for some $\lambda \in N$, then $\left\{a_1, a_2, a_1+a_2+2b, \frac{4b(a_1+b)(a_2+b)}{\lambda^2}\right\}$ is a quadruple with property $D(\lambda^2)$.
This result is used to construct quadruples consisting of polynomials with properties D(1), D(4), D(16), D(64) and D(256) which are presented in the sequel.

Theorem 7.1. *Let t be any integer* ≥ 0. *The following polynomial quadruples have property D(1):*
$\{4t+4, 16t+12, 36t+30, 9216t^3+23808t^2+20464t+5852\}$,
$\{3t^2+4t+1, 3t^2+10t+8, 12t^2+28t+15, 432t^6+3024t^5+8460t^4+12040t^3+9144t^2+3500t+528\}$,
$\{t^2+3t+2, t^2+7t+12, 4t^2+20t+24, 16t^6+240t^5+1456t^4+4560t^3+7756t^2+6780t+2380\}$,
Proof. Consider the quadruple $\{4t+4, 16t+12, 36t+30, 9216t^3+23808t^2+20464t+5852\}$.
We observe that $(4t+4).(16t+12)+1 = (8t+7)^2$,
$(4t+4).(36t+30)+1 = (12t+11)^2$,

$(4t+4).(9216t^3+23808t^2+20464t+5852)+1=(192t^2+344t+153)^2$,
$(16t+12).(36t+30)+1=(24t+19)^2$,
$(16t+12).(9216t^3+23808t^2+20464t+5852)+1=(384t^2+640t+265)^2$,
$(36t+30).(9216t^3+23808t^2+20464t+5852)+1=(576t^2+984t+419)^2$.
Therefore it follows that $\{4t+4, 16t+12, 36t+30, 9216t^3+23808t^2+20464t+5852\}$ is a P_1 set of size 4. Other results can be checked similarly.

Theorem 7.2. *The following polynomial quadruples have property D(4) for* $t \geq 0$*:*
$\{2t+1, 2t+5, 8t+12, 32t^3+144t^2+208t+96\}$,
$\{2t+5, 8t+12, 18t+33, 288t^3+1680t^2+3248t+2080\}$,
$\{5t^2+6t+1, 5t^2+16t+12, 20t^2+44t+21, 500t^6+3300t^5+8585t^4+11154t^3+7553t^2+2508t+320\}$.
Proof. Take the quadruple $\{2t+1, 2t+5, 8t+12, 32t^3+144t^2+208t+96\}$.
We have
$(2t+1).(2t+5)+4=(2t+3)^2$,
$(2t+1).(8t+12)+4=(4t+4)^2$,
$(2t+1).(32t^3+144t^2+208t+96)+4=(8t^2+20t+10)^2$,
$(2t+5).(8t+12)+4=(4t+8)^2$,
$(2t+5).(32t^3+144t^2+208t+96)+4=(8t^2+28t+22)^2$,
$(8t+12).(32t^3+144t^2+208t+96)+4=(16t^2+48t+34)^2$.
Consequently, it is seen that $\{2t+1, 2t+5, 8t+12, 32t^3+144t^2+208t+96\}$ is a P_4 set of size 4. Other results follow similarly.

Theorem 7.3. *The following polynomial quadruples have property D(16) for* $t \geq 0$*:*
$\{2t+1, 2t+9, 8t+20, 8t^3+60t^2+142t+105\}$,
$\{2t+9, 8t+20, 18t+57, 72t^3+732t^2+2462t+2737\}$,
$\{12t^2+20t+3, 12t^2+32t+16, 48t^2+104t+35, 1728t^6+11232t^5+27900t^4+33020t^3+19131t^2+5200t+528\}$.
Proof. Consider the quadruple $\{2t+1, 2t+9, 8t+20, 8t^3+60t^2+142t+105\}$.
On computation, we get
$(2t+1).(2t+9)+16=(2t+5)^2$,
$(2t+1).(8t+20)+16=(4t+6)^2$,
$(2t+1).(8t^3+60t^2+142t+105)+16=(4t^2+16t+11)^2$,
$(2t+9).(8t+20)+16=(4t+14)^2$,
$(2t+9).(8t^3+60t^2+142t+105)+16=(4t^2+24t+31)^2$,
$(8t+20).(8t^3+60t^2+142t+105)+16=(8t^2+40t+46)^2$.
Hence it follows that $\{2t+1, 2t+9, 8t+20, 8t^3+60t^2+142t+105\}$ is a P_{l6} set of size 4. Other results can be derived similarly.

Theorem 7.4. *The following polynomial quadruples have property D(64) for* $t \geq 0$*:*

$\{12t^2+20t+3, 12t^2+44t+35, 48t^2+128t+64, 432t^6+3456t^5+10728t^4+16256t^3+12459t^2+4552t+624\}$,
$\{80t^2+96t+16, 80t^2+136t+45, 320t^2+464t+117, 128000t^6+556800t^5+943760t^4+785784t^3+334901t^2+69774t+5621\}$,
$\{20t^2+44t+21, 20t^2+84t+85, 80t^2+256t+192, 2000t^6+19200t^5+74840t^4+151296t^3+167021t^2+95376t+22016\}$.

Proof: Take the quadruple
$\{12t^2+20t+3, 12t^2+44t+35, 48t^2+128t+64, 432t^6+3456t^5+10728t^4+16256t^3+12459t^2+4552t+624\}$.
It is observed that
$(12t^2+20t+3).(12t^2+44t+35)+64=(12t^2+32t+13)^2$,
$(12t^2+20t+3).(48t^2+128t+64)+64=(24t^2+52t+16)^2$,
$(12t^2+20t+3).(432t^6+3456t^5+10728t^4+16256t^3+12459t^2+4552t+624)+64=(72t^4+348t^3+542t^2+297t+44)^2$,
$(12t^2+44t+35).(48t^2+128t+64)+64=(24t^2+76t+48)^2$,
$(12t^2+44t+35).(432t^6+3456t^5+10728t^4+16256t^3+12459t^2+4552t+624)+64=(72t^4+420t^3+830t^2+631t+148)^2$,
$(48t^2+128t+64).(432t^6+3456t^5+10728t^4+16256t^3+12459t^2+4552t+624)+64=(144t^4+768t^3+1372t^2+928t+200)^2$.

Theorem 7.5. *The following polynomial quadruples have property D(256) with* $t \geq 0$*:*
$\{320t^2+304t+21, 320t^2+384t+64, 1280t^2+1376t+165, 2048000t^6+6604800t^5+7876160t^4+4212624t^3+993485t^2+103974t+3965\}$,
$\{48t^2+104t+35, 48t^2+152t+99, 192t^2+512t+256, 6912t^6+55296t^5+174240t^4+273920t^3+224859t^2+91720t+14640\}$,
$\{16t^2+88t+57, 16t^2+104t+105, 64t^2+384t+320, 256t^6+4608t^5+31456t^4+100992t^3+155905t^2+112902t+30889\}$.

Proof. Consider the quadruple $\{320t^2+304t+21, 320t^2+384t+64, 1280t^2+1376t+165, 2048000t^6+6604800t^5+7876160t^4+4212624t^3+993485t^2+103974t+3965\}$.
We observe that
$(320t^2+304t+21).(320t^2+384t+64)+256=(320t^2+344t+40)^2$,
$(320t^2+304t+21).(1280t^2+1376t+165)+256=(640t^3+648t+61)^2$,
$(320t^2+304t+21).(2048000t^6+6604800t^5+7876160t^4+4212624t^3+993485t^2+103974t+3965)+256=(25600t^4+53440t^3+33504t^2+5863t+289)^2$,
$(320t^2+384t+64).(1280t^2+1376t+165)+256=(640t^2+728t+104)^2$,
$(320t^2+384t+64).(2048000t^6+6604800t^5+7876160t^4+4212624t^3+993485t^2+103974t+3965)+256=(256001t^4+56640t^3+38664t^2+8112t+504)^2$,
$(1280t^2+1376t+165)(2048000t^6+6604800t^5+7876160t^4+4212624t^3+993485t^2+$

$103974t + 3965) + 256 = (51200t^4 + 110080t^3 + 72168t^2 + 13975t + 809)^2$.
Therefore we see that
$\{320t^2 + 304t + 21, 320t^2 + 384t + 64, 1280t^2 + 1376t + 165, 2048000t^6 + 6604800t^5 + 7876160t^4 + 4212624t^3 + 993485t^2 + 103974t + 3965\}$ is a P_{256} set of size 4. Other results are got similarly.

8. Ramanujan's number in Diophantine quadruples

The number 1729 has acquired a special status in mathematics. It is referred to as Ramanujan number. There is a famous anecdote about this number. Ramanujan made a statement to G. H. Hardy that 1729 is the smallest number that can be expressed as a sum of two cubes in two different ways. We have the two expressions $1729 = 9^3 + 10^3$ and $1729 = 1^3 + 12^3$. Thus 1729 is the smallest integral solution of the Diophantine equation $A^3 + B^3 = C^3 + D^3$ (see for e.g., [21]).

From our work we observe another property of this number, namely it shares property D(64) with 9, 33 and 80. This leads to two quadruples of polynomials with property D(k) as brought out in the following:

Theorem 8.1. *The quadruple* $\{t + 9, 9t + 33, 16t + 80, 9t^3 + 159t^2 + 919t + 1729\}$ *has the property D(64),* $t \geq 0$.
Proof. We have
$(t + 9).(9t + 33) + 64 = (3t + 19)^2$,
$(t + 9).(16t + 80) + 64 = (4t + 28)^2$,
$(t + 9).(9t^3 + 159t^2 + 919t + 1729) + 64 = (3t^2 + 40t + 125)^2$,
$(9t + 33).(16t + 80) + 64 = (12t + 52)^2$,
$(9t + 33).(9t^3 + 159t^2 + 919t + 1729) + 64 = (9t^2 + 96t + 239)^2$,
$(16t + 80).(9t^3 + 159t^2 + 919t + 1729) + 64 = (12t^2 + 136t + 372)^2$,

Theorem 8.2. *The quadruple* $\{4t^2 + 20t + 9, 4t^2 + 28t + 33, 16t^2 + 96t + 80, 16t^6 + 288t^5 + 1960t^4 + 6240t^3 + 9457t^2 + 6630t + 1729\}$ *has property D(64),* $t \geq 0$.
Proof. It is seen that
$(4t^2 + 20t + 9).(4t^2 + 28t + 33) + 64 = (4t^2 + 24t + 19)^2$,
$(4t^2 + 20t + 9).(16t^2 + 96t + 80) + 64 = (8t^2 + 44t + 28)^2$,
$(4t^2 + 20t + 9).(16t^6 + 288t^5 + 1960t^4 + 6240t^3 + 9457t^2 + 6630t + 1729) + 64 = (8t^4 + 92t^3 + 330t^2 + 377t + 125)^2$,
$(4t^2 + 28t + 33).(16t^2 + 96t + 80) + 64 = (8t^2 + 52t + 52)^2$,
$(4t^2 + 28t + 33).(16t^6 + 288t^5 + 1960t^4 + 6240t^3 + 9457t^2 + 6630t + 1729) + 64 = (8t^4 + 100t^3 + 402t^2 + 559t + 239)^2$,
$(16t^2 + 96t + 80).(16t^6 + 288t^5 + 1960t^4 + 6240t^3 + 9457t^2 + 6630t + 1729) + 64 = (16t^4 + 192t^3 + 732t^2 + 936t + 372)^2$.

9. Polynomials producing quadruples with property $D(k)$, k being a square

Now we consider sets with property D(k) where k is a polynomial taking square values.

Theorem 9.1. *The quadruple* $\{1, 4t^2+12t+9, 4t^6+36t^5+129t^4+234t^3+225t^2+108t+20, 4t^6+44t^5+197t^4+460t^3+590t^2+392t+105\}$ *has property* $D((2t^2+6t+4)^2)$ *for* $t \geq 0$.

Theorem 9.2. *The quadruple* $\{1, 36t^2+32t+7, 104976t^6+174960t^5+120528t^4+43920t^3+8892t^2+940t+40, 104976t^6+244944t^5+229392t^4+110448t^3+28932t^2+3924t+216\}$ *has property* $D((18t^2+16t+3)^2)$, $t \geq 0$.

Theorem 9.3. *The quadruple* $\{1, 36t^2+32t+7, 104976t^6+314928t^5+384912t^4+244368t^3+84612t^2+15084t+1080, 104976t^6+384912t^5+587088t^4+476784t^3+217404t^2+52756t+5320\}$ *has property* $D((18t^2+16t+3)^2)$ *for* $t \geq 0$.

10. Polynomials producing quadruples with property $D(k)$ where k is not a square

Theorem 10.1. *The quadruple* $\{t+1, 9t^3+33t^2+42t+18, 9t^3+39t^2+57t+29, 36t^3+144t^2+197t+93\}$ *has the property* $D(4t^2+10t+7)$, $t \geq 0$.

Proof. It is seen that

$(t+1).(9t^3+33t^2+42t+18)+(4t^2+10t+7) = (3t^2+7t+5)^2$,

$(t+1).(9t^3+39t^2+57t+29)+(4t^2+10t+7) = (3t^2+8t+6)^2$,

$(t+1).(36t^3+144t^2+197t+93)+(4t^2+10t+7) = (6t^2+15t+10)^2$,

$(9t^3+33t^2+42t+18).(9t^3+39t^2+57t+29)+(4t^2+10t+7) = (9t^3+36t^2+49t+23)^2$,

$(9t^3+33t^2+42t+18).(36t^3+144t^2+197t+93)+(4t^2+10t+7) = (18t^3+69t^2+91t+41)^2$,

$(9t^3+39t^2+57t+29).(36t^3+144t^2+197t+93)+(4t^2+10t+7) = (18t^3+75t^2+106t+52)^2$.

This theorem provides a generalization of the result contained in [20] concerning the quadruple $\{1, 18, 29, 93\}$ with property D(7).

Theorem 10.2. *The quadruple* $\{1, 3t^2+12t+13, 3t^2+20t+26, 12t^2+64t+77\}$ *has property* $D(13t^2+36t+23)$, $t \geq 0$.

Proof. We have

$1.(3t^2+12t+13)+(13t^2+36t+23) = (4t+6)^2$,

$1.(3t^2+20t+26)+(13t^2+36t+23) = (4t+7)^2$,

$1.(12t^2+64t+77)+(13t^2+36t+23) = (5t+10)^2$,

$(3t^2+12t+13).(3t^2+20t+26)+(13t^2+36t+23) = (3t^2+16t+19)^2$,

$(3t^2+12t+13).(12t^2+64t+77)+(13t^2+36t+23) = (6t^2+28t+32)^2$,

$(3t^2 + 20t + 26).(12t^2 + 64t + 77) + (13t^2 + 36t + 23) = (6t^2 + 36t + 45)^2$.

Theorem 10.3. *The quadruple* $\{1, 12t^2 + 144t + 433, 14t^2 + 168t + 484, 52t^2 + 624t + 1833\}$ *has property* $D(t^4 + 24t^3 + 182t^2 + 456t + 192)$, $t \geq 0$.

Theorem 10.4. *The quadruple* $\left\{1, \frac{t^3 + 11t}{3} + 6, \frac{4t^3 + 44t}{3} + 17, \frac{10t^3 + 110t}{3} + 45\right\}$ *has property* $D\left(\frac{t^6}{4} + \frac{11t^4}{2} + \frac{14t^3}{3} + \frac{121t^2}{4} + \frac{154t}{3} + 19\right)$, $t \geq 0$.
This generalizes a result in [18] concerning the quadruple $\{1, 6, 17, 45\}$ *with property D(19).*

Theorem 10.5. *The quadruple* $\{2t + 3, 19t^2 + 48t + 30, 57t^2 + 150t + 99, 152t^2 + 394t + 255\}$ *has property* $D(361t^4 + 1862t^3 + 3601t^2 + 3096t + 999)$, $t \geq 0$.

Theorem 10.6. *The quadruple* $\{2t+7, 3t^2+12t+10, 9t^2+42t+51, 24t^2+106t+115\}$ *has property* $D(9t^4 + 78t^3 + 253t^2 + 372t + 219)$, $t \geq 0$.

Theorem 10.7. *The quadruple* $\{2t + 15, t^2 + 6t + 3, 3t^2 + 24t + 54, 8t^2 + 58t + 99\}$ *has property* $D(t^4 + 14t^3 + 73t^2 + 192t + 279)$, $t \geq 0$.

Theorem 10.8. *The quadruple* $\{8t+4, 216t^2+128t+18, 432t^2+272t+44, 1296t^2+792t + 120\}$ *has property* $D(11664t^4 + 12960t^3 + 5112t^2 + 840t + 49)$, $t \geq 0$.

Theorem 10.9. *The quadruple* $\{8t + 20, 32t^2 + 72t + 34, 64t^2 + 160t + 108, 192t^2 + 456t + 264\}$ *has property* $D(256t^4 + 1024t^3 + 1248t^2 + 448t + 49)$, $t \geq 0$.

11. Conclusion

In this paper we have provided several quadruples with property D(k) in terms of polynomials which serve as a generalization of various specific cases. We have obtained expressions for various polynomials providing quadruples of regular and irregular types. These polynomials are expected to provide a basis for research workers in future to identify various interesting attributes of numbers with property D(k).

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