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ON A COMBINATORIAL INTERPRETATION OF THE BISECTIONAL PENTAGONAL NUMBER THEOREM

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Dedicated to Prof. A.K. Agarwal on his 70^{th} ***Birth Anniversary***

Abstract: In this paper, we invoke the bisectional pentagonal number theorem to prove that the number of overpartitions of the positive integer n into odd parts is equal to twice the number of partitions of n into parts not congruent to 0, 2, 12, 14, 16, 18, 20 or 30 mod 32. This result allows us to experimentally discover new infinite families of linear partition inequalities involving Euler's partition function $p(n)$. In this context, we conjecture that for $k > 0$, the theta series

$$(-q;-q)_\infty \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

has non-negative coefficients.

Keyword and Phrases: Partitions, overpartitions, pentagonal number theorem.

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1. Introduction

The 18^{th} century mathematician Leonard Euler discovered a simple formula for the limiting case $n \to \infty$ of the q-shifted factorial

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0 \end{cases}$$

when $a = q$. This famous identity is known as Euler's pentagonal number theorem and states that

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{G_n}, \qquad |q| < 1, \tag{1}$$

where the exponents G_n are called generalized pentagonal numbers, i.e.,

$$G_n = \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \left(3 \left\lceil \frac{n}{2} \right\rceil + (-1)^n \right).$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \leqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$ and we shall use the compact notation

$$(a_1, a_2 \dots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

The Euler partition function $p(n)$ gives the number of ways of writing the nonnegative integer n as a sum of positive integers, where the order of addends is not considered significant. This may be defined by the generating function

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Multiplying both sides of this expression by $(q;q)_\infty$ and using the pentagonal number theorem, we get Euler's recurrence relation for $p(n)$:

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} p(n - G_j) = \delta_{0,n}, \tag{2}$$

where $\delta_{i,j}$ is the Kronecker delta function, $p(n) = 0$ for any negative integer n and $p(0) = 1$. More details about these famous results in partition theory can be found in Andrews's book [1].

Linear inequalities involving Eulers partition function $p(n)$ have been the subject of recent studies [3, 4, 8, 16, 18, 20]. In [16], the author proved the inequality

$$p(n) - p(n-1) - p(n-2) + p(n-5) \leqslant 0, \qquad n > 0, \tag{3}$$

in order to provide the fastest known algorithm for the generation of the partitions of n. Subsequently, Andrews and Merca [3] considered Euler's pentagonal number theorem (1) and proved a truncated theorem on partitions: For $k \geqslant 1$,

$$\frac{(-1)^{k-1}}{(q;q)_\infty} \sum_{n=0}^{2k-1} (-1)^{\lceil n/2 \rceil} q^{G_n} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2} + (k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \tag{4}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

An immediate consequence owing to the positivity of the sum on the right hand side of (4) is given by the following infinite family of linear partition inequalities: For $n > 0$, $k \geqslant 1$,

$$(-1)^{k-1} \sum_{j=0}^{2k-1} (-1)^{\lceil j/2 \rceil} p(n - G_j) \geqslant 0, \tag{5}$$

with strict inequality if $n \geqslant G_{2k}$. One can easily verify that (5) reduces to (3) when $k = 2$.

In this paper, motivated by these results, we consider a known bisection of Euler's pentagonal number theorem to derive a new partition identity. This result allows us to experimentally discover new infinite families of linear partition inequalities involving $p(n)$.

2. The bisectional pentagonal number theorem

Recently, Merca [17] proved the following bisectional version of Euler's pentagonal number theorem:

$$\sum_{G_n \text{ even}} (-1)^{\lceil n/2 \rceil} q^{G_n} = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty$$

which can be restated as

$$\frac{(q;q)_\infty + (-q;-q)_\infty}{2} = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty.$$

Multiplying both sides of this relation by $\dfrac{1}{(q;q)_\infty}$, we obtain

$$\frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \overline{p_o}(n) q^n\right) = \frac{(q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty}{(q;q)_\infty}, \tag{6}$$

where we have invoked the generating function for the number of overpartitions of n into odd parts, $\overline{p_o}(n)$, namely,

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = \frac{(-q;-q)_\infty}{(q;q)_\infty}. \tag{7}$$

Recall that an overpartition of the nonnegative integer n is a partition of n where the first occurrence of parts of each size may be overlined. For example, the overpartitions of the integer 3 are:

$$3,\ \bar{3},\ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1 \text{ and } \bar{1}+1+1.$$

The properties of the overpartitions have been the subject of many recent studies of Andrews [2], Corteel and Lovejoy [5], Hirschhorn [7], Kim [9], Lovejoy [12, 13, 14], Mahlburg [15] and Merca [19]. From our example, we see that $\overline{p_o}(3) = 4$.

Considering the identity (6), we immediately deduce a new combinatorial interpretation of the bisectional pentagonal number theorem.

Theorem 2.1. *For any positive integer n, the number of overpartitions of n into odd parts is equal to twice the number of partitions of n into parts not congruent to* 0, 2, 12, 14, 16, 18, 20 *or* 30 mod 32.

Example 1. We have $\overline{p_o}(5) = 8$ because the overpartions into odd parts of five are:

$$5,\ \bar{5},\ 3+1+1,\ 3+\bar{1}+1,\ \bar{3}+1+1,\ \bar{3}+\bar{1}+1,\ 1+1+1+1+1,\ \bar{1}+1+1+1+1.$$

On the other hand, the integer 5 has four partitions into parts not congruent to 0, 2, 12, 14, 16, 18, 20 or 30 mod 32:

$$5,\ 4+1,\ 3+1+1,\ 1+1+1+1+1.$$

We remark that the number of overpartitions of n can be expressed in terms of Euler's partition function $p(n)$ considering the generalized pentagonal numbers G_n.

Corollary 2.2 *For $n \geqslant 0$,*

$$\overline{p_o}(n) = \sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil + G_j} p(n - G_j).$$

Proof. This identity follows easily from the generating function (7), considering pentagonal numbers theorem and the generating function of $p(n)$:

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = \left(\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} (-q)^{G_n} \right) \left(\sum_{n=0}^{\infty} p(n) q^n \right).$$

For any positive integer n, Euler's recurrence relation (2) for the partition function $p(n)$ can be rewritten as

$$\sum_{G_j \text{ even}} (-1)^{\lceil j/2 \rceil} p(n-G_j) = \sum_{G_j \text{ odd}} (-1)^{\lceil j/2 \rceil -1} p(n-G_j). \tag{8}$$

Moreover, considering this identity and Corollary 2.2, it is clear that the overpartitions function $\overline{p_o}(n)$ can be expressed in terms of Euler's partition function $p(n)$ using only the generalized pentagonal numbers of the same parity.

Corollary 2.3. *For $n > 0$,*

$$\overline{p_o}(n) = 2 \sum_{G_j \text{ even}} (-1)^{\lceil j/2 \rceil} p(n-G_j).$$

It is not difficult to prove that the nth generalized pentagonal number is even if and only if n is congruent to $\{0,2,5,7\}$ mod 8.

In the next section, we consider Corollaries 2.2 and 2.3 in order to provide new infinite families of partition inequalities for the Euler's partition function $p(n)$.

3. Linear partition inequalities

The Hardy-Ramanujan-Rademacher [6, 10, 11, 21] formula for $p(n)$ states that

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right),$$

where

$$A_k(n) = \sum_{\substack{0 \leqslant h < k \\ gcd(h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

and $\omega_{h,k}$ is a certain 24th root of unity. By taking only the first term of this expansion, we obtain the asymptotic behavior of $p(n)$,

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, \qquad n \to \infty,$$

which shows that the growth of $p(n)$ is subexponential.

For $k \geqslant 0$ and $r \in \{0,1,2\}$, it follows from the large n asymptotic behavior of the partition function $p(n)$ that an inequality of the form

$$(-1)^k \sum_{j=0}^{4k+r} (-1)^{\lceil j/2 \rceil + G_j} p(n-G_j) \geqslant 0$$

holds for all $n > N$, for an appropriately specified N. From an asymptotic point of view, this inequality can be considered trivial because the number of positive terms is different from the number of negative terms.

For $k \geqslant 0$, the asymptotic information provided by the partition function $p(n)$ is not useful when we consider an expression of the form

$$\sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} p(n - G_j) \tag{9}$$

in which the number of positive terms equals the number of negative terms. However, we remark the following infinite family of inequalities.

Theorem 3.1. *For $n, k \geqslant 0$ there is a non-negative integer N such that the inequality*

$$(-1)^k \sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} p(n - G_j) \geqslant 0,$$

holds for $n > N$.

Proof. First, we consider that

$$\begin{aligned}
&\sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} (-G_j) \\
&\quad = \sum_{j=0}^{2k+1} (-1)^{j + G_{2j}} (-G_{2j}) + \sum_{j=0}^{2k+1} (-1)^{j+1+G_{2j+1}} (-G_{2j+1}) \\
&\quad = \frac{1}{2} \sum_{j=1}^{2k+1} (-1)^{j(j-1)/2} (3j^2 + j) + \frac{1}{2} \sum_{j=1}^{2k+2} (-1)^{j(j-1)/2 + j} (3j^2 - j) \\
&\quad = \frac{1}{2} \sum_{j=1}^{2k+1} (-1)^{j(j-1)/2} \big(3j^2 + j + (-1)^j (3j^2 - j)\big) + (-1)^k (k+1)(6k+5) \\
&\quad = 12 \sum_{j=1}^{k} (-1)^{j+1} j^2 + \sum_{j=0}^{k} (-1)^j (2j+1) + (-1)^k (k+1)(6k+5) \\
&\quad = 6(-1)^{k+1}(k^2 + k) + (-1)^k (k+1) + (-1)^k (k+1)(6k+5) \\
&\quad = 6(-1)^k (k+1).
\end{aligned}$$

It is clear that

$$(-1)^k \sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} (-G_j) \geqslant 0.$$

According to [20, Theorem 1.2], there is a non-negative integer N such that for $n > N$, the expression

$$\sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} p(n - G_j)$$

has the same sign as the expression

$$\sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} (-G_j).$$

This concludes the proof.

Related to Theorem 3.1, we remark that it is still an open problem to give a characterization for the behavior of the sum (9) when $1 \leqslant n \leqslant N$. There is a substantial amount of numerical evidence to state the following conjectures.

Conjecture 1. *For $k, n \geqslant 0$,*

$$(-1)^{k-1} \left(\overline{p_o}(n) - \sum_{j=0}^{4k+3} (-1)^{\lceil j/2 \rceil + G_j} p(n - G_j) \right) \geqslant 0,$$

with strict inequality if $n \geqslant G_{4k+4}$. For example,

$$\begin{aligned} & p(n) + p(n-1) - p(n-2) - p(n-5) \geqslant \overline{p_o}(n), \\ & p(n) + p(n-1) - p(n-2) - p(n-5) \\ & \qquad - p(n-7) - p(n-12) + p(n-15) + p(n-22) \leqslant \overline{p_o}(n). \end{aligned}$$

Let

$$\{E_n\}_{n \geqslant 0} = \{0, 2, 12, 22, 26, 40, 70, 92, 100, 126, 176, 210, 222, 260, 330, \ldots\}$$

be the sequence of the even generalized pentagonal numbers. Inspired by Corollary 2.3, we get the following result.

Theorem 3.2. *For $n, k \geqslant 0$ there is a non-negative integer N such that the inequality*

$$(-1)^k \sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} p(n - E_j) \geqslant 0,$$

holds for $n > N$.

Proof. Taking into account that

$$E_{4k} = G_{8k}, \quad E_{4k+1} = G_{8k+2}, \quad E_{4k+2} = G_{8k+5} \quad \text{and} \quad E_{4k+3} = G_{8k+7},$$

we can write:

$$\begin{aligned}
\sum_{j=0}^{4k+1}(-1)^{\lceil j/2\rceil}(-E_j) &= \sum_{j=0}^{2k}(-1)^j E_{2j+1} - \sum_{j=0}^{2k}(-1)^j E_{2j} \\
&= \sum_{j=0}^{k} E_{4j+1} - \sum_{j=0}^{k-1} E_{4j+3} - \sum_{j=0}^{k} E_{4j} + \sum_{j=0}^{k-1} E_{4j+2} \\
&= \sum_{j=0}^{k}(G_{8j+2} - G_{8j}) - \sum_{j=0}^{k-1}(G_{8j+7} - G_{8j+5}) \\
&= 2\sum_{j=0}^{k}(6j+1) - 2\sum_{j=0}^{k-1}(6j+5) \\
&= \sum_{j=0}^{k-1}(-8) + 12k + 2 \\
&= 4k + 2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{4k+3}(-1)^{\lceil j/2\rceil}(-E_j) &= \sum_{j=0}^{2k+1}(-1)^j E_{2j+1} - \sum_{j=0}^{2k+1}(-1)^j E_{2j} \\
&= \sum_{j=0}^{k}(E_{4j+1} - E_{4j+3} - E_{4j} + E_{4j+2}) \\
&= \sum_{j=0}^{k}(G_{8j+2} - G_{8j} - G_{8j+7} + G_{8j+5}) \\
&= \sum_{j=0}^{k}(-8) \\
&= -8(k+1).
\end{aligned}$$

So we deduce that

$$\sum_{j=0}^{2k+1}(-1)^{\lceil j/2\rceil}(-E_j) = \begin{cases} 2(k+1), & \text{if } k \text{ is even,} \\ -4(k+1), & \text{if } k \text{ is odd.} \end{cases}$$

It is clear that

$$(-1)^k \sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} (-E_j) \geqslant 0.$$

According to [20, Theorem 1.2], there is a non-negative integer N such that for $n > N$, the expression

$$\sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} p(n - E_j)$$

has the same sign as the expression

$$\sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} (-E_j).$$

This concludes the proof.

In analogy with Conjecture 1, we propose the following infinite family of linear partition inequalities.

Conjecture 2. *For $k \geqslant 0$ and $n > 0$,*

$$(-1)^{k-1} \left(\frac{\overline{p_o}(n)}{2} - \sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} p(n - E_j) \right) \geqslant 0,$$

with strict inequality if $n \geqslant E_{2k+2}$. For example,

$$p(n) - p(n-2) \geqslant \overline{p_o}(n)/2,$$
$$p(n) - p(n-2) - p(n-12) + p(n-22) \leqslant \overline{p_o}(n)/2,$$
$$p(n) - p(n-2) - p(n-12) + p(n-22) + p(n-26) - p(n-40) \geqslant \overline{p_o}(n)/2.$$

It is still an open problem to give combinatorial interpretations for our sums in these conjectures.

4. Concluding remarks

A new partition identity (Theorem 2.1) has been introduced in this paper as a combinatorial interpretation of the bisectional pentagonal number theorem. Few connections between overpartitions into odd parts and Euler's partition function $p(n)$ are derived in this context.

Taking into account the generating function (7), we deduce that the overpartition function $\overline{p_o}(n)$ satisfies Euler's recurrence relation for the partition function

$p(n)$ unless n is a generalized pentagonal number.

Theorem 4.1. *For $n \geqslant 0$,*

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} \overline{p_o}(n - G_j) = \delta(n),$$

where

$$\delta(n) = \begin{cases} (-1)^{G_{m-(-1)^m}}, & \text{if } n = G_m,\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases}$$

In analogy with (5), we propose the following inequality.

Conjecture 3. *For $k, n \geqslant 0$,*

$$(-1)^k \left(\sum_{j=0}^{2k+1} (-1)^{\lceil j/2 \rceil} \overline{p_o}(n - G_j) - \delta(n) \right) \geqslant 0,$$

with strict inequality if $n \geqslant G_{2k+2}$. For example,

$$\begin{aligned} &\overline{p_o}(n) - \overline{p_o}(n-1) \geqslant \delta(n) \\ &\overline{p_o}(n) - \overline{p_o}(n-1) - \overline{p_o}(n-2) + \overline{p_o}(n-5) \leqslant \delta(n). \end{aligned}$$

It would be very appealing to have a combinatorial interpretation of the sum in this conjecture. By the truncated pentagonal number theorem (3), we deduce that Conjecture 3 can be rewritten as follows.

Conjecture 4. *For $k > 0$, the expression*

$$(-q;-q)_\infty \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

has non-negative coefficients.

According to Andrews and Merca [3], we have

$$\sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

where $M_k(n)$ is the number of partitions of n in which k is the least positive integer that is not a part and there are more parts $> k$ than there are $< k$. We can write a new combinatorial interpretation of Conjecture 4.

Conjecture 5. *For $k > 0$, $n \geqslant 0$,*

$$\sum_{j=0}^{2k-1} (-1)^{\lceil j/2 \rceil + G_j} M_k(n - G_j) \geqslant 0,$$

with strict inequality if $n \geqslant G_{2k}$.

On the other hand, there is a substantial amount of numerical evidence to state the following conjectures.

Conjecture 6. *For $k > 1$, $n \geqslant G_{2k}$,*

$$\sum_{j=0}^{2k-1} (-1)^{\lceil j/2 \rceil + G_j} M_k(n - G_j) \leqslant \overline{p_o}(n - G_{2k}),$$

with strict inequality if $n \geqslant G_{2k} + k + 2$.

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