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COMBINATORICS OF THIRD ORDER MOCK THETA FUNCTION $f(q)$ AND SIXTH ORDER MOCK THETA FUNCTIONS $\phi(q)$, $\psi(q)$

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Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: The third order mock theta function $f(q)$ is the generating function for the number of partitions with even rank minus the number of partitions with odd rank. In this paper, mock theta function $f(q)$ is interpreted in terms of n–color partitions which lead to a combinatorial proof of the above fact. The two sixth order mock theta functions $\phi(q)$ and $\psi(q)$ from Ramanujan's Lost Notebook are also interpreted in terms of n–color partitions by attaching weights.

Keyword and Phrases: Mock theta functions, (n + t)-color partitions, Generating functions.

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1. Introduction

The coefficients in the series representation of a mock theta function many times have a simple partition-theoretic interpretation. Fine's interpretation of the coefficients of series represented by the following third order mock theta function as the number of partitions into odd parts without gaps serves as an example to the above claim [3]:

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}. \tag{1.1}$$

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Bringmann and Ono [2] exhibit the following identity for third order mock theta function $f(q)$:

$$1+\sum_{n=1}^{\infty}(N_e(n)-N_o(n))q^n=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=f(q), \tag{1.2}$$

where $N_o(n)$ (resp. $N_e(n)$) denotes the number of partitions of n with odd (resp. even) rank. In Section 2, we study this third order mock theta function $f(q)$ in terms of n–color partitions by attaching weights which helps in establishing a combinatorial proof for the equation (1.2).

Definition 1.1. [1] *A n–color partition (also called a partition with "n copies of n") is a partition in which a part of size n, $n \geq 0$ can occur in n different colors denoted by $n_1, n_2, \ldots, n_n$.*

Definition 1.2. *The weighted difference of two parts $(a_i)_{b_i}$ and $(a_j)_{b_j}$ $(a_i \geq a_j)$ in an n–color partition $(a_1)_{b_1}+(a_2)_{b_2}+\cdots+(a_p)_{b_p}$ such that $(a_1)_{b_1} \geq (a_2)_{b_2} \geq \cdots \geq (a_p)_{b_p}$, is $a_i - b_i - a_j - b_j$ and denoted by $(((a_i)_{b_i}-(a_j)_{b_j}))$.*

In his Lost Notebook [4], Ramanujan defines the following two sixth order mock theta functions:

$$\phi(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \tag{1.3}$$

$$\psi(q)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1} q^{n^2}(q;q^2)_{n-1}}{(-q;q)_{2n-1}}. \tag{1.4}$$

These functions can also be interpreted in terms of n–color partitions by attaching weights. Section 3 of the present paper is devoted to the study of these two functions.

2. Third order mock theta function $f(q)$

Theorem 2.1. *Let $A_1(\nu)$ denote the number of n–color partitions of ν such that $(((a_i)_{b_i}-(a_{i+1})_{b_{i+1}})) \geq -(b_{i+1}-1)$. Then*

$$\sum_{n=0}^{\infty}A_1(\nu)q^{\nu}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2}.$$

Proof. In the expression $\frac{q^{n^2}}{(q;q)_n^2}$, q^{n^2} generates the following n–color partition

$$(2n-1)_1+\cdots+3_1+1_1. \tag{2.1}$$

The factor $\frac{1}{(q;q)_n}$ generates the multiples of $i = 1, 2, \ldots, n$. Let us say these are $l_1 \times 1, l_2 \times 2, \ldots, l_n \times n$, $l_i \geq 0$. Thus, to account for the factor $\frac{1}{(q;q)_n}$ the number being partitioned must increase by $l_1 \times 1 + l_2 \times 2 + \cdots + l_n \times n$. This is done by transforming the partition (2.1) to

$$((2n-1) + l_n + l_{n-1} + \cdots + l_1)_{1+l_1} + \cdots + (3 + l_n + l_{n-1})_{1+l_{n-1}} + (1 + l_n)_{1+l_n}. \quad (2.2)$$

Now, another factor $\frac{1}{(q;q)_n}$ generates, say $m_1 \times 1, m_2 \times 2, \ldots, m_n \times n$, $m_i \geq 0$. This factor transforms the n–color partition given by (2.2) to

$$\begin{aligned}((2n-1)+&l_n + l_{n-1} + \cdots + l_1 + m_n + m_{n-1} + \cdots + m_1)_{1+l_1} + \cdots + (3 + l_n + l_{n-1} + \\ &+ m_n + m_{n-1})_{1+l_{n-1}} + (1 + l_n + m_n)_{1+l_n}. \end{aligned} \quad (2.3)$$

Thus, the i^{th} and $(i+1)^{th}$ part in a partition generated in this manner are

$$a_i = 2(n-i+1) - 1 + l_n + \cdots + l_{i+1} + l_i + m_n + \cdots + m_{i+1} + m_i, \quad (2.4)$$

$$b_i = 1 + l_i, \quad (2.5)$$

$$a_{i+1} = 2(n-i) - 1 + l_n + \cdots + l_{i+1} + m_n + \cdots + m_{i+1}, \quad (2.6)$$

$$b_{i+1} = 1 + l_{i+1}. \quad (2.7)$$

Therefore, by examining the equations (2.4)–(2.7) it can be easily seen that the n–color partitions satisfy the condition stated in the theorem.

Now, suppose that each part of a n–color partition $((a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_p)_{b_p})$ enumerated by $A_1(\nu)$ is mapped to a 2×1 array in the following manner

$$\phi((a_i)_{b_i}) = \begin{pmatrix} a_i - b_i - b_{i+1} - \cdots - b_n \\ b_i + b_{i+1} + \cdots + b_n - 1 \end{pmatrix} \quad \textit{for } 1 \leq i < n.$$

Thus, a n–color partition with n parts is mapped to a $2 \times n$ array

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \end{pmatrix}$$

This array clearly satisfies $c_1 > c_2 > \cdots > c_n \geq 0$, $d_1 > d_2 > \cdots > d_n \geq 0$ and $\nu = n + \sum_{i=1}^n c_i + \sum_{i=1}^n d_i$. Thus, it is the Frobenius representation of an ordinary partition of ν. Conversely, every ordinary partition of ν can be mapped to a n–color partition by the following map:

$$\phi^{-1} \begin{pmatrix} c_i \\ d_i \end{pmatrix} = (c_i + d_i + 1)_{d_i - d_{i+1}} \quad \textit{for } 1 \leq i < n-1,$$

$$\phi^{-1} \begin{pmatrix} c_n \\ d_n \end{pmatrix} = (c_n + d_n + 1)_{d_n + 1}.$$

The n–color partitions generated in this manner are those enumerated by $A_1(\nu)$. Thus, ordinary partitions of ν are in one-one correspondence with partitions enumerated by $A_1(\nu)$ and this is a well known fact since $\frac{q^{n^2}}{(q;q)_n^2}$ is the generating function for ordinary partitions.
Now, if we attach a weight $(-1)^t$ to the count of each n–color partition enumerated by $A_1(\nu)$, where $t = a_1 - 2n + 1$, then we see that $f(q)$ is the generating function for this weighted number of n–color partitions. The corresponding value of t for ordinary partitions is $c_1 + d_1$. $(-1)^{c_1+d_1}$ gives the same result as $(-1)^{c_1-d_1}$. Therefore, the partitions with even rank have positive weight and partitions with odd rank have negative weight which leads to the establishment of the combinatorial proof of the equation (1.2).

3. Sixth order mock theta functions $\phi(q)$ and $\psi(q)$

Theorem 3.1 *Let $A_2(\nu)$ denote the number of n–color partitions of ν such that for $1 \le i \le p-1$*

$$(((a_i)_{b_i} - (a_{i+1})_{b_{i+1}})) \ge \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \ (mod\ 2), \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \ (mod\ 2), \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=0}^{\infty} A_2(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q)_{2n}}.$$

Proof. In the expression

$$\frac{q^{n^2}(q;q^2)_n}{(q;q)_{2n}} = \frac{q^{n^2}(q;q^2)_n}{(q;q^2)_n(q^2;q^2)_n},$$

q^{n^2} generates the following n–color partition

$$(2n-1)_1 + \cdots + 3_1 + 1_1. \tag{3.1}$$

The factor $\frac{1}{(q;q^2)_n}$ generates the multiples of $i = 1, 3, \ldots, 2n-1$. Let us say these are $k_1 \times 1, k_2 \times 3, \ldots, k_n \times (2n-1)$, $k_i \ge 0$. Thus, to account for the factor $\frac{1}{(q;q^2)_n}$ the number being partitioned must increase by $k_1 \times 1 + k_2 \times 3 + \cdots + k_n \times (2n-1)$. This is done by transforming the partition (3.1) to

$$((2n-1)+2(k_n+\cdots+k_2)+k_1)_{1+k_1}+\cdots+(3+2k_n+k_{n-1})_{1+k_{n-1}}+(1+k_n)_{1+k_n}. \tag{3.2}$$

Now, the factor $\frac{1}{(q^2;q^2)_n}$ generates, say $l_1 \times 2, l_2 \times 4, \ldots, l_n \times 2n$, $l_i \geq 0$. The factor $(-q;q^2)_n$ generates, say $m_1 \times 1 + m_2 \times 3 + \cdots + m_n \times (2n-1)$, $m_i = 0$ or 1. These factors transform the n–color partition given by (3.2) to

$$\begin{aligned}&((2n-1)+2(k_n+\cdots+k_2+l_n+\cdots+l_1+m_n+\cdots+m_2)+k_1+m_1)_{1+k_1}\\&+\cdots+(3+2(k_n+l_n+l_{n-1}+m_n)+k_{n-1}+m_{n-1})_{1+k_{n-1}}\\&+(1+2l_n+k_n+m_n)_{1+k_n}.\end{aligned} \tag{3.3}$$

Thus, the i^{th} and $(i+1)^{th}$ part in a partition generated in this manner are

$$\begin{aligned}a_i =&2(n-i+1)-1+2(k_n+\cdots+k_{i+1}+l_n+\cdots+l_i+m_n+\cdots+m_{i+1})\\&+k_i+m_i,\end{aligned} \tag{3.4}$$

$$b_i =1+k_i, \tag{3.5}$$

$$\begin{aligned}a_{i+1} =&2(n-i)-1+2(k_n+\cdots+k_{i+2}+l_n+\cdots+l_{i+1}+m_n+\cdots+m_{i+2})\\&+k_{i+1}+m_{i+1},\end{aligned} \tag{3.6}$$

$$b_{i+1} =1+k_{i+1}. \tag{3.7}$$

An examination of the equations (3.4)–(3.7) establishes the theorem.

Now, if we attach weight $(-1)^t$, where

$$t = \begin{cases} \frac{1}{2}(a_1+b_1+1) & \text{if } a_1 \not\equiv b_1 \ (mod\ 2), \\ \frac{1}{2}(a_1+b_1) & \text{if } a_1 \equiv b_1 \ (mod\ 2), \end{cases}$$

to the count of each n–color partition, then we see that $\phi(q)$ is the generating function for this weighted number of n–color partitions.

Theorem 3.2. *Let $A_3(\nu)$ denote the number of n–color partitions of ν such that $a_p = b_p$ and for $1 \leq i \leq p-1$*

$$(((a_i)_{b_i} - (a_{i+1})_{b_{i+1}})) \geq \begin{cases} 0 & \text{if } a_i \equiv b_i,\ a_{i+1} \equiv b_{i+1} \pmod 2, \\ 2 & \text{if } a_i \not\equiv b_i,\ a_{i+1} \not\equiv b_{i+1} \pmod 2, \\ 1 & \text{otherwise}. \end{cases}$$

Then

$$\sum_{n=0}^{\infty} A_3(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q)_{2n+1}}.$$

The proof of this theorem can also be accomplished in a similar manner as the previous one. The value of t for the attached weight to obtain the interpretation of $\psi(q)$ is given by

$$t = \begin{cases} \frac{1}{2}(a_1+b_1-1) & \text{if } a_1 \not\equiv b_1 \pmod 2, \\ \frac{1}{2}(a_1+b_1-2) & \text{if } a_1 \equiv b_1 \pmod 2. \end{cases}$$

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