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A NOTE ON THE CONSTANT TERM METHOD TO MOCK THETA FUNCTIONS

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Dedicated to Prof. A.K. Agarwal on his 70^{th} Birth Anniversary

Abstract: The main purpose of this paper is to show that certain mock theta functions can be expressed as constant terms in the Laurent series expansion of rational functions of theta functions. And, two identities can be proved by using the constant term method.

Keyword and Phrases: Mock theta function, Constant term method, Theta function, Hecke type series.

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1. Introduction

In his famous last letter to Hardy [20], Ramanujan introduced seventeen mock theta functions–four of order three, two groups five in each group of order five and three of order seven without giving an explicit definition. The mock theta functions are interpreted by Andrews and Hickerson [5] to mean a function $f(q)$ defined by a q-series which converges for $|q| < 1$ and satisfies the following two conditions:

(0) For every root of unity ξ, there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.

(1) There is no single theta function which works for all ξ; i.e., for every theta function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta_\xi(q)$ is unbounded as $q \to \xi$ radially.

In [24], Watson studied the third order mock theta functions and introduced three new one. Later, McIntosh [18], Andrews and Hickerson [5], Gordon and McIntosh [12], Choi [7,8] studied second-, sixth-, eighth-, tenth-order mock theta functions, respectively. Recently, Andrews [2] and Bringmann, Hikami and Lovejoy [6] obtained some new third-order mock theta functions.

There are many forms of representations for mock theta functions: Eulerian forms, Hecke-type double sums, Appell-Lerch sums, and Fourier coefficients of meromorphic Jacobi forms. To see the history of mock theta functions and their modern and classical developments, we recommend the survey papers [11, 13-15, 21, 25].

Nowadays, the constant term method play an important role in the study of mock theta functions, which relate to Hecke type identities. On the basis [1], Andrews [3] showed that eight of Ramanujan's fifth order mock theta functions [4] arise from constant term identities involving rational expressions of various $\theta(z, q)$. In [22], Srivastava showed some results of certain sixth order [5] and eighth order [23] mock theta functions by employing this method.

The remainder of the paper is structured as follows. Some useful tools and results about the basic hypergeometric series and mock theta functions is collected in Sect.2. In Sect.3, we prove that some mock theta functions can be expressed as constant terms in the Laurent series expansion of rational functions of theta functions. In Sect.4, we prove two identities by using the constant term method.

2. Preliminaries

In this paper, we adopt the standard notation for q-shifted factorials in [10]:

$$\begin{aligned}
&(a;q^k)_0 = 1,\\
&(a;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\cdots(1-aq^{(n-1)k}),\\
&(a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}).
\end{aligned}$$

when $k = 1$ we usually write $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

Recall that the Appell-Lerch sums are defined by

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r\in Z} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz},$$

The classical theta series are defined by

$$j(x;q) := \sum_{r=-\infty}^{\infty} (-1)^r x^r q^{\binom{r}{2}} = (x, q/x, q)_\infty.$$

For brevity, we write $J_{a,m} := j(q^a; q^m)$ with $\overline{J}_{a,m} := j(-q^a; q^m)$, and $J_m := J_{m,3m}$. We shall make use of the following mock theta functions:
"Second-order" mock theta functions [18]:

$$B(q) := \sum_{n\geq 0} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}, \qquad \mu(q) := \sum_{n\geq 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}.$$

Third-order" mock theta functions [2,6,15]:

$$\psi(q) := \sum_{n\geq 1} \frac{q^{n^2}}{(q;q^2)_n}, \qquad \nu(q) := \sum_{n\geq 0} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

$$\phi(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q^2;q^2)_n}, \qquad \zeta(q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}(q;q^2)_n}{(q^2;q^2)_n(-q)_{2n}},$$

$$\overline{\psi}_0(q) := \sum_{n\geq 0} \frac{q^{2n^2}}{(-q)_{2n}}, \qquad \overline{\psi}_1(q) := \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(-q)_{2n+1}},$$

$$\overline{\phi}_0(q) := \sum_{n\geq 0} q^n(-q)_{2n+1}, \qquad \overline{\phi}_1(q) := \sum_{n\geq 0} q^n(-q)_{2n}.$$

Eighth-order" mock theta function [12]:

$$T_1(q) := \sum_{n\geq 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}}.$$

Lemma 2.1. *[22, Lemma 1] In the annulus $1 < |z| < |q^\lambda|^{-1}$ the coefficient of z^0 in the Laurent series expansion of*

$$\frac{(q^B;q^B)_\infty (q^\lambda;q^\lambda)_\infty^2 \theta(\epsilon z^A q^C, q^B)}{\theta(\frac{1}{z}, q^\lambda)},$$

is

$$\sum_{r=0}^{\infty} \sum_{|Aj|\leq r} (-\epsilon)^j (-1)^{r+Aj} q^{B\binom{j}{2}+Cj+\lambda\binom{r+1}{2}-\lambda\binom{Aj+1}{2}}.$$

where $\theta(z,q) = (z, q/z)_\infty$.

Lemma 2.2. *[3, Lemma 1] In the annulus*

$$max(|a|, |a|^{-1}) < |z| < min(|aq^\lambda|^{-1}, |a^{-1}q^\lambda|^{-1}),$$

the coefficient of z^0 *in the Laurent series expansion of*

$$\frac{a(q^B;q^B)_\infty(q^\lambda;q^\lambda)_\infty^2\theta(\epsilon z^A q^C, q^B)\theta(z,q^\lambda)\theta(a^2,q^\lambda)}{\theta(z^{-1}a,q^\lambda)\theta(az,q^\lambda)\theta(a,q^\lambda)},$$

is

$$\sum_{r=0}^{\infty}\sum_{|Aj|\le r}(-\epsilon)^j(-1)^{r+Aj}a^{-r}q^{B\binom{j}{2}+Cj+\lambda\binom{r+1}{2}-\lambda\binom{Aj+1}{2}}(1+a^{2r+1}).$$

where B *and* λ *are positive real numbers and* A *is a nonzero integer.*

3. Mock theta functions as coefficient of z^0

Andrews [3] and Srivastava [22] showed that certain classical mock theta functions, corresponding to Hecke-type double sums, can be expressed as constant terms in the Laurent series expansion of rational functions of theta functions. In this section, we will obtain more results of classical mock theta functions by using Lemma 2.1 and Lemma 2.2.

Recently, Andrews [2] and Mortenson [19] deduced some mock theta functions in terms of Hecke-type double sums. In [9], Cui, Gu and Hao obtained Hecke-type double sums for the second and eighth order mock theta functions by using the Bailey pairs and Bailey lemma. we shall make use of these results as follows.

$$J_{1,2}\cdot B(q) = \sum_{n=0}^{\infty}(-1)^n q^{2n^2+2n}\sum_{j=-n}^{n}q^{-j^2},$$

$$\overline{J}_{2,8}\cdot \mu(q) = \sum_{n=0}^{\infty}q^{2n^2+n}(1-q^{2n+1})\sum_{j=-n}^{n}q^{-j^2},$$

$$J_1\cdot(1+2\psi(q)) = \sum_{n=0}^{\infty}(-1)^n q^{2n^2+n}(1+q^{2n+1})\sum_{j=-n}^{n}q^{-\binom{j+1}{2}},$$

$$J_1\cdot \nu(-q) = \sum_{n=0}^{\infty}(-1)^n q^{2n^2+2n}\sum_{j=-n}^{n}q^{-\binom{j+1}{2}},$$

$$J_2\cdot\overline{\psi_0}(q) = \sum_{n=0}^{\infty}q^{4n^2+n}(1-q^{6n+3})\sum_{j=-n}^{n}(-1)^j q^{-j^2},$$

$$J_2 \cdot \overline{\psi_1}(q) = \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

$$J_2 \cdot \zeta(q) = \sum_{n=0}^{\infty} q^{4n^2+2n}(1-q^{4n+2}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

$$\overline{J}_{1,4} \cdot \phi(q) = \sum_{n=0}^{\infty} (-1)^n q^{2n^2+n}(1+q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-3j^2/2+j/2},$$

$$\overline{J}_{1,4} \cdot \nu(q) = \sum_{n=0}^{\infty} (-1)^n q^{2n^2+2n} \sum_{j=-n}^{n} (-1)^j q^{-3j^2/2+j/2},$$

$$J_{1,2} \cdot (1+q\overline{\phi_0}(q)) = \sum_{n=0}^{\infty} q^{4n^2+n}(1-q^{6n+3}) \sum_{j=-n}^{n} (-1)^j q^{-3j^2-j},$$

$$J_{1,2} \cdot \overline{\phi_1}(q) = \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-3j^2-j},$$

$$J_{2,4} \cdot T_1(q) = \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-2j^2-j}.$$

Theorem 3.1. *We have*
1. $J_{1,2} \cdot B(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{(q^2;q^2)_\infty (q^4;q^4)_\infty^2 \theta(zq^3,q^2)}{\theta(\frac{1}{z},q^4)}. \tag{1}$$

in the annulus $1 < |z| < |q|^{-4}$.
2. $\overline{J}_{2,8} \cdot \mu(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q(q^2;q^2)_\infty (q^4;q^4)_\infty^2 \theta(zq^3,q^2)\theta(z,q^4)\theta(q^2,q^4)}{\theta(-z^{-1}q,q^4)\theta(-zq,q^4)\theta(-q,q^4)}. \tag{2}$$

in the annulus $|q|^{-1} < |z| < |q|^{-3}$.
3. $J_1 \cdot (1+2\psi(q))$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{q(q^3;q^3)_\infty (q^4;q^4)_\infty^2 \theta(zq^3,q^3)\theta(z,q^4)\theta(q^2,q^4)}{\theta(z^{-1}q,q^4)\theta(zq,q^4)\theta(q,q^4)}. \tag{3}$$

in the annulus $|q|^{-1} < |z| < |q|^{-3}$.

4. $J_1 \cdot \nu(-q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{(q^3;q^3)_\infty (q^4;q^4)_\infty^2 \theta(zq^3,q^3)}{\theta(\frac{1}{z},q^4)}. \tag{4}$$

in the annulus $1 < |z| < |q|^{-4}$.

5. $J_2 \cdot \overline{\psi_0}(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q^3(q^6;q^6)_\infty (q^8;q^8)_\infty^2 \theta(-zq^7,q^6)\theta(z,q^8)\theta(q^6,q^8)}{\theta(-z^{-1}q^3,q^8)\theta(-zq^3,q^8)\theta(-q^3,q^8)}. \tag{5}$$

in the annulus $|q|^{-3} < |z| < |q|^{-5}$.

6. $J_2 \cdot \overline{\psi_1}(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q(q^6;q^6)_\infty (q^8;q^8)_\infty^2 \theta(-zq^7,q^6)\theta(z,q^8)\theta(q^2,q^8)}{\theta(-z^{-1}q,q^8)\theta(-zq,q^8)\theta(-q,q^8)}. \tag{6}$$

in the annulus $|q|^{-1} < |z| < |q|^{-7}$.

7. $J_2 \cdot \zeta(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q^2(q^6;q^6)_\infty (q^8;q^8)_\infty^2 \theta(-zq^7,q^6)\theta(z,q^8)\theta(q^4,q^8)}{\theta(-z^{-1}q^2,q^8)\theta(-zq^2,q^8)\theta(-q^2,q^8)}. \tag{7}$$

in the annulus $|q|^{-2} < |z| < |q|^{-6}$.

8. $\overline{J}_{1,4} \cdot \phi(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{q(q)_\infty (q^4;q^4)_\infty^2 \theta(-zq^3,q)\theta(z,q^4)\theta(q^2,q^4)}{\theta(z^{-1}q,q^4)\theta(zq,q^4)\theta(q,q^4)}. \tag{8}$$

in the annulus $|q|^{-1} < |z| < |q|^{-3}$.

9. $\overline{J}_{1,4} \cdot \nu(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{(q)_\infty (q^4;q^4)_\infty^2 \theta(-zq^3,q)}{\theta(\frac{1}{z},q^4)}. \tag{9}$$

in the annulus $1 < |z| < |q|^{-4}$.

10. $J_{1,2} \cdot (1 + q\overline{\phi_0}(q))$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q^3(q^2;q^2)_\infty (q^8;q^8)_\infty^2 \theta(-zq^4,q^2)\theta(z,q^8)\theta(q^6,q^8)}{\theta(-z^{-1}q^3,q^8)\theta(-zq^3,q^8)\theta(-q^3,q^8)}. \tag{10}$$

in the annulus $|q|^{-3} < |z| < |q|^{-5}$.

11. $J_{1,2} \cdot \overline{\phi_1}(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q(q^2;q^2)_\infty(q^8;q^8)_\infty^2\theta(-zq^4,q^2)\theta(z,q^8)\theta(q^2,q^8)}{\theta(-z^{-1}q,q^8)\theta(-zq,q^8)\theta(-q,q^8)}. \tag{11}$$

in the annulus $|q|^{-1} < |z| < |q|^{-7}$.

12. $J_{2,4} \cdot T_1(q)$ *is the coefficient of* z^0 *in the Laurent series expansion of*

$$\frac{-q(q^4;q^4)_\infty(q^8;q^8)_\infty^2\theta(-zq^5,q^4)\theta(z,q^8)\theta(q^2,q^8)}{\theta(-z^{-1}q,q^8)\theta(-zq,q^8)\theta(-q,q^8)}. \tag{12}$$

in the annulus $|q|^{-1} < |z| < |q|^{-7}$.

Proof.

For (1), setting $\epsilon = 1, A = 1, B = 2, C = 3$ and $\lambda = 4$ in Lemma 2.1.

For (2), setting $\epsilon = 1, A = 1, B = 2, C = 3, a = -q$ and $\lambda = 4$ in Lemma 2.2.

For (3), setting $\epsilon = 1, A = 1, B = 3, C = 3, a = q$ and $\lambda = 4$ in Lemma 2.2.

For (4), setting $\epsilon = 1, A = 1, B = 3, C = 3$ and $\lambda = 4$ in Lemma 2.1.

For (5), setting $\epsilon = -1, A = 1, B = 6, C = 7, a = -q^3$ and $\lambda = 8$ in Lemma 2.2.

For (6), setting $\epsilon = -1, A = 1, B = 6, C = 7, a = -q$ and $\lambda = 8$ in Lemma 2.2.

For (7), setting $\epsilon = -1, A = 1, B = 6, C = 7, a = -q^2$ and $\lambda = 8$ in Lemma 2.2.

For (8), setting $\epsilon = -1, A = 1, B = 1, C = 3, a = q$ and $\lambda = 4$ in Lemma 2.2.

For (9), setting $\epsilon = -1, A = 1, B = 1, C = 3$ and $\lambda = 4$ in Lemma 2.1.

For (10), setting $\epsilon = -1, A = 1, B = 2, C = 4, a = -q^3$ and $\lambda = 8$ in Lemma 2.2.

For (11), setting $\epsilon = -1, A = 1, B = 2, C = 4, a = -q$ and $\lambda = 8$ in Lemma 2.2.

For (12), setting $\epsilon = -1, A = 1, B = 4, C = 5, a = -q$ and $\lambda = 8$ in Lemma 2.2.

4. Simple proofs of two Liu's identities

In fact, the constant term method not only can study mock theta functions but also prove some well known identities. In this section, we prove two Liu's identities that derived from an expansion formula for q-series. We believe that the method use here can be applied to prove many other identities.

Identity I. (Liu, [16, eq. (8.22)])

$$(q)_\infty(q^2;q^2)_\infty = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j(1-q^{2n+1})q^{2n^2+n-j^2}. \tag{13}$$

Proof. Taking $\epsilon = -1, A = 1, B = 2, C = 3, a = -q$ and $\lambda = 4$ in Lemma 2.2. Then the right side of (13) is the coefficient of z^0 in the Laurent series expansion of

$$\begin{aligned}
&\frac{-q(q^2;q^2)_\infty(q^4;q^4)_\infty^2\theta(-zq^3,q^2)\theta(z,q^4)\theta(q^2,q^4)}{\theta(-z^{-1}q,q^4)\theta(-zq,q^4)\theta(-q,q^4)}\\
&= \frac{-q(q^2;q^2)_\infty(q^2;q^4)_\infty^2(q^4;q^4)_\infty^2\theta(-zq^3,q^2)\theta(z,q^4)}{(-q;q^2)_\infty\theta(-z^{-1}q,q^4)\theta(-zq,q^4)}\\
&= \frac{-q(q)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty(-zq^3,-z^{-1}q^{-1};q^2)_\infty(z,z^{-1}q^4;q^4)_\infty}{(-z^{-1}q,-zq^3;q^4)_\infty(-zq,-z^{-1}q^3;q^4)_\infty}\\
&= -(q)_\infty(q^2;q^2)_\infty j(z;q^4)z^{-1}\\
&= -(q)_\infty(q^2;q^2)_\infty\sum_{n=-\infty}^{\infty}(-1)^n q^{4\binom{n}{2}}z^{n-1},
\end{aligned}$$

in the annulus $|q|^{-1} < |z| < |q|^{-3}$. We get the coefficient of z^0, when $n = 1$ and is

$$(q)_\infty(q^2;q^2)_\infty.$$

This completes the proof.

Identity II. (Liu, [17, eq.(3.19)])

$$\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty} = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n(1+q^{2n+1})q^{3n^2+2n-2j^2-j}. \tag{14}$$

Proof. Taking $\epsilon = 1, A = 1, B = 2, C = 3, a = q$ and $\lambda = 6$ in Lemma 2.2. Then the right side of (14) is the coefficient of z^0 in the Laurent series expansion of

$$\begin{aligned}
&\frac{q(q^2;q^2)_\infty(q^6;q^6)_\infty^2\theta(zq^3,q^2)\theta(z,q^6)\theta(q^2,q^6)}{\theta(z^{-1}q,q^6)\theta(zq,q^6)\theta(q,q^6)}\\
&= \frac{q(q^2;q^2)_\infty^2(q^3;q^6)_\infty(q^6;q^6)_\infty\theta(zq^3,q^2)\theta(z,q^6)}{(q;q^2)_\infty\theta(z^{-1}q,q^6)\theta(zq,q^6)}\\
&= \frac{q(q^2;q^2)_\infty^2(q^3;q^6)_\infty(q^6;q^6)_\infty(zq^3,z^{-1}q^{-1};q^2)_\infty(z,z^{-1}q^6;q^6)_\infty(zq^3,z^{-1}q^3;q^6)_\infty}{(q;q^2)_\infty(z^{-1}q,zq^5;q^6)_\infty(zq,z^{-1}q^5;q^6)_\infty(zq^3,z^{-1}q^3;q^6)_\infty}\\
&= -\frac{(q^2;q^2)_\infty^2(q^3;q^6)_\infty j(z;q^6)j(zq^3;q^6)_\infty z^{-1}}{(q;q^2)_\infty(q^6;q^6)_\infty}\\
&= -\frac{(q^2;q^2)_\infty^2(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{6\binom{n}{2}}z^{n-1}\sum_{m=-\infty}^{\infty}(-1)^m q^{6\binom{m}{2}}(zq^3)^m,
\end{aligned}$$

in the annulus $|q|^{-1} < |z| < |q|^{-5}$. We get the coefficient of z^0 when $m = -n+1$ and is

$$\begin{aligned}
&\frac{(q^2;q^2)_\infty^2(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty} q^{6n^2-9n+3}\\
&=\frac{(q^2;q^2)_\infty^2(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}q^3(-q^{-3},-q^{15},q^{12};q^{12})_\infty\\
&=\frac{(q^2;q^2)_\infty^2(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}(-q^9,-q^3,q^{12};q^{12})_\infty\\
&=\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty}.
\end{aligned}$$

This completes the proof.

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