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k-JACOBSTHAL SEQUENCE AND ITS CATALAN TRANSFORM

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Abstract: This Paper concerns k-Jacobsthal Sequence and its Catalan Transform to obtain special integer sequence after that we relate the Henkel Transform to catalan transform of k-Jacobsthal.

Keywords and Phrases: k-Jacobsthal Numbers, Catalan Numbers, Henkel Transform.

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1. Introduction

The Fibonacci numbers are used in as a variety in sciences like biology, demography or economy. In current situation Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas sequence were generalized for any positive real number k. Also the study of the k-Fibonacci, k-Lucas, k-Pell, k-Jacobsthal and k-jacobsthal-Lucas sequence appeared. Sergio Falco has given several formula for the some of Catalan transform of the k-Fibonacci sequence. Now we are present k-Jacobsthal sequence and its Catalan transform.

2. k-Jacobsthal Numbers

For some positive real number k, the k-Jacobsthal sequence is defined by

$$j_{k,n+1} = kj_{k,n} + 2j_{k,n-1}, \ n \geq 1$$

with the initial condition $j_{k,0} = 0, j_{k,1} = 1$. Binet formula in the Jacobsthal number agree to use the k-Jacobsthal numbers in the function of the roots α, β of the characteristic equation,

$$j_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

If α is a root of the characteristic equation

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2}.$$

The general term of the form

$$j_{k,n} = \frac{\alpha^n - (-\alpha)^n}{\alpha + \alpha^{-1}}$$

and it is quotient two terms

$$\lim_{n\to} \frac{j_{k,n+r}}{j_{k,n}} = \alpha^r.$$

In exacting iff $k = 1$ then α is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ in addition.

The common term of the k-Jacobsthal sequence

$$j_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{2i+1} k^{n-1-2i}(k^2 + 8)^i$$

or equally

$$j_{k,n} = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-i-1}{i} k^{n-1-2i} 2^i.$$

3. Catalan Numbers

Catalan number are describe by [1]

$$c_n = \frac{1}{n+1} \binom{2n}{n} \tag{1}$$

the generating function of the consequent Catalan sequence is given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Equation (1) can be

$$c_n = \frac{2n!}{(n+1)!n!}$$

to end with a recurrence relation for c(n) is obtained from

$$\frac{c_{n+1}}{c_n} = \frac{2(2n+1)}{n+2}.$$

The initial few Catalan numbers for $n = 0, 1, 2, ...$ are $\{1, 1, 2, 5, 14, 42, 132, ...\}$ on OEIS as A000108.

4. Catalan transform of the k-Jacobsthal numbers

Here we describe the Catalan transform of the k-Jacobsthal sequence $\{j_{k,n}\}$ as

$$cj_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} j_{k,i}$$

for $n \geq 1$ with $cj_{k,0} = 0$.
The initial value of the sequence, we can say Catalan transform of the k-Jacobsthal numbers are the polynomials in k:

$$cj_{k,1} = \sum_{1}^{1} \frac{i}{2-i} \binom{2-i}{1-i} j_{k,i} = 1$$

$$cj_{k,2} = \sum_{1}^{2} \frac{i}{4-i} \binom{4-i}{2-i} j_{k,i} = k+1$$

$$cj_{k,3} = \sum_{1}^{3} \frac{i}{6-i} \binom{6-i}{3-i} j_{k,i} = k^2+2k+4$$

$$cj_{k,4} = \sum_{1}^{4} \frac{i}{8-i} \binom{8-i}{4-i} j_{k,i} = k^3+3k^2+9k+11$$

$$cj_{k,5} = \sum_{1}^{5} \frac{i}{10-i} \binom{10-i}{5-i} j_{k,i} = k^4+4k^3+15k^2+30k+36.$$

We use the equation as the product of the lower triangular matrix C and $n \times 1$ matrix

$$\begin{pmatrix} cj_{k,1} \\ cj_{k,2} \\ cj_{k,3} \\ cj_{k,4} \\ cj_{k,5} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 5 & 5 & 3 & 1 & \\ 14 & 14 & 9 & 4 & 1 \\ \vdots & & & & \end{pmatrix} \begin{pmatrix} j_{k,1} \\ j_{k,2} \\ j_{k,3} \\ j_{k,4} \\ j_{k,5} \\ \vdots \end{pmatrix}$$

The entries of the matrix c confirm the recurrence relation $c_{i,j} = \sum_{r=j}^{i=1}$ and the first column is equal to the second column for $i > 1$ and they are Catalan numbers. The lower triangular matrix c, c_{n-1} is known as the Catalan triangle and its elements verify the formula

$$c_{n,n-i} = \frac{(2n-i)(i+1)}{(n-i)!(n+1)!}, \quad 0 \leq i \leq n$$

with the coefficients of the Catalan transform of k-Jacobsthal sequence following triangle

Cj_1	1				
Cj_2	1	1			
Cj_3	1	2	4		
Cj_4	1	3	9	11	
Cj_5	1	4	15	30	36
...	...				

The first diagonal sequence 1,1,4,11,36,117,393,....: A106640 and the next diagonal sequence is A277239 is the self assessment sequence A106640. From this point on, each diagonal sequence is the convolution of its proceeding diagonal and the first one. To end with, for k=1,2,3,... we obtain the subsequent Catalan transform of the k-Jacobsthal sequence.

$$\begin{aligned} Cj_1 &= \{0, 1, 2, 7, 24, 86, ...\} \\ Cj_2 &= \{0, 1, 3, 12, 40, 204, ...\} \\ Cj_3 &= \{0, 1, 4, 19, 92, 450, ...\} \\ Cj_4 &= \{0, 1, 5, 28, 159, 743, ...\} \\ Cj_5 &= \{0, 1, 6, 39, 256, 1686, ...\}. \end{aligned}$$

5. Henkel Transform

Let $A = \{a_0, a_1, a_2, ...\}$ be a sequence of real numbers. The Henkel transform of the sequence A is the sequence of determinants $H_n = \det[ai + j - 2]$, i.e.,

$$H_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

The Henkel determinant of order n of the sequence A is the upper-left $n \times n$sub determinant of H_n. Its Catalan sequence the $\{1, 1, 1, ...\}$ and its summation of successive generalized Catalan number is the bisection of standard k-Jacobsthal sequence.

Enchanting into relation the Catalan transform of k-Jacobsthal sequence of preceding part, we find out.

$$HCj_1 = \det[1] = 1$$

$$HCj_2 = \begin{vmatrix} 1 & k+1 \\ k+1 & k^2+2k+4 \end{vmatrix} = 3$$

$$HCj_3 = \begin{vmatrix} 1 & k+1 & k^2+2k+4 \\ k+1 & k^2+2k+4 & k^3+3k^2+9k+11 \\ k^2+2k+4 & k^3+3k^2+9k+11 & k^4+4k^3+15k^2+30k+36 \end{vmatrix} = 25.$$

References

[1] P. Barry, A Catalan transform and related transformations on integer sequence, J. Integer seq. 8 (2005), No.4. Article 05.4.5., 24pp.

[2] A. Cvetkovic, R. Rajkovic and M. Ivkovic, Catalan numbers, and Henkel Transform and Fibonacci numbers, J. Integer seq. 5 (2002), No.1 Article 02.1.3., 8pp.

[3] S. Falcon, Catalan Transform of the k-Fibonacci sequence, Commun. korean math. soc., 28 (2013), N0.4 pp 827-832.

[4] S. Uygun and H. Eldogan, Properties of k-Jacobsthal and k-Jacobsthal lucas sequence, Gen. math. Notes, 36 (2016) 34-47.

[5] G. Srividhya and T. Ragunathan, k-Jacobsthal and k-Jacobsthal lucas Numbers and their Associated Numbers, International journal of Research in Advent Technology, 7 (2019) 152-156.