

BINARY AND SEMI-FIBONACCI PARTITIONS

George E. Andrews

The Pennsylvania State University
University Park, PA 16802, USA

E-mail: gea1@psu.edu

Dedicated to Prof. A.K. Agarwal on his 70th Birth Anniversary

Abstract: It is proved that the partitions of n into powers of two with all parts appearing an odd number of times equals the number of Semi-Fibonacci partition of n . The parity of the number of such partitions is also exhibited.

Keyword and Phrases: Binary partitions, semi-Fibonacci partitions.

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1. Introduction

The set $\mathfrak{S}\mathfrak{F}(n)$ of semi-Fibonacci partitions is defined as follows: $\mathfrak{S}\mathfrak{F}(1) = \{1\}$, $\mathfrak{S}\mathfrak{F}(2) = \{2\}$. If $n > 2$ and even the $\mathfrak{S}\mathfrak{F}(n)$ consists of the partitions of $\mathfrak{S}\mathfrak{F}(\frac{n}{2})$ wherein each part has been multiplied by 2. If n is odd, $\mathfrak{S}\mathfrak{F}(n)$ arises from two sources: first a 1 is inserted in each partition of $n - 1$ and second a 2 is added to the single odd part of $\mathfrak{S}(n - 2)$ (note: it is easily seen by induction that semi-Fibonacci partitions have at most one odd part).

Thus here are the first seven $\mathfrak{S}\mathfrak{F}(n)$:

$$\mathfrak{S}\mathfrak{F}(1) = \{1\}$$

$$\mathfrak{S}\mathfrak{F}(2) = \{2\}$$

$$\mathfrak{S}\mathfrak{F}(3) = \{2 + 1, 3\}$$

$$\mathfrak{S}\mathfrak{F}(4) = \{4\}$$

$$\mathfrak{S}\mathfrak{F}(5) = \{4 + 1, 3 + 2, 5\}$$

$$\mathfrak{S}\mathfrak{F}(6) = \{4 + 2, 6\}$$

$$\mathfrak{S}\mathfrak{F}(7) = \{4 + 2 + 1, 6 + 1, 4 + 3, 5 + 2, 7\}$$

We now define

$$\text{sf}(n) = |\mathfrak{S}\mathfrak{F}(n)|.$$

Thus $\text{sf}(1) = \text{sf}(2) = 1$, $\text{sf}(3) = 2$, $\text{sf}(4) = 1$, $\text{sf}(5) = 3$, $\text{sf}(6) = 2$, $\text{sf}(7) = 5$.

From the definition of $\mathfrak{S}\mathfrak{F}(n)$, we see that $\text{sf}(-1) = 0$, $\text{sf}(0) = 1$, and for $n > 0$

$$\text{sf}(n) = \begin{cases} \text{sf}(\frac{n}{2}) & \text{if } n \text{ is even} \\ \text{sf}(n-1) + \text{sf}(n-2) & \text{if } n \text{ is odd} \end{cases} \quad (1.1)$$

It appears that Jonathan Vos Post is the first one to consider the semi-Fibonacci sequence $\text{sf}(n)$ [2]. Numerous properties are listed in [2] including a function equation for $g(x) = F(x) - 1$; however, both our theorems are not there.

George Beck [1] appears to be the first to consider semi-Fibonacci partitions, and in [1], he proves a nice theorem for a set of related polynomials. I thank George Beck for drawing my attention to semi-Fibonacci partitions.

Binary partitions are partitions into powers of 2. We let $\text{ob}(n)$ denote the number of binary partitions of n in which each part appears an odd number of times. Thus $\text{ob}(7) = 5$ because the relevant partitions are $4 + 2 + 1$, $4 + 1 + 1 + 1$, $2 + 2 + 2 + 1$, $2 + 1 + 1 + 1 + 1 + 1$, and $1 + 1 + 1 + 1 + 1 + 1 + 1$.

Theorem 1. *For each $n \geq 0$*

$$\text{sf}(n) = \text{ob}(n) \quad (1.2)$$

We shall also treat the parity of these sequences.

Theorem 2. *For each $n > 0$, $\text{sf}(n)$ is even if $3|n$ and odd otherwise.*

Section 2 is devoted to the proof of Theorem 1. Section 3 treats Theorem 2, and Section 4 considers open questions.

2. Proof of Theorem 1

We define

$$F(x) = \sum_{n \geq 0} \text{sf}(n)x^n. \quad (2.1)$$

Then

$$\begin{aligned}
F(x) &= \sum_{n \geq 0} \text{sf}(2n)x^{2n} + \sum_{n \geq 0} \text{sf}(2n+1)x^{2n+1} \\
&= \sum_{n \geq 0} \text{sf}(n)x^{2n} + \sum_{n \geq 0} (\text{sf}(2n) + \text{sf}(2n-1))x^{2n+1} \\
&= F(x^2)(1+x) + x^2 \sum_{n \geq 0} \text{sf}(2n+1)x^{2n+1} \\
&= F(x^2)(1+x) + \frac{x^2}{2}(F(x) - F(-x)).
\end{aligned}$$

Also

$$\begin{aligned}
\frac{1}{2}(F(x) + F(-x)) &= \sum_{n \geq 0} \text{sf}(2n)x^{2n} \\
&= \sum_{n \geq 0} \text{sf}(n)x^{2n} \\
&= F(x^2)
\end{aligned} \tag{2.2}$$

From (2.2) we deduce that

$$F(-x) = 2F(x^2) - F(x). \tag{2.3}$$

Substituting (2.3) into (2.1), we find

$$F(x) = (1+x)F(x^2) + \frac{x^2}{2}F(x) - x^2(F(x^2) - \frac{1}{2}F(x)) \tag{2.4}$$

Simplifying we obtain

$$(1-x^2)F(x) = (1+x-x^2)F(x^2), \tag{2.5}$$

or

$$F(x) = \frac{1+x-x^2}{1-x^2}F(x^2), \tag{2.6}$$

and iterating (2.6), we obtain

$$\begin{aligned}
 F(x) &= \prod_{n=0}^{\infty} \frac{1 + x^{2^n} - x^{2^{n+1}}}{1 - x^{2^{n+1}}} \\
 &= \prod_{n=0}^{\infty} \left(1 + \frac{x^{2^n}}{1 - x^{2^{n+1}}} \right) \\
 &= \prod_{n=0}^{\infty} \left(1 + \sum_{m=0}^{\infty} x^{2^n(2m+1)} \right) \\
 &= \sum_{m \geq 0} \text{ob}(m)x^m. \tag{2.7}
 \end{aligned}$$

Finally comparing coefficients in (2.1) and (2.7), we obtain Theorem 1.

3. Proof of Theorem 2

Here we must utilize the fact that every positive integer is uniquely the sum of distinct powers of 2. In terms of generating functions, this is the identity

$$\frac{1}{1-x} = \prod_{n=0}^{\infty} (1 + x^{2^n}). \tag{3.1}$$

We now proceed modulo 2 (where $-x \equiv x \pmod{2}$). Hence

$$\begin{aligned}
 \sum_{n \geq 0} \text{sf}(n)x^n &= \prod_{n=0}^{\infty} \frac{(1 + x^{2^n} - x^{2^{n+1}})}{1 - x^{2^{n+1}}} \\
 &\equiv \prod_{n=0}^{\infty} \frac{1 + x^{2^n} + x^{2^{n+1}}}{(1 - x^{2^n})^2} \pmod{2} \\
 &= \prod_{n=0}^{\infty} \frac{(1 - x^{3 \cdot 2^n})}{(1 - x^{2^n})^3} \\
 &\equiv \prod_{n=0}^{\infty} \frac{(1 + x^{3 \cdot 2^n})}{(1 + x^{2^n})^3} \pmod{2} \\
 &= \frac{(1-x)^3}{1-x^3}
 \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{-3x + 3x^2}{1 - x^3} \\
&\equiv 1 + \frac{x + x^2}{1 - x^3} \pmod{2} \\
&= 1 + \sum_{n=0}^{\infty} (x^{3n+1} + x^{3n+2}), \tag{3.2}
\end{aligned}$$

and Theorem 2 follows by comparing coefficients in the extremes of (3.2).

4. Conclusion

It would be nice to have combinatorial proofs of both theorems. In light of the recursive nature of the definition of $\text{sf}(n)$, this should be quite tractable. One only need note that the recurrence (1.1) for $\text{ob}(n)$ works as follows. The top line follows by multiplying each part in the partition of $\frac{n}{2}$ by 2. The bottom line arises by inserting a 1 in the partitions of $n - 1$ and inserting two ones in the partitions of $n - 2$.

References

- [1] G. Beck, <http://demonstrations.wolfram.com/SemiFibonacciPartitions/>
- [2] On Line Encyclopedia of Integer Sequences, sequence A030067.

