South East Asian J. of Mathematics and Mathematical Sciences
Vol. 15, No. 3 (2019), pp. 171-184

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

SOME REDUCTION FORMULAS FOR APPELL'S FUNCTION OF FOURTH KIND HAVING DIFFERENT ARGUMENT

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(**Received:** *Sep. 30, 2019* **Accepted:** *Dec. 25, 2019* **Published:** *Dec. 31, 2019)*

Abstract: The objective of this paper is to find some closed form of certain reduction formulas for Appell's hypergeometric function F_4 with suitable convergence conditions.

Keywords and Phrases: Gauss hypergeometric function; Appell's function of fourth kind; Kümmer's first, second and third summation theorems.

2010 Mathematics Subject Classification: Primary 33C65, 33C20; Secondary 33C05.

1. Introduction and Preliminaries

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} := \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$

$$\mathbb{Z}_0^- := \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- := \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers. In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 \ , & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\dots(\lambda+n-1) \ , & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ quotient exists

$$\int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha}, \tag{1.2}$$

$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \text{ or } \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

Gauss ordinary hypergeometric function is defined as

$${}_2F_1\left[\begin{array}{cc} \lambda, \mu; & \\ & z \\ \sigma; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n z^n}{(\sigma)_n \ n!} \tag{1.3}$$

(1.3a) The infinite series (1.3) is always convergent when $|z| < 1$ and $\lambda, \mu, \sigma \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

(1.3b) The infinite series (1.3) is absolutely convergent when $|z| = 1$, $\Re(\sigma - \mu - \lambda) > 0$ and $\lambda, \mu, \sigma \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

(1.3c) The infinite series (1.3) is conditionally convergent when $|z| = 1$, $z \neq 1$, $-1 < \Re(\sigma - \mu - \lambda) \leq 0$ and $\lambda, \mu, \sigma \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

(1.3d) The infinite series (1.3) is divergent when $|z| = 1$, $\Re(\sigma - \mu - \lambda) \leq -1$ and $\lambda, \mu, \sigma \in \mathbb{C} \backslash \mathbb{Z}_0^-$.

(1.3e) When any one or both numerator parameters of Gauss series (1.3) is zero or a negative integer and denominator parameter is neither zero nor a negative integer then series (1.3) terminates and the questions of convergence does not enter the discussion.

Legendre's duplication formula

$$\sqrt{(\pi)}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big), \qquad \big(2z \in \mathbb{C} \backslash \mathbb{Z}_0^-\big). \tag{1.4}$$

Classical Gauss summation theorem [10, p.30,Equation 1.2(7)]

$$ {}_2F_1\left[\begin{array}{c} a, b; \\ d; \end{array} 1\right] = \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}\ , \quad \Big(\Re(d-a-b)>0,\ d\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big). \tag{1.5} $$

Kümmer's first, second and third summation theorems [4, p.134]

$$ {}_2F_1\left[\begin{array}{c} a,\quad b; \\ 1+a-b; \end{array} -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}, \tag{1.6} $$

$$ \Big(\Re(b)<1, 1+a-b\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big)\ . $$

$$ {}_2F_1\left[\begin{array}{c} a,\ b; \\ \frac{1+a+b}{2}; \end{array} \frac{1}{2}\right] = \frac{\sqrt{\pi}\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})}, \qquad \left(\frac{1+a+b}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{1.7} $$

$$ {}_2F_1\left[\begin{array}{c} a,\ 1-a; \\ c; \end{array} \frac{1}{2}\right] = \frac{2^{1-c}\sqrt{\pi}\Gamma(c)}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})} = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2})\Gamma(\frac{c+1-a}{2})}, \tag{1.8} $$

$$ \Big(c\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big). $$

Some generalizations of Kümmer's second summation theorem

Summation theorem recorded by Prudnikov *et al.* [5, p.491, Entry (7.3.7.2)]

$$ {}_2F_1\left[\begin{array}{c} a,\quad b; \\ \frac{a+b+1-p}{2}; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{a+b+1-p}{2})}{\Gamma(b)} \sum_{k=0}^{p}\left\{\binom{p}{k}\frac{\Gamma(\frac{b+k}{2})}{\Gamma(\frac{a+1+k-p}{2})}\right\}, \tag{1.9} $$

$$ \left(b, \tfrac{1+a+b-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad p\in\mathbb{N}_0\right). $$

In the year 2011, following summation theorem was given by Rakha-Rathie [8, p.827, Theorem (1)]

$$ {}_2F_1\left[\begin{array}{c} a,\quad b; \\ \frac{a+b+1+p}{2}; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{a+b+1+p}{2})\Gamma(\frac{a-b+1-p}{2})}{\Gamma(b)\Gamma(\frac{a-b+1+p}{2})} \sum_{k=0}^{p}\left\{\binom{p}{k}\frac{(-1)^k\Gamma(\frac{b+k}{2})}{\Gamma(\frac{a+1+k-p}{2})}\right\}\ , \tag{1.10} $$

$$ \left(b, \tfrac{a+b+1+p}{2}, \tfrac{a-b+1-p}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad p\in\mathbb{N}_0\right). $$

In the year 2016, two more summation theorems were given by Qureshi-Baboo [6, p.48, Equations (3.1) and (3.3)]

$$ {}_2F_1\left[\begin{matrix} a, & b; \\ \frac{a+b-m}{2}; & \end{matrix} \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{a+b-m}{2})}{\Gamma(a)} \sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}, \tag{1.11} $$

$$ \left(a,\ \tfrac{a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right). $$

and

$$ {}_2F_1\left[\begin{matrix} a, & b; \\ \frac{a+b+m}{2}; & \end{matrix} \frac{1}{2}\right] = \frac{2^{a-1}\Gamma(\frac{a+b+m}{2})\Gamma(\frac{b-a-m}{2})}{\Gamma(a)\Gamma(\frac{b-a+m}{2})} $$

$$ \times\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}, \tag{1.12} $$

$$ \left(a,\ \tfrac{a+b+m}{2},\ \tfrac{b-a-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right). $$

Some generalizations of Kümmer's third summation theorem

In the year 2011, following summation theorems were given by Rakha-Rathie [8, p.828, Theorems (6, 5)]

$$ {}_2F_1\left[\begin{matrix} a, 1-a-m; \\ c; \end{matrix} \frac{1}{2}\right] = \frac{2^{1-m-c}\Gamma(\frac{1}{2})\Gamma(c)}{\Gamma(\frac{c-a}{2})\Gamma(\frac{c-a+1}{2})} \sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{c-a+r}{2})}{\Gamma(\frac{c+a+r}{2})}\right\}, \tag{1.13} $$

$$ \left(c,\ c-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right). $$

$$ {}_2F_1\left[\begin{matrix} a, 1-a+m; \\ c; \end{matrix} \frac{1}{2}\right] = \frac{2^{1+m-c}\Gamma(\frac{1}{2})\Gamma(c)\Gamma(a-m)}{\Gamma(a)\Gamma(\frac{c-a}{2})\Gamma(\frac{c-a+1}{2})} \sum_{r=0}^{m}\left\{(-1)^r\binom{m}{r}\frac{\Gamma(\frac{c-a+r}{2})}{\Gamma(\frac{c+a+r}{2}-m)}\right\}, \tag{1.14} $$

$$ \left(c, a, a-m, c-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\right). $$

In the year 2016, some summation theorems were given by Qureshi-Baboo [7, p.144, Equation (3.3) and p.145, Equation (3.5)]

$$ {}_2F_1\left[\begin{matrix} a, & -a-m; \\ & c; \end{matrix} \frac{1}{2}\right] = \frac{2^{-a-m-1}\Gamma(c)}{\Gamma(c-a)} \sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{c-a+r}{2})}{\Gamma(\frac{c+a+r}{2})}+\frac{\Gamma(\frac{c-a+r+1}{2})}{\Gamma(\frac{c+a+r+1}{2})}\right]\right\}, \tag{1.15} $$

$$\left(c, c-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right).$$

$$ {}_2F_1\left[\begin{array}{ccc} a, & -a+m; & \frac{1}{2} \\ & c; & \end{array}\right] = \frac{2^{-a+m-1}\Gamma(c)\Gamma(a-m)}{\Gamma(a)\Gamma(c-a)} \times $$

$$\sum_{r=0}^{m}\left\{\binom{m}{r}(-1)^r\left[\frac{\Gamma(\frac{c-a+r}{2})}{\Gamma(\frac{c+a+r-2m}{2})} + \frac{\Gamma(\frac{c-a+r+1}{2})}{\Gamma(\frac{c+a+r+1-2m}{2})}\right]\right\}, \tag{1.16}$$

$$\left(a, c, a-m, c-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right).$$

Appell's double hypergeometric series

In the year 1880 Appell's defined the following double hypergeometric series ([1, p,73, Equation 4] see also [9, p,211. Equation 8.1.6])

$$F_4\left[\alpha, \beta; \gamma, \delta; x, y\right] = F^{2:0;0}_{0:1;1}\left[\begin{array}{c} \alpha,\ \beta : —; —; \\ — : \gamma\ ;\ \delta\ ; \end{array}\ x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n}\frac{x^m y^n}{m!n!} \tag{1.17}$$

Convergence conditions of Appell's double series F_4

(1.17a) Appell's series F_4 is convergent when $\sqrt{|x|}+\sqrt{|y|} < 1\ ;\ \alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

(1.17b) Appell's series F_4 is absolutely convergent when $\sqrt{|x|} + \sqrt{|y|} = 1\ ;\ x \neq 0, y \neq 0\ ;\ \alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re[(\alpha+\beta)-(\gamma+\delta)] < -1$.

(1.17c) When α or β or α, β both are negative integers and $\gamma, \delta \in \mathbb{C}\backslash\mathbb{Z}_0^-$, then Appell's series F_4 will be a polynomial.

For absolute convergence (1.17b) of above series see a paper of Hái *et al.* [3, p.105, Equation 1.3 and p.107, Theorem 3 (Equations 1.7, 1.8 and 1.9)].

In each section of this paper, any values of parameters and variables leading to the results which do not make sense, are tacitly excluded.

2. Application in $F_4\left[A, B; B, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right]$

Consider the following reduction formula ([2], p.238,Equation 7, see also [1], p.102,Question 20(iv))

$$F_4\left[A, B; B, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right]$$

$$= [(1-x)(1-y)]^{A}\, {}_2F_1\left[\begin{array}{r} A, 1+A-B; \\ B; \end{array} xy\right], \tag{2.1}$$

$$\left(|xy| < 1\, ; \sqrt{\left|\frac{-x}{(1-x)(1-y)}\right|} + \sqrt{\left|\frac{-y}{(1-x)(1-y)}\right|} < 1\, ; A, B, 1+A-B \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Putting $y = \frac{1}{x}$ in reduction formula (2.1) and applying classical Gauss summation theorem (1.5), we get

$$F_4\left[A, B; B, B; \frac{x^2}{(1-x)^2}, \frac{1}{(1-x)^2}\right] = \left[\frac{-(1-x)^2}{x}\right]^{A} \frac{\Gamma(B)\Gamma(2B-2A-1)}{\Gamma(B-A)\Gamma(2B-A-1)}, \tag{2.2}$$

$$\left(\Re(A-B) < -1\ ;\ \sqrt{\left|\frac{x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1}{(1-x)^2}\right|} = 1 \text{ for } -\infty < \Re(x) < 0\ ;\right.$$

$$\left. A, B,\ 2B-2A-1 \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Here $\Re(A-B) < -1$ is the common convergence condition (see the absolute convergence condition (1.3b) of Gauss series and absolute convergence condition (1.17b) for Appell's function F_4).

Put $y = \frac{1}{2x}$ in reduction formula (2.1), we get

$$F_4\left[A, B; B, B; \frac{2x^2}{(1-x)(1-2x)}, \frac{1}{(1-x)(1-2x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^{A} {}_2F_1\left[\begin{array}{r} A, 1+A-B; \\ B; \end{array} \frac{1}{2}\right], \tag{2.3}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0\ ;\right.$$

$$\left. A, B, 1+A-B \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Now putting $A = a$, $B = \frac{2(a+1)}{3}$ in reduction formula (2.3) and using Kummer's second summation theorem (1.7), we get

$$F_4\left[a, \frac{2(a+1)}{3}; \frac{2(a+1)}{3}, \frac{2(a+1)}{3}; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{a+4}{6})}, \tag{2.4}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ; \right.$$

$$\left. a,\ \frac{2(a+1)}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^- \right).$$

In the reduction formula (2.3) putting $A = a$, $B = \frac{2a+2-m}{3}$ and using summation theorem (1.9), we get

$$F_4\left[a, \frac{2a+2-m}{3}; \frac{2a+2-m}{3}, \frac{2a+2-m}{3}; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{2^{a-1}\Gamma(\frac{2a+2-m}{3})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{a+r}{2})}{\Gamma(\frac{4+a-2m+3r}{6})} \right\}, \tag{2.5}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ; \right.$$

$$\left. a, \frac{2a+2-m}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0 \right).$$

In reduction formula (2.3) putting $A = a$, $B = \frac{2a+2+m}{3}$ and using summation theorem (1.10), we get

$$F_4\left[a, \frac{2a+2+m}{3}; \frac{2a+2+m}{3}, \frac{2a+2+m}{3}; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{2^{a-1}\Gamma(\frac{2a+2+m}{3})\Gamma(\frac{2-a-2m}{3})}{\Gamma(a)\Gamma(\frac{2-a+m}{3})} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{4+a-4m+3r}{6})} \right\}, \tag{2.6}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0; \right.$$

$$\left. a, \frac{2a+2+m}{3}, \frac{2-a-2m}{3}, \frac{2-a+m}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0 \right).$$

In reduction formula (2.3) putting $A = a$, $B = \frac{2a+1-m}{3}$ and using summation theorem (1.11), we get

$$F_4\left[a, \frac{2a+1-m}{3}; \frac{2a+1-m}{3}, \frac{2a+1-m}{3}; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{2^{a-1}\Gamma(\frac{2a+1-m}{3})}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{a+r}{2})}{\Gamma(\frac{2+a+3r-2m}{6})} + \frac{\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{5+a+3r-2m}{6})}\right]\right\}, \tag{2.7}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \;; \right.$$

$$\left. a, \frac{2a+1-m}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^- \;;\; m \in \mathbb{N}_0 \right).$$

In reduction formula (2.3) putting $A = a$, $B = \frac{2a+1+m}{3}$ and using summation theorem (1.12), we get

$$F_4\left[a, \frac{2a+1+m}{3}; \frac{2a+1+m}{3}, \frac{2a+1+m}{3}; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{2^{a-1}\Gamma(\frac{2a+1+m}{3})\Gamma(\frac{1-a-2m}{3})}{\Gamma(a)\Gamma(\frac{1-a+m}{3})} \times$$

$$\sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{2+a-4m+3r}{6})} + \frac{(-1)^r\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{5+a-4m+3r}{6})}\right]\right\}, \tag{2.8}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \;; \right.$$

$$\left. a, \frac{2a+1+m}{3}, \frac{1-a-2m}{3}, \frac{1-a+m}{3} \in \mathbb{C}\backslash\mathbb{Z}_0^- \;;\; m \in \mathbb{N}_0 \right).$$

Now putting $A = a$, $B = 2a$ in reduction formula (2.3) and using Kummer's third summation theorem (1.8), we get

$$F_4\left[a, 2a; 2a, 2a; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(a)\Gamma(\frac{2a+1}{2})}{\Gamma(\frac{3a}{2})\Gamma(\frac{a+1}{2})}, \tag{2.9}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|}+\sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{for } -\infty < \Re(x) < 0; 2a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Now putting $A = a$ and $B = 2a+m$ in reduction formula (2.3) and using summation theorem (1.13), we get

$$F_4\left[a, 2a+m; 2a+m, 2a+m; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(2a+m)}{2^{a+m}\Gamma(a+m)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{a+m+r}{2})}{\Gamma(\frac{3a+m+r}{2})} \right\}, \tag{2.10}$$

$$\left(\sqrt{\left|\tfrac{2x^2}{(1-x)(1-2x)}\right|}+\sqrt{\left|\tfrac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ; \ a \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0\right).$$

Now putting $A = a$, $B = 2a - m$ in reduction formula (2.3) and using summation theorem (1.14), we get

$$F_4\left[a, 2a-m; 2a-m, 2a-m; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(2a-m)}{2^{a-m}\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r\Gamma(\frac{a-m+r}{2})}{\Gamma(\frac{3a-3m+r}{2})} \right\}, \tag{2.11}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|}+\sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ;\right.$$

$$\left. a, \ 2a-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0\right).$$

Now putting $A = a$, $B = 1+2a+m$ in reduction formula (2.3) and using summation theorem (1.15), we get

$$F_4\left[a, 1+2a+m; 1+2a+m, 1+2a+m; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(1+2a+m)}{2^{a+m+1}\Gamma(1+a+m)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{1+a+m+r}{2})}{\Gamma(\frac{1+3a+m+r}{2})} + \frac{\Gamma(\frac{2+a+m+r}{2})}{\Gamma(\frac{2+3a+m+r}{2})} \right] \right\}, \tag{2.12}$$

$$\left(\sqrt{\left|\tfrac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\tfrac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ; \ a \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0 \right).$$

Now putting $A = a$, $B = 1+2a-m$ in reduction formula (2.3) and using summation theorem (1.16), we get

$$F_4\left[a, 1+2a-m; 1+2a-m, 1+2a-m; \frac{2x^2}{(1-2x)(1-x)}, \frac{1}{(1-2x)(1-x)}\right]$$

$$= \left[\frac{(1-x)(2x-1)}{2x}\right]^a \frac{\Gamma(1+2a-m)}{2^{a-m+1}(a-m)\Gamma(a)} \times$$

$$\sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r\Gamma(\frac{1+a-m+r}{2})}{\Gamma(\frac{1+3a-3m+r}{2})} + \frac{(-1)^r\Gamma(\frac{2+a-m+r}{2})}{\Gamma(\frac{2+3a-3m+r}{2})} \right] \right\}, \tag{2.13}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)(1-2x)}\right|} + \sqrt{\left|\frac{1}{(1-x)(1-2x)}\right|} < 1 \text{ for } -\infty < \Re(x) < 0 \ ; \right.$$

$$\left. a \neq m \ ; \ a, \ 1+2a-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \ m \in \mathbb{N}_0 \right).$$

3. Application in $F_4\left[A, B; 1+A-B, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right]$

Consider the following reduction formula ([2], p.238,Equation 8, see also [1], p.102,Question 20(v))

$$F_4\left[A, B; 1+A-B, B; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right]$$

$$= [(1-y)]^A \, {}_2F_1\left[\begin{matrix} A, B; \\ 1+A-B; \end{matrix} \ \frac{-x(1-y)}{(1-x)}\right], \tag{3.1}$$

$$\left(\left|\frac{-x(1-y)}{(1-x)}\right| < 1 \ ; \ \sqrt{\left|\frac{-x}{(1-x)(1-y)}\right|} + \sqrt{\left|\frac{-y}{(1-x)(1-y)}\right|} < 1 \right.$$

$$; \ A, \ B, 1+A-B \in \mathbb{C}\backslash\mathbb{Z}_0^-\Big).$$

Putting $y = \frac{1}{x}$ in reduction formula (3.1) and applying the classical Gauss summation theorem (1.5), we get

$$F_4\left[A, B; 1+A-B, B; \frac{x^2}{(1-x)^2}, \frac{1}{(1-x)^2}\right] = \left[\frac{(x-1)}{x}\right]^A \frac{\Gamma(1+A-B)\Gamma(1-2B)}{\Gamma(1-B)\Gamma(1+A-2B)}, \tag{3.2}$$

$$\Bigg(\Re(B) < 0; \sqrt{\left|\frac{x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1}{(1-x)^2}\right|} = 1$$

$$\text{for } -\infty < \Re(x) < 0; A, B, 1+A-B \in \mathbb{C}\backslash\mathbb{Z}_0^-\Bigg).$$

Here $\Re(B) < 0$ is the common convergence condition (see absolute convergence condition (1.3b) of Gauss function and absolute convergence condition (1.17b) for Appell's function F_4).

Again put $y = \frac{1+x}{2x}$ in (3.1), we get

$$F_4\left[A, B; 1+A-B, B; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right] = \left[\frac{x-1}{2x}\right]^A {}_2F_1\left[\begin{array}{r} A, B; \\ 1+A-B; \end{array} \frac{1}{2}\right], \tag{3.3}$$

$$\Bigg(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1$$

$$\text{for } -1 < \Re(x) < 0 \ ; \ A, B, 1+A-B \in \mathbb{C}\backslash\mathbb{Z}_0^-\Bigg).$$

Now putting $A = a$, $B = \frac{a+1}{3}$ in (3.3) and using Kümmer's second summation theorem (1.7), we get

$$F_4\left[a, \frac{a+1}{3}; \frac{2a+2}{3}, \frac{a+1}{3}; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right] = \left[\frac{x-1}{2x}\right]^a \frac{\Gamma(\frac{1}{2})\Gamma(\frac{2a+2}{3})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{a+4}{6})}, \tag{3.4}$$

$$\Bigg(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1$$

$$\text{for } -1 < \Re(x) < 0 \ ; \ a, \frac{a+1}{3}, \in \mathbb{C}\backslash\mathbb{Z}_0^-\Bigg).$$

Now putting $A = a$, $B = 1 - a$ and using Kümmer's third summation theorem (1.8) in (3.3), we get

$$F_4\left[a, 1-a; 2a, 1-a; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right] = \left[\frac{x-1}{2x}\right]^a \frac{\Gamma(a)\Gamma(\frac{2a+1}{2})}{\Gamma(\frac{3a}{2})\Gamma(\frac{a+1}{2})}, \tag{3.5}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1 \text{ for } -1 < \Re(x) < 0 \; ; \; a, 1-a \in \mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Now putting $A = a$, $B = 1 - a - m$ in (3.3) and using summation theorem (1.13) given by Rakha-Rathie, we get

$$F_4\left[a, 1-a-m; 2a+m, 1-a-m; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right]$$

$$= \left[\frac{x-1}{2x}\right]^a \frac{\Gamma(2a+m)}{2^{a+m}\Gamma(a+m)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{\Gamma(\frac{a+m+r}{2})}{\Gamma(\frac{3a+m+r}{2})} \right\}, \tag{3.6}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1 \text{ for } -1 < \Re(x) < 0 \; ; \; a, 1-a-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \; ; \quad m \in \mathbb{N}_0\right).$$

Now putting $A = a$, $B = 1 - a + m$ in (3.3) and using summation theorem (1.14), we get

$$F_4\left[a, 1-a+m; 2a-m, 1-a+m; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right]$$

$$= \left[\frac{x-1}{2x}\right]^a \frac{2^{m-a}\Gamma(2a-m)}{\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \frac{(-1)^r\Gamma(\frac{a-m+r}{2})}{\Gamma(\frac{3a-3m+r}{2})} \right\}, \tag{3.7}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1 \text{ for } -1 < \Re(x) < 0 \; ;$$

$$a, \; a-m, \; 2a-m, 1-a+m \in \mathbb{C}\backslash\mathbb{Z}_0^- \; ; \quad m \in \mathbb{N}_0\right).$$

Now putting $A = a$, $B = -a - m$ in (3.3) and using summation theorem (1.15), we get

$$F_4\left[a, -a-m; 1+2a+m, -a-m; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right]$$

$$= \left[\frac{x-1}{2x}\right]^a \frac{2^{-a-m-1}\Gamma(1+2a+m)}{\Gamma(1+a+m)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{\Gamma(\frac{a+1+m+r}{2})}{\Gamma(\frac{3a+m+1+r}{2})} + \frac{\Gamma(\frac{2+a+m+r}{2})}{\Gamma(\frac{2+3a+m+r}{2})} \right] \right\}, \tag{3.8}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1 \text{ for } -1 < \Re(x) < 0 \ ; \ a, -a-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \quad m \in \mathbb{N}_0 \right).$$

Now putting $A = a$, $B = -a+m$ in (3.3) and using summation theorem (1.16), we get

$$F_4\left[a, -a+m; 1+2a-m, -a+m; \frac{2x^2}{(1-x)^2}, \frac{(1+x)}{(1-x)^2}\right]$$

$$= \left[\frac{x-1}{2x}\right]^a \frac{\Gamma(1+2a-m)\Gamma(a-m)}{2^{a-m+1}\Gamma(1+a-m)\Gamma(a)} \sum_{r=0}^{m} \left\{ \binom{m}{r} \left[\frac{(-1)^r\Gamma(\frac{a+1-m+r}{2})}{\Gamma(\frac{3a-3m+1+r}{2})} + \frac{(-1)^r\Gamma(\frac{2+a-m+r}{2})}{\Gamma(\frac{2+3a-3m+r}{2})} \right] \right\}, \tag{3.9}$$

$$\left(\sqrt{\left|\frac{2x^2}{(1-x)^2}\right|} + \sqrt{\left|\frac{1+x}{(1-x)^2}\right|} < 1 \text{ for } -1 < \Re(x) < 0 \ ; \right.$$

$$\left. a, a-m, -a+m, 1+2a-m \in \mathbb{C}\backslash\mathbb{Z}_0^- \ ; \quad m \in \mathbb{N}_0 \right).$$

Remark: We have verified the convergence conditions related to arguments and parameters numerically (using scientific calculator) in left hand and right hand sides of all above results of each section.

We conclude our present investigation by observing that several reduction formulas can be derived in an analogous manner using known summation theorems.

Acknowledgment: The authors are thankful to the referee for his valuable suggestions about the convergence conditions.

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