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A GENERAL MATHEMATICAL AND STATISTICAL MODEL

A.M. Mathai

Emeritus Professor of Mathematics and Statistics,
McGill University, CANADA

E-mail: directorcms458@gmail.com, a.mathai@mcgill.ca

and
Director, Centre for Mathematical and Statistical Sciences, INDIA

Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: A general model in the elliptically contoured family of functions and under the idea of a pathway model is introduced here. Through a pathway parameter b one will be able to go from one family of functions to two other families of functions, thus three different families of functions. A standard form of the model belongs to spherically symmetric family of functions. As particular cases and applications, it is shown that some real multivariate extensions of the basic models of generalized type-1 beta, type-2 beta, gamma, chisquare, Student-t, F, Cauchy, Maxwell-Boltzmann, Raleigh, Gaussian and related densities are available from the general model introduced. Reliability analysis concepts are introduced for the multivariate cases and some properties of the general model are also discussed. Then the model is extended to the complex multivariate case. In the complex case also, various connections and applications are pointed out.

Keyword and Phrases: Multivariate distributions, ellipsoid of concentration, random volumes, multivariate reliability analysis, generalized entropy optimization, generalized Gaussian, Maxwell-Boltzmann, Raleigh distributions, H-function.

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1. Introduction

In this paper the following general notation will be used. Real scalar variables, mathematical as well as random, will be denoted by small letters $x, y, \ldots$. Vector /matrix variables, mathematical and random, will be denoted by capital letters $X, Y, \ldots$. Constant scalars will be denoted by $a, b, \ldots$ and constant matrices by $A, B, \ldots$. Variables in the complex domain will be denoted with a tilde such as $\tilde{x}, \tilde{y}, \tilde{X}, \tilde{Y}, \ldots$. Constants in the complex domain will be written without the tilde. Let X be a $p \times 1$ vector of real scalar variables $x_1, \ldots, x_p, X' = (x_1, \ldots, x_p)$, a prime denoting the transpose. Let $f(X)$ be a real-valued scalar function of X and integrable so that $\int_X f(X)\mathrm{d}X < \infty$ where $\mathrm{d}X = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_p = \mathrm{d}X'$ is the wedge product of all differentials in X or X'. Let A be a $p \times p$ real positive definite constant matrix. Then $X'AX = c > 0$ describes an ellipsoid, centered at the origin $X = O$. If x_j's, $j = 1, \ldots, p$ are real scalar random variables and if the covariance matrix in X is $\Sigma > O$, that is, $\mathrm{Cov}(X) = \Sigma > O$ (positive definite), then a generalized distance of X from the origin or a generalized norm in X is $(\Sigma^{-\frac{1}{2}}X)'(\Sigma^{-\frac{1}{2}}X) = X'\Sigma^{-1}X$, and $X'\Sigma^{-1}X = c > 0$ is known as the ellipsoid of concentration for $f(X)$ if $f(X)$ is a multivariate statistical density. Consider the following pathway form of an elliptically contoured multivariate or p-variate density:

$$f_1(X) = c_1(X'AX)^{\gamma}[1 - ab(X'AX)^{\delta}]^{\frac{\eta}{b}}, a > 0, \eta > 0, \delta > 0, b \neq 0 \tag{1.1}$$

for $1 - ab(X'AX)^{\delta} > 0$, and zero elsewhere. For $b > 0$, (1.1) is in the form of a generalized type-1 beta density for the positive definite quadratic form $X'AX$. If b in (1.1) is replaced by $-b$, with $b > 0$ then the model in (1.1) switches into the model

$$f_2(X) = c_2(X'AX)^{\gamma}[1 + ab(X'AX)^{\delta}]^{-\frac{\eta}{b}}, a > 0, \eta > 0, \delta > 0, b > 0. \tag{1.2}$$

For $b \to 0$ the models in (1.1) and (1.2) go to the model

$$f_3(X) = c_3(X'AX)^{\gamma}\mathrm{e}^{-a\eta(X'AX)^{\delta}}, a > 0, \eta > 0, \delta > 0. \tag{1.3}$$

Thus, one can go from (1.1) to (1.2) and (1.3) or from (1.2) to (1.1) and (1.3) or all the three families of functions are available through the parameter b. Hence b will be called the pathway parameter here. The model in (1.1) and its pathway forms (1.2),(1.3) can act as a multivariate mathematical model or as a statistical model. If $f_j(X), j = 1, 2, 3$ are statistical densities then $c_j, j = 1, 2, 3$ are the normalizing constants there. In a physical situation, if (1.3) is the stable or idealized situation then the unstable neighborhoods are described by (1.1) and (1.2) and the transitional stages are also described by the pathway parameter b. Pathway idea for the general matrix-variate case may be seen from Mathai (2005). Let $Y = A^{\frac{1}{2}}X$ where $A^{\frac{1}{2}}$ is the real positive definite square root of the real positive definite constant matrix A. Then $\mathrm{d}Y = |A|^{\frac{1}{2}}\mathrm{d}X$ where $|(\cdot)| = \det(\cdot)$ denotes the determinant of $(\cdot)$ and the connection between the wedge product of differentials

$\mathrm{d}X$ and $\mathrm{d}Y$ or the Jacobian in the transformation $Y = A^{\frac{1}{2}}X$ may be seen from Mathai (1997). Under the above transformation the models in (1.1) to (1.3) change to the following:

$$f_4(Y) = c_4(y_1^2 + ... + y_p^2)^{\gamma}[1 - ab(y_1^2 + ... + y_p^2)^{\delta}]^{\frac{\eta}{b}} \tag{1.4}$$

$$f_5(Y) = c_5(y_1^2 + ... + y_p^2)^{\gamma}[1 + ab(y_1^2 + ... + y_p^2)^{\delta}]^{-\frac{\eta}{b}} \tag{1.5}$$

$$f_6(Y) = c_6(y_1^2 + ... + y_p^2)^{\gamma}\mathrm{e}^{-a\eta(y_1^2+...+y_p^2)^{\delta}} \tag{1.6}$$

for $a > 0, b > 0, \eta > 0, \delta > 0$, in (1.4) $1 - ab(y_1^2 + ... + y_p^2)^{\delta} > 0$, and $c_j|A|^{-\frac{1}{2}} = c_{j+3}, j = 1, 3$. How do we evaluate c_5 to c_6 when $f_4(Y)$ to $f_6(Y)$ are densities? For example, if $f_4(Y)$ is a density then $\int_Y f_4(Y)\mathrm{d}Y = 1$. But $f_4(Y)$ contains the quanity $Y'Y = y_1^2 + ... + y_p^2 = s$ or it is a spherically symmetric distribution, invariant under orthonormal transformations. We can go from the wedge product $\mathrm{d}Y$ to $\mathrm{d}s$, where Y is $p \times 1$ whereas s is a scalar quantity. This can be achieved either through a general polar coordinate transformation or through Jacobians of matrix transformations. To this end, we will take a result form Mathai (1997), which will be stated as a lemma here.

Lemma 1.1. *Let $Z = (z_{ij})$ be a $p \times n, n \geq p$ matrix of distinct real scalar variables z_{ij}'s and of full rank p. Let $S = ZZ'$ so that $S = (s_{ij})$ is $p \times p$ and positive definite, $S = S' > O$ and of $p(p+1)/2$ distinct real variables s_{ij}'s. In Z there are pn distinct real variables. Then a connection between $\mathrm{d}Z$ and $\mathrm{d}S$ is available after integrating out over the Stiefel manifold and the connection is the following:*

$$\mathrm{d}Z = \frac{\pi^{\frac{np}{2}}}{\Gamma_p(\frac{n}{2})}|S|^{\frac{n}{2}-\frac{p+1}{2}}\mathrm{d}S \tag{1.7}$$

where $\Gamma_p(\alpha)$ is the real matrix-variate gamma, given by,

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})...\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2} \tag{1.8}$$

where $\Re(\cdot)$ means the real part of $(\cdot)$, and $|S|$ is the determinant of S.

Now, apply the result (1.7) to the case $s = Y'Y$ by taking n as p and p as 1. Then

$$\mathrm{d}Y = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}s^{\frac{p}{2}-1}\mathrm{d}s. \tag{1.9}$$

Then

$$1=\int_Y f_4(Y)\mathrm{d}Y=c_4\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\int_{s=0}^{\infty}s^{\gamma+\frac{p}{2}-1}[1-abs^{\delta}]^{\frac{\eta}{b}}\mathrm{d}s \quad \text{(i)}$$

$$=c_4\frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}\frac{1}{\delta}(ab)^{-\frac{1}{\delta}(\gamma+\frac{p}{2})}\frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))\Gamma(\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\frac{\eta}{b}+1)},\Re(\gamma)>-\frac{p}{2}. \quad \text{(ii)}$$

The integral in (i) is evaluated by using a type-1 beta integral. Hence for $\delta>0,\eta>0,a>0$

$$c_1=|A|^{\frac{1}{2}}\delta\frac{\Gamma(\frac{p}{2})}{\pi^{\frac{p}{2}}}(ab)^{\frac{1}{\delta}(\gamma+\frac{p}{2})}\times\frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))\Gamma(\frac{\eta}{b}+1)},\Re(\gamma)>-\frac{p}{2}. \quad (1.10)$$

Evaluating c_2 by using a type-2 beta integral and c_3 by using a gamma integral we have

$$c_2=|A|^{\frac{1}{2}}\delta\frac{\Gamma(\frac{p}{2})}{\pi^{\frac{p}{2}}}(ab)^{\frac{1}{\delta}(\gamma+\frac{p}{2})}\frac{\Gamma(\frac{\eta}{b})}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))\Gamma(\frac{\eta}{b}-\frac{1}{\delta}(\gamma+\frac{p}{2}))} \quad (1.11)$$

for $\delta>0,a>0,\eta>0,\Re(\gamma)>-\frac{p}{2},\frac{\eta}{b}-\frac{1}{\delta}(\Re(\gamma)+\frac{p}{2})>0$, and

$$c_3=|A|^{\frac{1}{2}}\delta\frac{\Gamma(\frac{p}{2})}{\pi^{\frac{p}{2}}}(a\eta)^{\frac{1}{\delta}(\gamma+\frac{p}{2})}\frac{1}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))},\Re(\gamma)>-\frac{p}{2}. \quad (1.12)$$

2. Some Special Cases and Connection to Other Areas

Multivariate type-1,type-2, gamma and other models: For $X'AX=Y'Y=y_1^2+...+y_p^2$, (1.4),(1.5) and (1.6) give a real multivariate version of the type-1,type-2, and gamma densities as given in Mathai and Princy (2017a). This also includes real multivariate versions of Gaussian for $\gamma=0,\delta=1$; Student-t, Cauchy, F-density, gamma, exponential, chisquare, Weibull for $\gamma=\delta-1$, Maxwell-Boltzmann density in physical sciences for $\delta=1,\gamma=1$, Raleigh density for $\delta=1,\gamma=\frac{1}{2}$, stellar and solar models (Mathai and Haubold, 1988). Exponentiation in type-2 beta form will produce logistic, q-logistic (Mathai and Provost, 2006), Fermi-Dirac density and related densities.

Random points in geometrical probability: The models (1.4)-(1.6) for $\gamma=0,\delta=1$ are respectively the isotropic type-1 beta, type-2 beta and gamma distributed random points considered by Miles (1971) and Ruben (1979). They deal with random volumes of parallelotopes generated by such statistically independently distributed isotropic random points. Ruben (1979) and Mathai (1982) also deal with asymptotic normality of such random volumes. Models in (1.1)-(1.3) describe more general situations of random points. General situations of random volumes generated by random points, which need not be isotropic or statistically independent, are considered in Mathai (1999a,b).

Multivariate reliability models: A popular reliability model in the real scalar case is a type-1 beta form $f_x(x) = a_1\beta x^{\alpha-1}(1-a_1x)^{\beta-1}$ for $\alpha = 1$. A multivariate form of this model for $\alpha = 1$ is of the form

$$f_X(X) = c[1-a(X'AX)]^{\beta-1}, X' = (x_1,...,x_p), a > 0.$$

When extending real scalar variable case to the multivariate case, probability inequalities break down. Let X and T be $p \times 1$ vectors, $X' = (x_1,...,x_p), T' = (t_1,...,t_p)$. If X is a vector random variable and T is a fixed vector then what is the meaning of the probability statement $Pr\{X \geq T\}$? If $X \geq T$ is interpreted as $x_j \geq t_j, j = 1,...,p$, that is, element-wise bigger or equal, then further operations on this inequality is limited. For a $q \times p$ matrix B, $X \geq T$ element-wise need not imply that $BX \geq BT$. Even if B is $p \times p$ and real positive definite still $X \geq T$ element-wise need not imply $BX \geq BT$. Also, note that $X \geq T$ element-wise need not imply $X'X \geq T'T$. Hence reliability concepts cannot be computed in the multivariate case if $X \geq T$ is interpreted as element-wise bigger or equal when X and T are vectors or general matrices of the same orders. In order to overcome this difficulty, Mathai and Princy (2017b) interpreted $X \geq T$ to mean a norm of X is bigger or equal to the corresponding norm in T or $\|X\| \geq \|T\|$. When X is a real vector random variable then in order to take care of the covariance structure in X, a generalized norm can be taken. Then $X \geq T$ will be interpreted as $X'\Sigma^{-1}X \geq T'\Sigma^{-1}T$ where $\Sigma = \mathrm{Cov}(X)$. In general one may take a real positive definite matrix $A > O$ and take $X \geq T$ to mean $X'AX \geq T'AT$ for a specific $A > O$ relevant to the problem under consideration. Consider the model in (1.1) with $b = 1, \eta = \beta - 1$ so that (1.1) becomes

$$f_1(X) = c_1(X'AX)^{\gamma}[1-a(X'AX)^{\delta}]^{\beta-1}, 1-a(X'AX)^{\delta} > 0, \delta > 0, \beta > 0, a > 0. \tag{2.1}$$

In order to compute the survival function $Pr\{X \geq T\}$ we may compute $Pr\{X'AX \geq T'AT\}$. For the model in (2.1)

$$Pr\{X'AX \geq T'AT\} = |A|^{-\frac{1}{2}}\int_{Y'Y\geq d} c_1(Y'Y)^{\gamma}[1-a(Y'Y)^{\delta}]^{\beta-1}\mathrm{d}Y$$

where $Y = A^{\frac{1}{2}}X, A^{\frac{1}{2}}T = U, U'U = d$ and c_1 is given in (1.10). Let $s = Y'Y$. Then from Lemma 1.1,

$$\mathrm{d}Y = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} s^{\frac{p}{2}-1}\mathrm{d}s.$$

Then

$$\begin{aligned} Pr\{X \geq T\} &= Pr\{X'AX \geq T'AT\} \\ &= |A|^{-\frac{1}{2}} \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} c_1 \int_{s \geq d} s^{\gamma+\frac{p}{2}-1}[1 - as^{\delta}]^{\beta-1}\mathrm{d}s. \end{aligned} \tag{i}$$

Let $z = as^{\delta}$. Then

$$Pr\{X \geq T\} = c_1|A|^{-\frac{1}{2}} \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} \frac{a^{-\frac{1}{\delta}(\gamma+\frac{p}{2})}}{\delta} \int_{z \geq ad^{\delta}} z^{\frac{1}{\delta}(\gamma+\frac{p}{2})-1}[1 - z]^{\beta-1}\mathrm{d}s.$$

Substituting for c_1 we have the following for $b = 1, \eta = \beta - 1$

$$Pr\{X \geq T\} = \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \beta)}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))\Gamma(\beta)} \int_{z \geq ad^{\delta}} z^{\frac{1}{\delta}(\gamma+\frac{p}{2})-1}[1 - z]^{\beta-1}\mathrm{d}z. \tag{ii}$$

For $\gamma = -\frac{p}{2} + \delta$ the integral in (ii) is available as

$$\begin{aligned} Pr\{X \geq T\} &= \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \beta)}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))\Gamma(\beta)} \frac{1}{\beta}(1 - ad^{\delta})^{\beta} \\ &= \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \beta)}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))\Gamma(\beta)} \frac{1}{\beta}[1 - a(T'AT)^{\delta}]^{\beta} \end{aligned} \tag{2.2}$$

for $\Re(\gamma) > -\frac{p}{2}, \delta > 0, \beta > 0$. From here one can derive all other reliability concepts for $X \geq T$ interpreted as $X'AX \geq T'AT$. If $\gamma \neq -\frac{p}{2} + \delta$, or in the general case, we may proceed as follows:

$$Pr\{X \geq T\} = \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \beta)}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))\Gamma(\beta)} \int_{z \geq ad^{\delta}} z^{\frac{1}{\delta}(\gamma+\frac{p}{2})-1}(1 - z)^{\beta-1}\mathrm{d}z. \tag{iii}$$

Expanding $(1 - z)^{\beta-1} = (1 - z)^{-(1-\beta)}$ by using a binomial expansion we have the following:

$$(1 - z)^{-(1-\beta)} = \sum_{r=0}^{\infty} \frac{(1 - \beta)_r}{r!} z^r \tag{iv}$$

where, for example,

$$(a)_r = a(a+1)...(a+r-1), (a)_0 = 1, a \neq 0 \qquad (v)$$

is the Pochhammer symbol. Then the integral part reduces to the following:

$$\begin{aligned}\int_{z=ad^\delta}^{1} z^{\frac{1}{\delta}(\gamma+\frac{p}{2})+r-1}\mathrm{d}z &= \left[\frac{z^{\frac{1}{\delta}(\gamma+\frac{p}{2})+r}}{\frac{1}{\delta}(\gamma+\frac{p}{2})+r}\right]_{ad^\delta}^{1}\\ &= \frac{1}{\frac{1}{\delta}(\gamma+\frac{p}{2})+r}\{1-(ad^\delta)^{\frac{1}{\delta}(\gamma+\frac{p}{2})+r}\}\\ &= \frac{1}{\frac{1}{\delta}(\gamma+\frac{p}{2})}\frac{[\frac{1}{\delta}(\gamma+\frac{p}{2})]_r}{[\frac{1}{\delta}(\gamma+\frac{p}{2})+1]_r}\{1-[a(T'AT)^\delta]^{\frac{1}{\delta}(\gamma+\frac{p}{2})}[a(T'AT)^\delta]^r\}. \qquad (\mathrm{vi})\end{aligned}$$

$$\begin{aligned}\Pr\{X\geq T\} = \frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\beta)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+1)\Gamma(\beta)}\{&{}_2F_1(1-\beta,\frac{1}{\delta}(\gamma+\frac{p}{2});\frac{1}{\delta}(\gamma+\frac{p}{2})+1;1)\\ &-[a(T'AT)^\delta]^{\frac{1}{\delta}(\gamma+\frac{p}{2})}{}_2F_1(1-\beta,\frac{1}{\delta}(\gamma+\frac{p}{2});\frac{1}{\delta}(\gamma+\frac{p}{2})+1;a(T'AT)^\delta)\}\end{aligned}$$

for $a(T'AT)^\delta < 1$. But the ${}_2F_1$ with argument 1 can be opened up by using the formula

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (2.3)$$

whenever the gammas are defined. Therefore

$${}_2F_1(1-\beta,\frac{1}{\delta}(\gamma+\frac{p}{2});\frac{1}{\delta}(\gamma+\frac{p}{2})+1;1) = \frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+1)\Gamma(\beta)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\beta)}.$$

Then

$$\begin{aligned}Pr\{X\geq T\} = 1 - \frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\beta)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+1)\Gamma(\beta)}&[a(T'AT)^\delta]^{\frac{1}{\delta}(\gamma+\frac{p}{2})}\\ \times {}_2F_1(1-\beta,\frac{1}{\delta}(\gamma+\frac{p}{2});\frac{1}{\delta}(\gamma+\frac{p}{2})+1;a(T'AT)^\delta).& \qquad (2.4)\end{aligned}$$

Observe that $0 < a(T'AT)^\delta < 1$ from the starting model itself.

General Maxwell-Boltzmann and Raleigh Densities: From (1.3) we have a generalization of the Maxwell-Boltzmann and Raleigh densities for a multivariate case. For the real scalar variable case, the Maxwell-Boltzmann velocity density and Raleigh density are the following, denoted by $f_7(x)$ and $f_8(x)$ respectively.

$$f_7(x) = a_1 x^2 \mathrm{e}^{-a_2 x^2}, a_1 > 0, a_2 > 0, x \geq 0 \tag{2.5}$$

$$f_8(x) = b_1 x \mathrm{e}^{-b_2 x^2}, b_1 > 0, b_2 > 0, x \geq 0. \tag{2.6}$$

A multivariate version for (2.5) and (2.6), in the pattern of (1.3), are the following, denoted by $f_9(X)$ and $f_{10}(X)$:

$$f_9(X) = a_3(X'AX)\mathrm{e}^{-a_4(X'AX)}, X' = (x_1, ..., x_p), a_3 > 0, a_4 > 0 \tag{2.7}$$

$$f_{10}(X) = b_3(X'AX)^{\frac{1}{2}}\mathrm{e}^{-b_4(X'AX)}, b_3 > 0, b_4 > 0 \tag{2.8}$$

for $-\infty < x_j < \infty, j = 1, ..., p, X'AX > 0$. Then the standard forms are given by the following:

$$f_{11}(Y) = a_3|A|^{-\frac{1}{2}}(y_1^2 + ... + y_p^2)\mathrm{e}^{-a_4(y_1^2+...+y_p^2)} \tag{2.9}$$

$$f_{12}(Y) = b_3|A|^{-\frac{1}{2}}(y_1^2 + ... + y_p^2)^{\frac{1}{2}}\mathrm{e}^{-b_4(y_1^2+...+y_p^2)} \tag{2.10}$$

for $-\infty < y_j < \infty, j = 1, ..., p$. Much more general forms of (2.7) and (2.8) are the density in (1.3) for general parameters.

3. Some Properties

What is the h-th moment of $X'AX$ in (1.1) for a general h? This can be evaluated by replacing γ by $\gamma + h$ and then taking the ratio of the normalizing constant c_1 in (1.10) because when $E[X'AX]^h$ is taken, the only change is γ is replaced by $\gamma + h$, where $E(\cdot)$ denotes the expected value of $(\cdot)$. Then

$$E[X'AX]^h = \frac{\Gamma(\frac{1}{\delta}(\gamma + h + \frac{p}{2}))}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))} \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \frac{\eta}{b} + 1)}{\Gamma(\frac{1}{\delta}(\gamma + h + \frac{p}{2}) + \frac{\eta}{b} + 1)} \frac{1}{(ab)^{\frac{h}{\delta}}} \tag{3.1}$$

for $\Re(\gamma + h) > -\frac{p}{2}$. That is,

$$E[(ab)^{\frac{1}{\delta}}(X'AX)]^h = \frac{\Gamma(\frac{1}{\delta}(\gamma + h + \frac{p}{2}))}{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}))} \frac{\Gamma(\frac{1}{\delta}(\gamma + \frac{p}{2}) + \frac{\eta}{b} + 1)}{\Gamma(\frac{1}{\delta}(\gamma + h + \frac{p}{2} + \frac{\eta}{b} + 1)}. \tag{3.2}$$

For $h = s - 1$, where s is a complex variable, $E(X'AX)^h$ gives the Mellin transform of the density of $X'AX$. Then from (3.2) one can note that for $\delta = 1$, $(ab)(X'AX)$ is real scalar type-1 beta distributed with the parameters $(\gamma + \frac{p}{2}, \frac{\eta}{b} + 1)$. For a general case if $u_1 = (ab)^{\frac{1}{\delta}}(X'AX), \delta > 0$ and if the density of u_1 is denoted by $f_{u_1}(u_1)$, then for $i = \sqrt{(-1)}$

$$\begin{aligned} f_{u_1}(u_1) &= \frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\frac{s}{\delta}+\frac{1}{\delta}(\gamma-1+\frac{p}{2}))}{\Gamma(\frac{s}{\delta}+\frac{1}{\delta}(\gamma-1+\frac{p}{2})+\frac{\eta}{b}+1)}u_1^{-s}\mathrm{d}s \\ &= \frac{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2})+\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))} \\ &\times H_{1,1}^{1,0}\left[u_1\big|_{(\frac{1}{\delta}(\gamma-1+\frac{p}{2}),\frac{1}{\delta}),(\frac{1}{\delta}(\gamma-1+\frac{p}{2})+\frac{\eta}{b}+1,\frac{1}{\delta})}\right] \end{aligned} \tag{3.3}$$

for $0 \le u_1 \le 1$ where the c in the contour is such that $c > -(\gamma - 1 + \frac{p}{2})$ and $H(\cdot)$ is the H-function. For the theory and properties of the H-function, see Mathai et al. (2010). From the normalizing constant (1.11), for $b < 0$ case, the h-th moment of $u_2 = (ab)^{\frac{1}{\delta}}X'AX$, is available as the following:

$$E[(ab)^{\frac{1}{\delta}}(X'AX)]^h = \frac{\Gamma(\frac{1}{\delta}(\gamma+h+\frac{p}{2}))}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))}\frac{\Gamma(\frac{\eta}{b}-\frac{1}{\delta}(\gamma+h+\frac{p}{2}))}{\Gamma(\frac{\eta}{b}-\frac{1}{\delta}(\gamma+\frac{p}{2}))} \tag{3.4}$$

for $\Re(\gamma+h) > -\frac{p}{2}, \Re(\gamma+h) < -\frac{p}{2}+\frac{\delta\eta}{b}, \delta > 0, b > 0, \eta > 0$. For $\delta = 1$, the structure in (3.4) is that of the h-th moment of a real scalar type-2 beta random variable with the parameters $(\gamma+\frac{p}{2}, \frac{\eta}{b}-(\gamma+\frac{p}{2})$ for real $b > 0, \eta > 0$. For a general δ, the model in (3.4) is a H-function. Hence if the density of u_2 is denoted as $f_{u_2}(u_2)$, then

$$\begin{aligned} f_{u_2}(u_2) &= \frac{1}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))\Gamma(\frac{\eta}{b}-\frac{1}{\delta}(\gamma+\frac{p}{2}))} \\ &\times H_{1,1}^{1,1}\left[u_2\big|_{(\frac{1}{\delta}(\gamma-1+\frac{p}{2}),\frac{1}{\delta}),(1-\frac{\eta}{b}+\frac{1}{\delta}(\gamma-1+\frac{p}{2}),\frac{1}{\delta})}\right] \end{aligned} \tag{3.5}$$

for $0 \le u_2 < \infty$ where the c in the contour is such that $-\Re(\gamma+\frac{p}{2}) < c < -\Re(\gamma) - \frac{p}{2} + \frac{\eta\delta}{b}, \delta > 0, b > 0, \eta > 0$. Let $u_3 = (a\eta)^{\frac{1}{\delta}}X'AX$. Then from (1.12)

$$E[u_3^h] = E[(a\eta)^{\frac{1}{\delta}}(X'AX)]^h = \frac{\Gamma(\frac{1}{\delta}(\gamma+h+\frac{p}{2}))}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))}, \Re(\gamma+h) > -\frac{p}{2}. \tag{3.6}$$

Hence for a general δ, u_3 is a real scalar generalized gamma random variable with the parameters $(\gamma+\frac{p}{2}, 1, \delta)$ with the exponent δ or the density of u_3, denoted by $f_{u_3}(u_3)$, is the following:

$$f_{u_3}(u_3) = \frac{\delta}{\Gamma(\frac{1}{\delta}(\gamma+\frac{p}{2}))}u_3^{\gamma+\frac{p}{2}-1}\mathrm{e}^{-u_3^{\delta}}, 0 \le u_3 < \infty. \tag{3.7}$$

Observe that (3.7) can create a density for $\delta < 0$ also.

If the $p\times 1$ real vector X has the density in (1.1) or (1.2) or (1.3) what can we say about the density of a positive definite quadratic form $u = X'BX, B > O, B \neq A$?. Let $A^{\frac{1}{2}}X = Y$. Then the densities in (1.1),(1.2) and (1.3) become spherically symmetric in $Y'Y$ and $X'BX$ becomes $Y'A^{-\frac{1}{2}}BA^{-\frac{1}{2}}Y$. Since $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ is symmetric, there exists an orthonormal matrix $Q, Q'Q = I, QQ' = I, Q'A^{-\frac{1}{2}}BA^{-\frac{1}{2}}Q = D = \mathrm{diag}(\lambda_1, ..., \lambda_p)$, where $\lambda_1, ..., \lambda_p$ are the eigenvalues of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ which are real and positive. Then if $Y = QZ$ then $u = X'BX = \lambda_1 z_1^2 + ... + \lambda_p z_p^2$ and the densities in (1.1) to (1.3) are spherically symmetric in $Z'Z$. Since $X'BX$ is a linear function of z_j^2's, it will be difficult to evaluate the density of $u = X'BX$ when $B \neq kA$ where k is a scalar constant. Even if $b \to 0$ and $\delta = 1$ then u is a linear function of chisquare variables but still the density of u is complicated, see for example Mathai and Provost (1992).

4. Model through Entropy Optimization

In physical sciences, usually a model for a physical situation is derived by optimizing an appropriate measure of entropy or a measure of uncertainty in the corresponding probability scheme. Discussion of entropy started with Shannon entropy for a real scalar variable. Let x be a real scalar continuous variable with density function $f(x) \geq 0, \int_x f(x)\mathrm{d}x = 1$. Then Shannon entropy is given by the formula

$$S(f) = -c \int_x f(x) \ln f(x)\ \mathrm{d}x \tag{4.1}$$

or it is the expected value of $\ln f(x)$ where c is a positive constant. Several types of generalizations of the basic Shannon entropy are available in the literature, details of these generalizations and their characterizations may be seen from Mathai and Rathie (1975). A variant of one of the generalized entropies appearing in Mathai and Rathie (1975) is used by Tsallis in (1988) to come up with Tsallis' statistics and developed the area of non-extensive statistical mechanics. Superstatistics of Beck and Cohen (2003) also belongs to this area. Several generalizations were considered by this author and his coworkers in recent years. Here we consider a new entropy measure which when optimized will provide the model introduced in (1.1)-(1.3). Let X be a vector or a general $m \times n$ matrix and let $f(X)$ be a real-valued scalar function of X. Let $f(X)$ be such that $f(X) \geq 0$ for all X and $\int_X f(X)\mathrm{d}X = 1$ where $\mathrm{d}X$ stands for the wedge product of all differentials $\mathrm{d}x_{ij}$'s in $X = (x_{ij})$. Then

$f(X)$ is a statistical density. Consider the following generalized entropy measure

$$M_b(f) = \frac{1 - \int_X [f(X)]^{1+\frac{b}{\eta}} \mathrm{d}X}{b}, b \neq 0, \eta > 0. \tag{4.2}$$

Observe that the numerator in $M_b(f)$ is nothing but one minus the expected value of $[f(X)]^{\frac{b}{\eta}}$ or $1 - E[f(X)]^{\frac{b}{\eta}}, \eta > 0$. Then when $b \to 0$,

$$M_b(f) \to -\frac{1}{\eta} \int_X f(X) \ln f(X) \mathrm{d}X. \tag{4.3}$$

This is of the form of Shannon entropy for the density $f(X)$. Let us consider the generalized entropy $M_b(f)$ in (4.2) for a $p \times 1$ real vector random variable with density function $f(X)$ or we are considering the special case of (4.2) for $m = 1, n = p$. Consider the optimization of (4.2) in the $p \times 1$ vector variable case under the following restrictions, which are certain moments of the real positive definite quadratic form $X'AX, A > O$:

$$E[(X'AX)^{\delta+\frac{\gamma\eta}{b}}] = \text{ fixed over all functional } f \text{ for } \delta > 0, \eta > 0, b \neq 0 \tag{i}$$

$$E[(X'AX)^{\frac{\gamma\eta}{b}}] = \text{ fixed over all functional } f \text{ for } \eta > 0, b \neq 0 \tag{ii}$$

where γ is a fixed parameter. Observe that the restrictions (i) and (ii) say that two moments of the quadratic form $X'AX, A > O$ are given quantities over all functional f. When $A = \Sigma^{-1}, \Sigma = \mathrm{Cov}(X)$ one has the ellipsoid of concentration of $f(X)$ in the quadratic form $X'AX$ and hence the restrictions may be taken as the behavior of the ellipsoid of concentration in the density $f(X)$. Then what is that f for which $M_b(f)$ is optimized over all possible density f? If we apply calculus of variation in deriving f then the Euler equation in this case is $\frac{\partial g}{\partial f} = 0$ where

$$g = f^{1+\frac{b}{\eta}} - \lambda_1 (X'AX)^{\frac{\gamma\eta}{b}} f - \lambda_2 (X'AX)^{\delta+\frac{\gamma\eta}{b}} f \tag{iii}$$

where λ_1 and λ_2 are Lagrangian multipliers. Then

$$\frac{\partial g}{\partial f} = 0 \Rightarrow (1 + \frac{b}{\eta}) f^{\frac{b}{\eta}} = \lambda_1 (X'AX)^{\frac{\gamma\eta}{b}} [1 - \frac{\lambda_2}{\lambda_1} (X'AX)^{\delta}]. \tag{iv}$$

Then f can be written as the following:

$$f = \lambda_3 (X'AX)^{\gamma} [1 - ab(X'AX)^{\delta}]^{\frac{\eta}{b}} \tag{4.4}$$

for $b \neq 0, \lambda_3$ is a constant and $\frac{\lambda_2}{\lambda_1}$ is taken as ab where $a > 0$ is a scalar constant. The model in (4.4) is nothing but our model in (1.1) with $\lambda_3 = c_1$. Thus, our model (1.1), from where (1.2) and (1.3) are available, is obtained by optimizing the entropy $M_b(f)$ of (4.2), which is a generalized Shannon entropy for a real matrix-variate case of the density. Note that the measure in (4.2) can be the same in the complex domain also. In this case, denoting the complex matrix with a tilde, $\tilde{X}$ will be a $m \times n$ matrix in the complex domain but $\tilde{f}(\tilde{X})$ will be a real-valued scalar function of $\tilde{X}$ such that $\tilde{f}(\tilde{X}) \geq 0$ for all $\tilde{X}$ and $\int_{\tilde{X}} \tilde{f}(\tilde{X})\mathrm{d}\tilde{X} = 1$ where $\mathrm{d}\tilde{X} = \mathrm{d}X_1 \wedge \mathrm{d}X_2, \tilde{X} = X_1 + iX_2, i = \sqrt{(-1)}, X_1, X_2$ are real $m \times n$ matrices. Then the entropy measure corresponding to (4.2) in the complex case will be denoted by $\tilde{M}_b(\tilde{f})$. Then optimizing $\tilde{M}_b(\tilde{f})$ under the restrictions (i) and (ii) by replacing $X'AX$ by $\tilde{X}^*A\tilde{X}$ we obtain the model in the complex domain corresponding to (1.1) to (1.3), which will be discussed in the next section, where $\tilde{X}^*$ denotes the conjugate transpose of $\tilde{X}$ and $A = A^* > O$ is a Hermitian positive definite matrix.

5. The Proposed Model in the Complex Domain

Let $\tilde{X}$ be a $p \times 1$ vector in the complex domain or $\tilde{X}' = (\tilde{x}_1, ..., \tilde{x}_p)$, $\tilde{x}_j = x_{j1} + ix_{j2}, i = \sqrt{(-1)}, x_{j1}, x_{j2}$ real scalar variables. Let the conjugate transpose of $\tilde{X}$ be $\tilde{X}^* = (\tilde{x}_1^*, ..., \tilde{x}_p^*), \tilde{x}_j^* = x_{j1} - ix_{j2}$. Consider the Hermitian form $\tilde{X}^*A\tilde{X}, A = A^*$ where A^* is the conjugate transpose of the $p \times p$ constant matrix A. Consider the model

$$\tilde{f}_1(\tilde{X}) = \tilde{c}_1(\tilde{X}^*A\tilde{X})^\gamma[1 - ab(\tilde{X}^*A\tilde{X})^\delta]^{\frac{\eta}{b}}, \eta > 0, a > 0, A = A^* > O, b \neq 0 \quad (5.1)$$

for $1 - ab(\tilde{X}^*A\tilde{X})^\delta > 0, \delta > 0$ and $b > 0$ where $a > 0, \eta > 0, b > 0, \delta > 0, \gamma$ are scalar parameters. If $\tilde{f}_1(\tilde{X})$ is treated as a statistical density then $\tilde{c}_1$ is the normalizing constant there. For $b < 0$ the model in (5.1) switches into the model

$$\tilde{f}_2(\tilde{X}) = \tilde{c}_2(\tilde{X}^*A\tilde{X})^\gamma[1 + ab(\tilde{X}^*A\tilde{X})^\delta]^{-\frac{\eta}{b}}, b > 0, \eta > 0, a > 0, \delta > 0. \quad (5.2)$$

When $b \to 0$, (5.1) and (5.2) go to the model

$$\tilde{f}_3(\tilde{X}) = \tilde{c}_3(\tilde{X}^*A\tilde{X})^\gamma \mathrm{e}^{-a\eta(\tilde{X}^*A\tilde{X})^\delta}, a > 0, \eta > 0, \delta > 0. \quad (5.3)$$

If $\tilde{f}_j(\tilde{X}), j = 1, 2, 3$ are statistical densities then the normalizing constants $\tilde{c}_j, j = 1, 2, 3$ can be evaluated by using the following procedure: Let $A^{\frac{1}{2}}$ be the Hermitian positive definite square root of the Hermitian positive definite matrix A. Consider the transformation $\tilde{Y} = A^{\frac{1}{2}}\tilde{X}$ then from Mathai (1997)

$$\mathrm{d}\tilde{Y} = |\det(A^{\frac{1}{2}})|^2\mathrm{d}\tilde{X} = |\det(A)|\mathrm{d}\tilde{X} \quad (5.4)$$

where $|\det(\cdot)|$ is the absolute value of the determinant of $(\cdot)$. Then the models in (5.1)-(5.3) reduce to the following:

$$\tilde{f}_4(\tilde{Y}) = \tilde{c}_1|\det(A)|^{-1}[\tilde{Y}^*\tilde{Y}]^{\gamma}[1-ab(\tilde{Y}^*\tilde{Y})^{\delta}]^{\frac{\eta}{b}}, 1-ab(\tilde{Y}^*\tilde{Y})^{\delta}>0; \tag{5.5}$$

$$\tilde{f}_5(\tilde{Y}) = \tilde{c}_2|\det(A)|^{-1}[\tilde{Y}^*\tilde{Y}]^{\gamma}[1+ab(\tilde{Y}^*\tilde{Y})^{\delta}]^{-\frac{\eta}{b}}; \tag{5.6}$$

$$\tilde{f}_6(\tilde{Y}) = \tilde{c}_3|\det(A)|^{-1}[\tilde{Y}^*\tilde{Y}]^{\gamma}\mathrm{e}^{-a\eta(\tilde{Y}^*\tilde{Y})} \tag{5.7}$$

where

$$\tilde{Y}^*\tilde{Y} = |\tilde{y}_1|^2 + ... + |\tilde{y}_p|^2 = (y_{11}^2 + y_{12}^2) + ... + (y_{p1}^2 + y_{p2}^2)$$

where the y_{ij}'s are real or $\tilde{y}_j = y_{j1} + iy_{j2}, i = \sqrt{(-1)}, y_{j1}, y_{j2}$ real. In order to integrate out $\tilde{f}_j(\tilde{Y}), j = 4, 5, 6$ we will try to write $\mathrm{d}\tilde{Y}$ in terms of $\mathrm{d}\tilde{s}$ where $\tilde{s} = \tilde{Y}^*\tilde{Y}$ and observe that $\tilde{s} = s$ is real. To this end, we will use a result from Mathai (1997) which will be stated here as a lemma.

Lemma 5.1. *Let $\tilde{Z}$ be a $p \times n, n \geq p$ matrix in the complex domain of full rank p. Let $\tilde{S} = \tilde{Z}^*\tilde{Z}$ which is a $p \times p$ matrix and Hermitian positive definite. Let $\mathrm{d}\tilde{Z}$ and $\mathrm{d}\tilde{S}$ be the wedge product of differentials in the elements of $\tilde{Z}$ and $\tilde{S}$ respectively, where there are pn distinct complex elements in $\tilde{Z}$ and $p(p+1)/2$ distinct elements in $\tilde{S}$. Then, after integrating out over the Stiefel manifold*

$$\mathrm{d}\tilde{Z} = \frac{\pi^{np}}{\tilde{\Gamma}_p(n)}|\det(\tilde{S})|^{n-p}\mathrm{d}\tilde{S} \tag{5.8}$$

where, for example, $\tilde{\Gamma}_p(\alpha)$ is the complex matrix-variate gamma given by

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1)...\Gamma(\alpha-p+1), \Re(\alpha) > p-1. \tag{5.9}$$

With the help of (5.8) we can establish many results in the complex case. If (5.4) is a density then $\int_{\tilde{Y}} \tilde{f}_4(\tilde{Y})\mathrm{d}\tilde{Y} = 1$. Then

$$1 = \tilde{c}_1|\det(A)|^{-1}\int_{\tilde{Y}}[\tilde{Y}^*\tilde{Y}]^{\gamma}[1-ab(\tilde{Y}^*\tilde{Y})^{\delta}]^{\frac{\eta}{b}}\mathrm{d}\tilde{Y} \tag{i}$$

$$= \tilde{c}_1|\det(A)|^{-1}\frac{\pi^p}{\Gamma(p)}\int_s s^{\gamma+p-1}[1-abs^{\delta}]^{\frac{\eta}{b}}\mathrm{d}s. \tag{ii}$$

Observe that s is real and positive and $\tilde{\Gamma}(p) = \Gamma(p)$. Now, integrating out (ii) with the help of a real type-1 beta integral we have

$$1 = \tilde{c}_1|\det(A)|^{-1}\frac{\pi^p}{\Gamma(p)}\frac{1}{\delta}(ab)^{-\frac{1}{\delta}(\gamma+p)}\frac{\Gamma(\frac{1}{\delta}(\gamma+p))\Gamma(\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+p)+\frac{\eta}{b}+1)}. \tag{5.10}$$

Therefore

$$\tilde{c}_1 = |\det(A)|\frac{\Gamma(p)}{\pi^p}\delta(ab)^{\frac{1}{\delta}(\gamma+p)}\frac{\Gamma(\frac{1}{\delta}(\gamma+p)+\frac{\eta}{b}+1)}{\Gamma(\frac{1}{\delta}(\gamma+p))\Gamma(\frac{\eta}{b}+1)} \tag{5.11}$$

for $A = A^* > O, \delta > 0, a > 0, b > 0, \Re(\gamma) > -p, \eta > 0$. Similarly

$$\tilde{c}_2 = |\det(A)|\frac{\Gamma(p)}{\pi^p}\delta(ab)^{\frac{1}{\delta}(\gamma+p)}\frac{\Gamma(\frac{\eta}{b})}{\Gamma(\frac{1}{\delta}(\gamma+p))\Gamma(\frac{\eta}{b}-\frac{1}{\delta}(\gamma+p))} \tag{5.12}$$

with the same conditions on the parameters above except that $-p < \Re(\gamma) < \frac{\eta\delta}{b} - p$, and

$$\tilde{c}_3 = |\det(A)|\frac{\Gamma(p)}{\pi^p}\delta(a\eta)^{\frac{1}{\delta}(\gamma+p)}\frac{1}{\Gamma(\frac{1}{\delta}(\gamma+p))}, \Re(\gamma) > -p, \delta > 0, \eta > 0, a > 0. \tag{5.13}$$

5.1. Special Cases and Applications in Other Areas

Statistical models in the complex case Observe that (5.1) and (5.5) give generalized multivariate type-1 beta model in the elliptically contoured form and in the spherically symmetric form respectively. Similarly (5.2) and (5.6) give type-2 beta form, (5.3) and (5.7) give the gamma form in complex domain. Observe that the type-2 beta forms in (5.2) and (5.6) also provide complex multivariate analogues of Student-t, F, Cauchy and related models. Also, (5.3) and (5.7) provide multivariate analogues of Gaussian, gamma, generalized gamma, chisquare, exponential and related models. For $\gamma = 0$ and for a general δ, one has a generalization of the complex Gaussian model in (5.3) and (5.7). Complex Gaussian is the case $\delta = 1, \gamma = 0, a = 1, \eta = 1, A = \Sigma^{-1}, \Sigma = \Sigma^* > O$ where Σ the covariance matrix of the $p \times 1$ complex vector variable $\tilde{X}$.

Complex Maxwell-Boltzmann and Raleigh Densities (5.3) and (5.7) for $\delta = 1, a = 1$ and $\gamma = 1$ give a complex multivariate version of Maxwell-Boltzmann density with the normalizing constant in (5.13). A complex multivariate Raleigh case is available from (5.3) and (5.7) for $\delta = 1, a = 1, \gamma = \frac{1}{2}$ with the corresponding normalizing constant in (5.13).

Random Volumes Models in (5.1),(5.2),(5.3) provide elliptically contoured complex version of random points of generalized type-1 beta, type-2 beta types for $b = 1$ in (5.1) and (5.2), and gamma type from (5.3). The corresponding standard forms are available from (5.5) and (5.6) for $b = 1$, and (5.7). These random points in the complex domain and the corresponding random volumes do not seem to have been discussed in the literature. Also, reliability concepts for the complex case can be described by using the concept $Pr\{\tilde{X} \geq \tilde{T}\}$ implies $Pr\{\tilde{X}^* A \tilde{X} \geq \tilde{T}^* A \tilde{T}\}$ for $A = A^* > O$. Reliability concepts in the complex domain do not seem to have been discussed in the literature.

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