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A NOTE ON ROGER'S FINE IDENTITY

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Abstract: In this paper, making use of Rogers-Fine identity and certain known results, interesting and useful results have been established.

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1. Introduction and Definitions

For q real or complex $|q| < 1$, a basic hypergeometric series is given by

$$ {}_r\Phi_s \left[\begin{array}{c} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s; q^\lambda \end{array} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n} q^{\lambda n(n-1)/2}, \tag{1.1} $$

where $[a_1, a_2, ..., a_r; q]_n = [a_1; q]_n [a_2; q]_n ... [a_r; q]_n$,

$$ [a; q]_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), & \text{if } n = 1, 2, 3, ... \end{cases} $$

and

$$ [a; q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r). \tag{1.2} $$

For the convergence of the series we need $|q| < 1$ and $|z| < \infty$ if $\lambda > 0$ or $\max(|q||z|) < 1$ if $\lambda = 0$, provided no zeros appear in denominator. The identity

$$\sum_{n=0}^{\infty} \frac{[\alpha;q]_n}{[\beta;q]_n} z^n = \sum_{n=0}^{\infty} \frac{[\alpha;q]_n [\alpha z q/\beta;q]_n \beta^n z^n q^{n^2-n}(1-\alpha z q^{2n})}{[\beta;q]_n [z;q]_{n+1}} \tag{1.3}$$

[1, (9.1.1) p. 223]

was first proved by L.J. Rogers [4] and was rediscovered independently by N.J. Fine [2]. In this paper we shall deduce the special cases of (1.3) and then making use of some known results we shall establish the continued fractional representations of certain series.

2. Main Results

(i) Putting $\frac{zq}{\alpha}$ for z and then taking $\alpha \to \infty$ and $\beta \to 0$ in (1.3) we get,

$$\sum_{n=0}^{\infty} (-1)^n z^n q^{n(n+1)/2} = \sum_{n=0}^{\infty} (1 - zq^{2n+1}) q^{2n^2+n} z^{2n}. \tag{2.1}$$

Comparing (2.1)with [Andrews and Berndt 1; (6.2.29), p. 152], viz

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} = \frac{1}{1+} \frac{aq}{1+} \frac{a(q^2-q)}{1+} \frac{aq^3}{1+} \frac{a(q^4-q^2)}{1+\ldots} \tag{2.1(i)}$$

we have

$$\sum_{n=0}^{\infty} (1 - zq^{2n+1}) q^{n(2n+1)} z^{2n} = \frac{1}{1+} \frac{zq}{1+} \frac{z(q^2-q)}{1+} \frac{zq^3}{1+} \frac{z(q^4-q^2)}{1+\ldots} \tag{2.2}$$

(ii) Replacing z by β/α and then taking $\alpha \to \infty$ in (1.3) we have

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n-1)/2} \beta^n}{[\beta;q]_n} = \sum_{n=0}^{\infty} \frac{(-)^n [q;q]_n (1-\beta q^{2n}) \beta^{2n} q^{3n(n-1)/2}}{[\beta;q]_n} \tag{2.3}$$

For $\beta = q$ in (2.3) yields,

$$\sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+1)/2}}{[q;q]_n} = \sum_{n=0}^{\infty} (-)^n (1 - q^{2n+1}) q^{n(3n+1)/2}. \tag{2.4}$$

For $\beta = -q$, (2.3) yields

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{[-q;q]_n} = \sum_{n=0}^{\infty} \frac{(-)^n [q;q]_n (1-q^{2n+1}) q^{n(3n+1)/2}}{[-q;q]_n}. \tag{2.5}$$

If we take $\beta = -1$ in (2.3) we have,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[-1;q]_n} = \sum_{n=0}^{\infty} \frac{(-)^n [q;q]_n (1+q^{2n}) q^{3n(n-1)/2}}{[-1;q]_n}. \tag{2.6}$$

(iii) For $\beta = \alpha$, (1.3) yields

$$\sum_{n=0}^{\infty} (1-\alpha z q^{2n}) q^{n^2-n} (\alpha z)^n = 1. \tag{2.7}$$

Again, for $\beta = \alpha q$, (1.3) yields

$$\sum_{n=0}^{\infty} \frac{z^n}{1-\alpha q^n} = \sum_{n=0}^{\infty} \frac{(\alpha z)^n q^{n^2} (1-\alpha z q^{2n})}{(1-\alpha q^n)(1-z q^n)}, \tag{2.8}$$

which is representation of a Lambert series into generalized Lambert series. If we replace z by zq^i and α by αq^j in (2.8) we have

$$\sum_{n=0}^{\infty} \frac{z^n q^{in}}{1-\alpha q^{n+j}} = \sum_{n=0}^{\infty} \frac{(\alpha z)^n q^{n^2+(i+j)n} (1-\alpha z q^{2n+i+j})}{(1-\alpha q^{n+j})(1-z q^{n+i})}. \tag{2.9}$$

(iv) Replacing q by q^2 and then taking $\alpha = -q$ and $\beta = -q^2$ in (1.3) we get,

$$\sum_{n=0}^{\infty} \frac{[-q;q^2]_n z^n}{[-q^2;q^2]_n} = \sum_{n=0}^{\infty} \frac{[-q;q^2]_n (-z)^n q^{2n^2} (1+z q^{4n+1}) [zq;q^2]_n}{[-q^2;q^2]_n [z;q^2]_{n+1}}. \tag{2.10}$$

For $z = -q$, (2.10) yields

$$\sum_{n=0}^{\infty} \frac{[-q;q^2]_n (-q)^n}{[-q^2;q^2]_n} = \sum_{n=0}^{\infty} q^{2n^2+n} (1-q^{2n+1}). \tag{2.11}$$

Comparing (2.11) and [Andrews and Berndt 1; (9.3.1) p. 227] we have

$$\sum_{n=0}^{\infty} \frac{[-q;q^2]_n (-q)^n}{[-q^2;q^2]_n} = \sum_{n=0}^{\infty} (1-q^{2n+1}) q^{2n^2+n} = \sum_{n=0}^{\infty} (-)^n q^{n(n+1)/2}. \tag{2.12}$$

Again, comparing (2.12) with (2.1(i)) we have

$$\sum_{n=0}^{\infty}\frac{[-q;q^2]_n(-q)^n}{[-q^2;q^2]_n}=\frac{1}{1+}\frac{q}{1+}\frac{q^2-q}{1+}\frac{q^3}{1+}\frac{q^4-q^2}{1+...} \tag{2.13}$$

(v) Replacing q by q^2, z by $-z/\alpha$ and then putting $\beta=q^2$ in (1.3) we get,

$$\sum_{n=0}^{\infty}\frac{[\alpha;q^2]_n(-)^n(z/\alpha)^n}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}\frac{[\alpha;q^2]_n[-z;q^2]_n(-)^n(z/\alpha)^nq^{2n^2}(1+zq^{4n})}{[q^2;q^2]_n[-z/\alpha;q]_{n+1}} \tag{2.14}$$

As $\alpha\to\infty$, (2.14) yields

$$\sum_{n=0}^{\infty}\frac{q^{n^2-n}z^n}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}q^{3n^2-n}\frac{[-z;q^2]_nz^n(1+zq^{4n})}{[q^2;q^2]_n}, \tag{2.15}$$

Putting $z=q$ in (2.15) we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}q^{3n^2}\frac{[-q;q^2]_n(1+q^{4n+1})}{[q^2;q^2]_n}. \tag{2.16}$$

For $z=-q$, (2.15) yields

$$\sum_{n=0}^{\infty}\frac{(-)^nq^{n^2}}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}q^{3n^2}\frac{(-)^n[q;q^2]_n(1-q^{4n+1})}{[q^2;q^2]_n}. \tag{2.17}$$

Also, for $z=-q^2$, (2.15) yields

$$\sum_{n=0}^{\infty}\frac{(-)^nq^{n(n+1)}}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}(-)^nq^{n(3n+1)}(1-q^{4n+2}), \tag{2.18}$$

which is [Andrews and Berndt 1;(9.4.2), p. 232] if we replace q by q^2 and $a=1$ in it.

Again, taking $z=q^2$ in (2.15) we have

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}q^{n(3n+1)}\frac{[-q^2;q^2]_n(1+q^{4n+2})}{[q^2;q^2]_n}. \tag{2.19}$$

If we take $a=-1$ in [Andrwes and Berndt 1; (6.2.30), p. 152] we get,

$$\sum_{n=0}^{\infty}\frac{(-)^nq^{n^2}}{[q^2;q^2]_n}=\frac{1}{[-q;q]_\infty}. \tag{2.20}$$

Comparing (2.17) and (2.20) we have

$$\sum_{n=0}^{\infty}(-)^n q^{3n^2}\frac{[q;q^2]_n}{[q^2;q^2]_n}(1-q^{4n+1}) = \frac{1}{[-q;q]_\infty} = [q;q^2]_\infty. \tag{2.21}$$

Taking $a=-1$ in [Andrews and Berndt 1; (6.2.32) p. 152] we get,

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{[q^2;q^2]_n} = \frac{1}{[-q;q]_\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{\Psi(q)}{[-q;q]_\infty}, \tag{2.22}$$

where $\Psi(q)$ is Ramanujan's theta function defined in [Andrews and Berndt 1; (3.3.1), p.94].
Comparing (2.18) and (2.22) we have,

$$\sum_{n=0}^{\infty}(-)^n q^{n(3n+1)}(1-q^{4n+2}) = \frac{\Psi(q)}{[-q;q]_\infty} = [q;q^2]_\infty\Psi(q). \tag{2.23}$$

If we take $a=-1$ in [Andrews and Berndt 1; (6.2.29), p. 152] we get,

$$\Psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{1-}\frac{q}{1-}\frac{q^2-q}{1-}\frac{q^3}{1-}\frac{q^4-q^2}{1-...} \tag{2.24}$$

thus, dividing (2.23) by (2.21) we have,

$$\frac{\displaystyle\sum_{n=0}^{\infty}(-)^n q^{n(3n+1)}(1-q^{4n+2})}{\displaystyle\sum_{n=0}^{\infty}(-)^n q^{3n^2}\frac{[q;q^2]_n}{[q^2;q^2]_n}(1-q^{4n+1})} = \Psi(q) = \frac{1}{1-}\frac{q}{1-}\frac{q^2-q}{1-}\frac{q^3}{1-}\frac{q^4-q^2}{1-...}. \tag{2.25}$$

Also, dividing (2.22) by (2.20) we get,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)}}{[q^2;q^2]_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}}{[q^2;q^2]_n}} = \Psi(q) = \frac{1}{1-}\frac{q}{1-}\frac{q^2-q}{1-}\frac{q^3}{1-}\frac{q^4-q^2}{1-...}. \tag{2.26}$$

In the Ph.D. thesis of Lilawati Pal [3], she has established

$$[-q;q]_\infty = 1+\frac{q}{0+}\frac{1-q}{1+}\frac{q^2}{0+}\frac{1-q^2}{1+}\frac{q^3}{0+}\frac{1-q^3}{1+...}. \tag{2.27}$$

[3, chapter 3 (3.14)]

From (2.20) and (2.27) we have

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}}{[q^2;q^2]_n}=1+\frac{q}{0+}\frac{1-q}{1+}\frac{q^2}{0+}\frac{1-q^2}{1+}\frac{q^3}{0+}\frac{1-q^3}{1+...}. \tag{2.28}$$

Also, from (2.22), (2.24) and (2.27) we have,

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)}}{[q^2;q^2]_n}=\sum_{n=0}^{\infty}(-)^n q^{n(3n+1)}(1-q^{4n+2})$$

$$=\left(\frac{1}{1-}\frac{q}{1-}\frac{q^2-q}{1-}\frac{q^3}{1-}\frac{q^4-q^2}{1-...}\right)\left(\frac{1}{1+}\frac{q}{0+}\frac{1-q}{1+}\frac{q^2}{0+}\frac{1-q^2}{1+}\frac{q^3}{0+}\frac{1-q^3}{1+...}\right) \tag{2.29}$$

(vi) In identity (1.3), replacing q by q^2, then setting $\beta=-aq^3$, $\alpha=a^2q^2/z$, and letting $z\to\infty$ we get

$$\sum_{n=0}^{\infty}\frac{(-)^n a^{2n}q^{n(n+1)}}{[-aq;q^2]_{n+1}}=\sum_{n=0}^{\infty}a^{3n}q^{3n^2+2n}(1-aq^{2n+1}). \tag{2.30}$$

Taking $a=-1$ in (2.30) and the comparing it with [Andrews and Berndt 1; (6.2.39), p. 155] we get,

$$\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)}}{[q;q^2]_{n+1}}=\sum_{n=0}^{\infty}(-)^n q^{3n^2+2n}(1+q^{2n+1})$$

$$=\frac{1}{1+}\frac{q^2-q}{1+}\frac{q^4-q^2}{1+}\frac{q^6-q^3}{1+}\frac{q^8-q^4}{1+...}. \tag{2.31}$$

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