South East Asian J. of Mathematics and Mathematical Sciences
Vol. 15, No. 3 (2019), pp. 129-140

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

A STUDY OF MIXED TREE DOMINATION POLYNOMIALS IN SOME CLASS OF GRAPHS

Rafia Yoosuf and Preethi Kuttipulackal*

Department of Mathematics,
M E S Mampad College, Kerala, INDIA

E-mail : rafiafiroz@gmail.com

*Department of Mathematics,
University of Calicut, Kerala, INDIA

E-mail : pretikut@rediffmail.com

(**Received:** *Jul. 10, 2019* **Accepted:** *Dec. 12, 2019* **Published:** *Dec. 31, 2019)*

Abstract: The mixed tree domination polynomial of a connected graph G of order n is the polynomial $P(G,x) = \sum_{i=\gamma_{mt}(G)}^{2n-1} p(i)x^i$, where $p(i)$ is the number of mixed tree dominating sets of G of cardinality i and $\gamma_{mt}(G)$ is the mixed tree domination number of G. We analyse the specifications of the polynomial. Also $P(G,x)$ is determined for cycles, complete graphs and stars, and the roots of $P(G,x)$ are studied.

Keywords and Phrases: Domination, Mixed tree domination, domination polynomial, Mixed tree domination polynomial ($mtd-$ polynomial).

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng in [5]. Preethi and Raji introduced the concept of mixed tree dominating set in connected graph [2, 3, 4]. While imitating the concept of domination polynomial in view of mixed tree dominating set, we came across with many interesting relations with the coefficients of the polynomial and the graph parameters.

Also, the coefficients of the polynomial of some important class of graphs have attractive patterns and the roots of the polynomial have interesting nature. This paper includes the basic results in this study.

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighbourhood* of S is $N(S) = \cup_{v\in S} N(v)$ and the *closed neighbourhood* of S is $N[S] = N(S) \cup S$. A set$S \subseteq V$ is a *dominating set* of G, if $N[S] = V$, or equivalently every vertex in $V \backslash S$ is adjacent to atleast one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a $\gamma - set$. For the basic concepts in graph theory we refer mainly Bondy and Murthy[1] and the concepts in domination theory and mixed tree domination are followed from Preethi[3].

Saeid Alikhani and Yee-hock Peng introduced the concept of domination polynomial of a graph as the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ denotes the number of dominating sets of cardinality i. *A mixed dominating set* of G is a subset K of $V \cup E$ such that every element in $(V \cup E)\backslash K$ is either adjacent or incident to an element of K.

By *the graph formed by a subset* A of $V \cup E$, we mean the subgraph whose edge set is $A \cap E$ and the vertex set consists the vertices in A together with the ends of the edges in A.

A mixed dominating set $S \subseteq V \cup E$ of a connected graph $G(V, E)$ is a *mixed tree dominating set*$(mtd - set)$ if the graph formed by S is tree. *The mixed tree domination number* $\gamma_{mt}(G)$ is the minimum cardinality of a mixed tree dominating set in G.

Considering the polynomial idea of Alikhani et.al., we introduced the $mtd-$ polynomial[6] of a connected graph and study the information about the graph that we can obtain from the polynomial. The graphs considered here are all connected and simple of order n.

The following results of Preethi and Raji [2, 3, 4] are used in this paper.

Theorem 1. *The mixed tree domination number of the graphs P_n, C_n, K_n and $K_{1,t}$ are given by*

1.

$$\gamma_{mt}(Pn) = \begin{cases} 1, & if \quad n = 1 \text{ or } 3 \\ n - 1, & otherwise \end{cases}$$

2.
$$\gamma_{mt}(Kn) = \begin{cases} 1, & if\ n = 1 \\ n-1, & otherwise \end{cases}$$

3. $\gamma_{mt}(Cn) = n - 1$

4. $\gamma_{mt}(K_{1,t}) = 1$

2. Mixed Tree Domination Polynomial

As the elements of a $mtd-$ set form a tree, the maximum cardinality of an $mtd-$ set is $2n-1$, and the minimum cardinality is $\gamma_{mt}(G)$.

Definition 1. *Let G be a connected graph of order n. The mixed tree domination polynomial of G is the polynomial $P(G,x) = \sum_{i=\gamma_{mt}(G)}^{2n-1} p(i)x^i$, where $p(i)$ is the number of mixed tree dominating sets of G of cardinality i and $\gamma_{mt}(G)$ is the mixed tree domination number of G.*

In this paper we include the studies on mixed tree domination polynomials of cycles, complete graphs and stars, and their roots. If S is an $mtd-set$, as the elements of S form a tree, each vertex in S must be the end of some edge in S. So that in any $mtd-set$ S of a graph the number of edges cannot exceed $n-1$. And in K_n, C_n, P_n, the number of edges in an $mtd-set$ cannot be less than $n-2$.

Example 1. Consider the path $P_3 = v_1v_2v_3$. It has only one γ_{mt} set. The mixed tree dominating sets are $\{v_2\}$, $\{v_1v_2, v_2v_3\}$, $\{v_1v_2, v_2\}$, $\{v_2, v_2v_3\}$, $\{v_1v_2, v_2v_3, v_1\}$, $\{v_1v_2, v_2v_3, v_2\}$, $\{v_1v_2, v_2v_3, v_3\}$, $\{v_1, v_2, v_1v_2\}$, $\{v_2, v_3, v_2v_3\}$, $\{v_1v_2, v_2v_3, v_1, v_2\}$, $\{v_1v_2, v_2v_3, v_1, v_3\}$, $\{v_1v_2, v_2v_3, v_3, v_2\}$ and $\{v_1, v_2, v_3, v_1v_2, v_2v_3\}$. So that the polynomial is

$$x + 3x^2 + 5x^3 + 3x^4 + x^5.$$

The following results are immediate consequences of the definition.

Theorem 2. [6]

1. *$x^{\gamma_{mt}(G)}$ is a factor of $P(G,x)$ that means 0 is a root of multiciplicity $\gamma_{mt}(G)$.*

2. *$mtd-$ polynomial of any graph is an odd degree polynomial .*

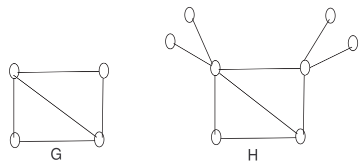
3. *The constant term is zero. The coefficient of x is 1 if and only if G is a star $K_{1,t};\ t > 1$.*

4. *The leading coefficient of $P(G, x)$ is $\tau(G)$ - the number of spanning trees of G.*

5. *The leading coefficient is 1 ie $P(G, x)$ is monic if and only if G is tree.*

Theorem 3. [6] *If G is hamiltonian then the leading coefficient is greater than or equal to n. But the converse not true.*

The following example shows the converse of Theorem 3. is not true. The number of spanning trees of G is 8. The graph H is formed from G by adding 4 pendant vertices, so that it has order 8 and all the pendent vertices and the edge incident to them must be part of any spanning tree. That is $\tau(H) = \tau(G) = n$, but H is not hamiltonian.



The following table shows the mixed tree domination polynomial of various paths. ($|\,| = Cardinality,\ vx = vertex,\ eg = edge$)

Paths	$\|\,\|$ of *mtd*-sets	No. of subsets	Specifications	Polynomial
P_1	1	1	The only vx	x
P_2	1 2 3	3 2 1	• any vx or the eg • one vx and one eg • $V \cup E$	$3x + 2x^2 + x^3$
P_3	1 2 3 4 5	1 3 2+3 $\binom{3}{2}$ 1	• The central vx • central vx and an eg or two eg • one eg and its ends or two eg and one vx • 2 eg and 2 vx • $V \cup E$	$x + 3x^2 + 5x^3 + 3x^4 + x^5$
P_4	3 4 5 6 7	1+2+1 4+$\binom{4}{1}$ 2+$\binom{4}{2}$ $\binom{4}{3}$ 1	• 1 eg and 2 vx or 2 eg and 1 vx or 3 eg • 2 eg and 2 vx or 3 eg and 2 vx • 2 eg and 3 vx or 3 eg and 2 vx • 3 eg and 3 vx • 3 eg and 4 vx	$4x^3 + 8x^4 + 8x^5 + 4x^6 + x^7$
P_5	4 5 6 7 8 9	1+2+1 $1 + 2 \times 3 + 5$ $2 \times (\binom{3}{2}+5)$ 2+$\binom{5}{3}$ $\binom{5}{4}$ 1	• 2 eg and 2 vx or 3 eg and 1 vx or 4 eg • 2 eg and 3 vx or 3 eg and 2 vx or 1 vx 4 eg • 3 eg and 3 vx or 2 vx and 4 eg • 3 eg and 4 vx or 4 eg and 3 vx • 4 eg and 4 vx • 5 vx 4 eg	$4x^4 + 12x^5 + 16x^6 + 12x^7 + 5x^8 + x^9$

Theorem 4. [6] *For $n \geq 4$, the mixed tree domination polynomial of the path P_n is*

$$\sum_{i=n-1}^{2n-1} [\binom{n}{i-n+1}+2\binom{n-2}{i-n+1}+\binom{n-4}{i-n+1}]x^i = x^{n-1}(x+1)^{n-4}(x^2+2x+2)^2,$$

where $\binom{r}{k} = 0$ *for* $k > r$.

The interesting fact is that the coefficients of these polynomials follow the pattern of Pascal's triangle.

P_2 :	1	2	3						
P_3 :	1	3	5	3	1				
P_4 :	1	4	8	8	4				
P_5 :	1	5	12	16	12	4			
P_6 :	1	6	17	28	28	16	4		
P_7 :	1	7	23	45	56	44	20	4	
P_8 :	1	8	30	68	101	100	64	24	4

$P(P_n, x) = x^{n-1}(x+1)^{n-4}(x^2+2x+2)^2 = x^{n-1}(x+1)^{n-4}[(x-(-1+i))(x+(1+i))]^2$, for $n \geq 4$. So that the roots are 0 with multiplicity $n-1$, -1 with multiplicity $n-4$ and $-1+i$, $-1-i$ are two roots 0f multiplicity 2.

Theorem 5. *For $n \geqslant 3$, the mixed tree domination polynomial of the cycle C_n is*

$$P(C_n, x) = \sum_{i=n-1}^{2n-1} [\binom{n}{i-(n-1)} + 2\binom{n-2}{i-(n-1)} - \binom{n-3}{i-n}]nx^i$$

$$= x^{n-1}(x+1)^{n-3}(x^3+3x^2+4x+3)^2,$$

where $\binom{j}{k} = 0$, $j < k$ *or* $k < 0$.

Proof. The $mtd-$ number of the cycle C_n is $\gamma_{mt}(C_n) = n-1$. Let C_n be the cycle $v_1v_2......v_nv_1$. Here we analyse the $mtd-$ sets of various cardinalities. First consider the mixed tree dominating sets of cardinality $n-1$.

There are two types of $mtd-$ sets of cardinality $n-1$; a set containing $n-1$ edges or a set containing $n-2$ edges and one vertex. Since any set of $n-1$ edges of a cycle form a tree, we have $\binom{n}{n-1}=\binom{n}{1}=n$ such sets. Now for an $mtd-$ set containing $n-2$ edges and one vertex, the edges must be consecutive and hence must form a path P_{n-1}. So choosing $n-2$ consecutive edges is equivalent to choose $n-1$ vertices, there are $\binom{n}{n-1}$ choices. To dominate the n^{th} vertex we have to include one of the two end vertices of this path P_{n-1}. Hence there are $2\binom{n}{n-1} = 2n$ $mtd-$sets of this type. Therefore the coefficient of x^{n-1} is $\binom{n}{n-1} + 2\binom{n}{n-1}$.

Now, there are two types of $mtd-$ sets of cardinality n. In the first type, there are $n-1$ edges and one vertex and in the second type, there are $n-2$ edges and 2 vertices. In the former type there are $n\binom{n}{n-1}$ chices. In the latter type, there are $\binom{n}{n-1}$ choices for the $n-2$ edges, one vertex must be the end vertex(which has 2 choices) of the path induced by these edges, and the other vertex has $\binom{n-2}{1}$ choices. The $mtd-$ set containing both the end vertices of this P_{n-1} is counted twice. It follows that the number f $mtd-$ sets in this case is given by $[2\binom{n-2}{1} - 1]\binom{n}{n-1}$.

In general for $i \geq n-1$, there are only two possible types of $mtd-$ sets of i elements. As above, we have $\binom{n}{n-1}\binom{n}{i-(n-1)}$ choices for an $mtd-$ set containing $n-1$ edges and $i-(n-1)$ vertices. Now consider an $mtd-$ set of $n-2$ edges and $i-(n-2)$ vertices. The $n-2$ edges must form a path P_{n-1}, and the vertices must be choosed from this path. So that $i-(n-2) \leq n-1$ or $i \leq 2n-3$. There are $\binom{n}{n-1}$ choices for $n-2$ edges and 2 choices for the end vertex. Remaining $i-(n-2)-1 = i-n+1$ vertices can be selected from the remaining $n-2$ vertices of this P_{n-1}, that yields $2\binom{n-2}{i-(n-1)}$ choices. If the selected $i-n+1$ vertices include the other end, that will be counted twice because that set will be included in both choice of the end vertex. The number of such sets, including both the end vertices is $\binom{n-3}{i-n}$. Therefore number of such $mtd-$ sets is $\binom{n}{n-1}[2\binom{n-2}{i-(n-1)} - \binom{n-3}{i-n}]$. Therefore there are $\binom{n}{n-1}[2\binom{n-2}{i-(n-1)} - \binom{n-3}{i-n} + \binom{n}{i-(n-1)}]$ $mtd-$ sets of i elements exists for $i \leq 2n-3$.

For $i = 2n-2$, the only possibility is an $mtd-$ set consisting of $n-1$ edges and $n-1$ vertices. There are n choices for choosing $n-1$ edges. As these $n-1$ edges constitute a path on n vertices, there are n choices for selecting $n-1$ vertices from each of these paths. So that the total number of such $mtd-$ sets is $p(2n-2) = n^2$.

For $i = 2n-1$, the $mtd-$ set must contain $n-1$ edges and n vertices, so that $p(2n-1) = n$. Therefore under the assumption $\binom{j}{k} = 0$ for $j < k$, $p(i)$ can be expressed as $p(i) = n[2\binom{n-2}{i-(n-1)} - \binom{n-3}{i-n} + \binom{n}{i-(n-1)}]$ for all i, $n-1 \leq i \leq 2n-1$.

Therefore,

$$P(C_n, x) = \sum_{i=n-1}^{2n-1} [\binom{n}{i-(n-1)} + 2\binom{n-2}{i-(n-1)} - \binom{n-3}{i-n}]nx^i,$$

where $\binom{j}{k} = 0$, $j < k$ or $k < 0$.
Note that $P(C_n, x) = x^{n-1}(x+1)^{n-3}(x^3+3x^2+4x+3)^2$ for $n \geq 4$. So that the roots are, 0 is a root of multiplicity $n-1$, -1 is a root of multiplicity $n-3$, -1.68233, $.65884 + i(1.16134)$ and $.65884 - i(1.16134)$ are the roots of multiplicity 1.

Theorem 6. *For $n \geqslant 1$, the mtd− polynomial of the complete graph K_n is given by*
$$P(K_n, x) = \sum_{i=n-1}^{2n-1} \{\binom{n}{i-(n-1)} n^{n-2} + n(n-1)^{n-3}\binom{n-1}{i-(n-2)}\}x^i.$$

Proof. We have $\gamma_{mt}(K_n) = n-1$. *mtd*− sets of cardinality n-1 fall in to two categories,

Case(1) n-1 edges: Since they form a spanning tree, the number of ways n-1 edges can be choosen is the number of spanning trees of K_n that is $\tau(K_n) = n^{n-2}$.

Case(2) n-2 edges and 1 vertex: Since any two vertices are adjacent in K_n, any n-2 edges form a tree of order n-1. There are $\binom{n}{n-1}$ choices for choosing n-1 vertices. If v is the vertex in K_n, which is not in the tree, number of such trees is $\tau(K_n - v) = \tau(K_{n-1})=(n-1)^{n-3}$. So there are $\binom{n}{n-1}(n-1)^{n-3}$ choices for $n-2$ edges. These $n-2$ edges dominate all the edges of K_n and all vertices except v. Therefore to dominate v, any one vertex other than v can be chosen. Therefore $\binom{n-1}{1}\binom{n}{n-1}(n-1)^{n-3}$ choices in category (2).

Therefore,
$$p(n-1) = n^{n-2} + \binom{n-1}{1}\binom{n}{n-1}(n-1)^{n-3}.$$

In general, for *mtd*− sets of i elements, we have the following choices.

Case(1) $n-1$ edges and $i-(n-1)$ vertices: In this case we must have $i-(n-1) \leq n$. That is $i \leq 2n-1$. We have seen that there are n^{n-2} choices for choosing $n-1$ edges. Since these edges induce a spanning tree of K_n, any of $i-(n-1)$ vertex will serve the purpose. So there are $\binom{n}{i-(n-1)}$ n^{n-2} choices for an *mtd*− set of $n-1$ edges and $i-(n-1)$ vertices.

Case(2) $n-2$ edges and $i-(n-2)$ vertices: We should have $i-(n-2) \leq n-1$ or $i \leq 2n-3$. There are $n(n-1)^{n-3}$ choices for $n-2$ edges, and the $i-(n-2)$ vertices can be selected from the ends of these $n-2$ edges in $\binom{n-1}{i-(n-2)}$ ways. Therefore there are $n(n-1)^{n-3}[\binom{n-1}{i-(n-2)}]$ *mtd*− sets in this case. Hence we can conclude that $p(i) = \binom{n}{i-(n-1)}n^{n-2} + n(n-1)^{n-3}[\binom{n-1}{i-(n-2)}]$ for $i \leq 2n-3$.
All *mtd*− sets of $2n-2$ or $2n-1$ elements falls in case(1). Hence for $2n-2 \leq i \leq 2n-1$, $p(i) = n^{n-2}\binom{n}{i-(n-1)}$. Thus we can conclude that,
$$P(K_n, x) = \sum_{i=n-1}^{2n-1} \{\binom{n}{i-(n-1)} n^{n-2} + n(n-1)^{n-3}\binom{n-1}{i-(n-2)}\}x^i.$$

Theorem 7. *The mtd− polynomial of the star graph $K_{1,t}$, $t > 1$ is* $P(K_{1,t}, x) = \sum_{i=1}^{2t+1} p(i)x^i$, *where p(i) is given by the following equations. For* $1 \leqslant i \leqslant t-1$,

$$p(i) = \begin{cases} \binom{t}{i-1} + \binom{t}{i-2}\binom{i-2}{1} + \binom{t}{i-3}\binom{i-3}{2} + ... + \binom{t}{\frac{i-1}{2}}\binom{\frac{i-1}{2}}{\frac{i-1}{2}}, & if \quad i \text{ is odd.} \\ \binom{t}{i-1} + \binom{t}{i-2}\binom{i-2}{1} + \binom{t}{i-3}\binom{i-3}{2} + ... + \binom{t}{\frac{i}{2}}\binom{\frac{i}{2}}{\frac{i-2}{2}}, & if \quad i \text{ is even.} \end{cases}$$

For $t \leqslant i \leqslant 2t+1$,

$$p(i) = \begin{cases} \binom{t+1}{i-t} + \binom{t}{t-1}\binom{t-1}{i-t} + \binom{t}{t-2}\binom{t-2}{i-t+1} + ... + \binom{t}{\frac{i-1}{2}}\binom{\frac{i-1}{2}}{\frac{i-1}{2}}, & if \quad i \text{ is odd.} \\ \binom{t+1}{i-t} + \binom{t}{t-1}\binom{t-1}{i-t} + \binom{t}{t-2}\binom{t-2}{i-t+1} + ... + \binom{t}{\frac{i}{2}}\binom{\frac{i}{2}}{\frac{i-2}{2}}, & if \quad i \text{ is even} \end{cases}$$

with the assumption $\binom{t}{j} = 0$ *if* $j < 0$.

Proof. Consider$K_{1,t}$, where $t > 1$. For $K_{1,t}$, $t > 1$ all $mtd-$ sets which do not contain all the t edges must contain the vertex of degree t, say the center vertex and denote it as v. There is only one $mtd-$ sets of cardinality 1. Let us count the $mtd-$ sets of cardinality i, when $2 \leq i \leq t-1$.

$Table(vx = vertex,\ eg = edge)$

Cases	Specifications	No. of mtd-sets
$i = 2$	The center vx v and any eg	$\binom{t}{1}$
$i = 3$	•1 vx, 2eg(v and any 2 eg) •2 vx 1 eg(v,any one eg and its other end)	• $\binom{t}{2}$ • $\binom{t}{1}$
$i = t-2$	•1 vx t-3 eg • 2 vx t-4 eg ⋮ • $\frac{t-1}{2}$ vx, $t - \frac{t+3}{2}$ eg(t odd) • $\frac{t}{2} - 1$ vx, $t - (\frac{t}{2} + 1)$ eg(t even)	•$\binom{t}{t-3}$ • $\binom{t}{t-4}\binom{t-4}{1}$ ⋮ • $\binom{t}{t-\frac{t+3}{2}}\binom{t-\frac{t+3}{2}}{\frac{t-1}{2}-1}$ • $\binom{t}{t-\frac{t}{2}+1}\binom{t-\frac{t}{2}+1}{\frac{t}{2}-2}$
$i = t-1$	•1 vx, t-2 eg • 2 vx, t-3 eg ⋮ • $\frac{t}{2}$ vx and $\frac{t}{2} - 1$ eg(t even) • $\frac{t-1}{2}$ vx and $\frac{t-1}{2}$ eg (t odd)	• $\binom{t}{t-2}$ •$\binom{t}{t-3}\binom{t-3}{1}$ ⋮ • $\binom{t}{\frac{t}{2}-1}$ • $\binom{t}{\frac{t-1}{2}}\binom{\frac{t-1}{2}}{\frac{t-1}{2}-1}$

In general for 1⩽ i⩽t-1, an $mtd-set$ of cardinality i may be one of the following type. Note that it must contain the center vertex.

First we consider the $mtd-$ set consisting of 1 vertex and $i-1$ edges, there are $\binom{t}{i-1}$ such $mtd-$ sets exist. An $mtd-$ set consisting of 2 vertices and $i-2$ edges, in this case the second vertex must be the end of some edge in the set, so that $\binom{i-2}{1}$ choices for 1 vertex. Therefore total number of such $mtd-$ sets is $[\binom{t}{i-2}][\binom{i-2}{1}]$. Continuing in this way we get,

Case(1) When i is odd, $\frac{i+1}{2}$ vertices and $\frac{i-1}{2}$edges: There are $\binom{t}{\frac{i-1}{2}}$ choices for edges. Except the center vertex all other vertices must be the end of some edge in the set, $\binom{\frac{i-1}{2}}{\frac{i-1}{2}}$ choices for vertices. Therefore total number of such $mtd-$ sets is $[\binom{t}{\frac{i-1}{2}}][\binom{\frac{i-1}{2}}{\frac{i-1}{2}}]$.

Case(2) When i is even, $\frac{i}{2}$ vertices and $\frac{i}{2}$ edges: There are $\binom{t}{\frac{i}{2}}$ choices for edges and $\binom{\frac{i}{2}}{\frac{i-2}{2}}$ choices for vertices. Therefore total number of such $mtd-$ sets is $[\binom{t}{\frac{i}{2}}][\binom{\frac{i}{2}}{\frac{i-2}{2}}]$.

If $\binom{t}{j}$ is taken as 0, for$j < 0$ then the coefficients of x^i, $1 \leq i \leq t-1$, is given by

$$p(i) = \begin{cases} \binom{t}{i-1} + \binom{t}{i-2}\binom{i-2}{1} + \binom{t}{i-3}\binom{i-3}{2} + ... + \binom{t}{\frac{i-1}{2}}\binom{\frac{i-1}{2}}{\frac{i-1}{2}}, & if \text{ i } is\ odd. \\ \binom{t}{i-1} + \binom{t}{i-2}\binom{i-2}{1} + \binom{t}{i-3}\binom{i-3}{2} + ... + \binom{t}{\frac{i}{2}}\binom{\frac{i}{2}}{\frac{i-2}{2}}, & if \text{ i } is\ even. \end{cases}$$

For $t \leq i \leq 2t+1$, we have the following table

Cases	Specifications	No. of mtd-sets
$i=t$	• t eg •1 vx, t-1 eg •2 vx, t-2 eg •3 vx, t-3 eg ⋮ • $\frac{t}{2}$ vx $\frac{t}{2}$ eg(t even) • $\frac{t+1}{2}$ vx $\frac{t-1}{2}$ eg (t odd)	• $1=\binom{t+1}{0}$ •$\binom{t}{t-1}$ • $\binom{t}{t-2}\binom{t-2}{1}$ •$\binom{t}{t-3}\binom{t-3}{2}$ ⋮ •$\binom{t}{\frac{t}{2}}\binom{\frac{t}{2}}{\frac{t}{2}-1}$ • $\binom{t}{\frac{t-1}{2}}$
$i=t+1$	•1 vx, t eg •2 vx, t-1 eg •3 vx and t-2 eg ⋮ • $\frac{t}{2}+1$ vx and t-$\frac{t}{2}$ eg(t even) • $\frac{t+1}{2}$ vx t-$\frac{t-1}{2}$ eg(t odd)	• $\binom{t+1}{1}$ •$\binom{t}{t-1}\binom{t-1}{1}$ •$\binom{t}{t-2}\binom{t-2}{2}$ ⋮ • $\binom{t}{t-\frac{t}{2}}\binom{t-\frac{t}{2}}{\frac{t}{2}}$ • $\binom{t}{t-\frac{t-1}{2}}\binom{t-\frac{t-1}{2}}{\frac{t-1}{2}}$
$i=t+2$	•2 vx t eg • 3 vx t-1 eg • 4 vx and t-2 eg ⋮ •$\frac{t+2}{2}$ vx,$\frac{t+2}{2}$eg(t even) •$\frac{t+3}{2}$ vx, $\frac{t+1}{2}$ eg(t odd)	•$\binom{t+1}{2}$ • $\binom{t}{t-1}\binom{t-1}{2}$ • $\binom{t}{t-2}\binom{t-2}{3}$ ⋮ • $\binom{t}{\frac{t+2}{2}}\binom{\frac{t+2}{2}}{\frac{t+2}{2}-1}$ •$\binom{t}{\frac{t+1}{2}}\binom{\frac{t+1}{2}}{\frac{t+1}{2}}$
$i=2n-4=2t-2$	• $t-2$ vx, t eg • $t-1$ vx, $t-1$ eg	• $\binom{t+1}{t-2}$ •$\binom{t}{t-1}\binom{t-1}{t-2}$
$i=2n-3=2t-1$	• t-1 vx, t eg • t vx and $t-1$ eg	•$\binom{t+1}{t-1}$ • $\binom{t}{t-1}\binom{t-1}{t-1}$
$i=2n-2=2t$	•t vx, t eg	• $\binom{t+1}{t}$
$i=2n-1=2t+1$	•t+1 vx, t eg	• $\binom{t+1}{t+1}$

In general, an $mtd-$ set of cardinality i, t⩽ i⩽2t+1 may consists of,

For an $mtd-$ set consisting of $i-t$ vertices and t edges, there are $\binom{t}{t}$ choices for edges and $\binom{t+1}{i-t}$ choices for vertices. Therefore total number of such $mtd-$ sets is [$\binom{t}{t}$] [$\binom{t+1}{i-t}$]. Now we consider an $mtd-$ set consisting of $(i-t)+1$ vertices and $t-1$ edges, in this case there are $\binom{t}{t-1}$ choices for edges, as one edge is not included, we must include the center vertex to dominate its ends. Also, in order to form a tree,the remaining vertices included must be from the other ends of the t-1 edges in the set. So there are $\binom{t-1}{i-t})$ choices. Therefore total number of such $mtd-$ sets is$[\binom{t}{t-1}]$ $[\binom{t-1}{i-t}]$. Continuing in this way get,

Case(1) i is odd, $\frac{i+1}{2}$ vertices and $\frac{i-1}{2}$ edges: By choosing $\frac{i-1}{2}$ edges arbitrarily(that yields $\binom{t}{\frac{i-1}{2}}$ choices), the center vertex and $\frac{i+1}{2}-1$ vertices from the ends of the $\frac{i-1}{2}$ edges chosen, we have total number of such $mtd-$ sets is $[\binom{t}{\frac{i-1}{2}}][\binom{\frac{i-1}{2}}{\frac{i-1}{2}}]$.

Case(2) i is even,$\frac{i}{2}$ vertices and $\frac{i}{2}$ edges: By choosing $\frac{i}{2}$ edges arbitrarily(that yields $\binom{t}{\frac{i}{2}}$ choices), the center vertex and $\frac{i}{2}-1$ vertices from the ends of the $\frac{i}{2}$ edges chosen, we have total number of such $mtd-$ sets is $[\binom{t}{\frac{i}{2}}][\binom{\frac{i}{2}}{\frac{i-2}{2}}]$. If $\binom{t}{j}$ is taken as 0, for$j<0$ then the coefficients of x^i, $t \leq i \leq 2t+1$, is given by

$$p(i)=\begin{cases} \binom{t+1}{i-t}+\binom{t}{t-1}\binom{t-1}{i-t}+\binom{t}{t-2}\binom{t-2}{i-t+1}+...+\binom{t}{\frac{i-1}{2}}\binom{\frac{i-1}{2}}{\frac{i-1}{2}}, & if \text{ i } is\ odd. \\ \binom{t+1}{i-t}+\binom{t}{t-1}\binom{t-1}{i-t}+\binom{t}{t-2}\binom{t-2}{i-t+1}+...+\binom{t}{\frac{i}{2}}\binom{\frac{i}{2}}{\frac{i-2}{2}}, & if \text{ i } is\ even. \end{cases}$$

3. Conclusion

We have studied the $mtd-$ polynomial of many important classes of graphs, and continue for other classes of graphs, and derived graphs, using graphs operations like join, corona,etc. In this area our main task is to relate the $mtd-$ polynomial of the derived graph with that of its underlying graphs. Also we are trying to find whether we can extract the important features of a graph from the parameters related to its $mtd-$ polynomial. For example, we have noticed that the leading coefficient of the $mtd-$ polynomial of G is $\tau(G)$, the number of spanning trees of G.

References

[1] J. A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.

[2] P. Raji, K. Preethi, On Mixed Tree Domination Number, Proceedings of the International Conference on Number Theory, P.D.E. and Geometry, Department of Mathematics, University of Calicut, (2009) 249-255.

[3] P. Raji, K. Preethi, On Mixed Tree Domination Number, Advances and Applications in Discrete Mathematics, Pushpa Publishers, Vol.9, No.1, (2012)45-55.

[4] K. Preethi, PhD Thesis, Mixed Tree Domination in graphs, University of Calicut, 2012.

[5] Saeid Alikhani and Yee-hock Peng, Introduction to domination polinomial of a graph, arXiv: 0905, 225 1 V 1 (2009).

[6] Preethi Kuttipulackal and Rafia Yoosuf, The mixed Tree Domination Polynomial, Contemporary Studies In Discrete Mathematics, CSDM Publishers, Vol.2, No.3(2018)15-20