

ON CERTAIN CAYLEY-ORR IDENTITIES OF SPECIAL TYPE

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(Received: Jul. 05, 2019 Accepted: Sep. 12, 2019 Published: Dec. 31, 2019)

Abstract: In this paper, making use of certain known transformations of poly-basic hypergeometric series we establish some Cayley-Orr type of identities which are of number theoretic importance.

Keywords and Phrases: Cayley- Orr identities, poly- basic, transformations, generalized hypergeometric functions, q-series.

2010 Mathematics Subject Classification: 33A30, 33A15.

1. Notations and Definitions

We shall follow the following notations and definitions in this paper.
 We define a generalized hypergeometric function, ${}_rF_s$ as,

$${}_rF_s \left[\begin{matrix} (a_r); z \\ (b_s) \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n z^n}{(1)_n (b_1)_n (b_2)_n \dots (b_s)_n} \quad (1.1)$$

where the symbol

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1); & n > 0 \\ 1; & n = 0 \end{cases}$$

The series (1.1) is valid for $|z| \leq 1$ when $r = s + 1$; for all values of $|z|$ when $r \leq s + 1$ and for $z = 0$ when $r \geq s + 1$.

Again for any α and q , ($|q| < 1$) real or complex, we shall write

$$(\alpha; q)_n \equiv (\alpha)_n = \begin{cases} (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \dots (1 - \alpha q^n); & n > 0 \\ 1; & n = 0 \end{cases}$$

and,

$$(\alpha; q)_{-n} \equiv (\alpha)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{(\alpha)^n \left(\frac{q}{\alpha}\right)_n} \quad (1.2)$$

Now, we define a basic hypergeometric function

$${}_r\phi_s \left[\begin{matrix} (a_r); q; z \end{matrix} \right] \equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n z^n}{(q)_n (b_1)_n (b_2)_n \dots (b_s)_n}, |z| < 1 \quad (1.3)$$

We also define a basic bilateral hypergeometric function

$${}_r\psi_r \left[\begin{matrix} (a_r); q; z \end{matrix} \right] \equiv {}_r\psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_r \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n z^n}{(q)_n (b_1)_n (b_2)_n \dots (b_r)_n}, \quad (1.4)$$

valid for $|b_1 b_2 \dots b_r / a_1, a_2, \dots, a_r| < |z| < 1$.

We define a poly-basic hypergeometric series,

$$\begin{aligned} & \phi \left[\begin{matrix} a_1, a_2, a_3, \dots; b_1, b_2, b_3, \dots; c_1, c_2, c_3, \dots; p_1, p_2, p_3, \dots; \dots; z \\ d_1, d_2, d_3, \dots; e_1, e_2, e_3, \dots; f_1, f_2, f_3, \dots; \dots \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{[a_1, a_2, a_3, \dots; p_1]_n [b_1, b_2, b_3, \dots; p_2]_n [c_1, c_2, c_3, \dots; p_3]_n \dots z^n}{[p_1]_n [d_1, d_2, d_3, \dots; p_1]_n [e_1, e_2, e_3, \dots; p_2]_n [f_1, f_2, f_3, \dots; p_3]_n \dots} \end{aligned} \quad (1.5)$$

for $|z|, |p_1|, |p_2|, \dots, < 1$.

We also introduce a bilateral poly-basic hypergeometric series,

$$\begin{aligned} & \psi \left[\begin{matrix} a_1, a_2, a_3, \dots; b_1, b_2, b_3, \dots; c_1, c_2, c_3, \dots; p_1, p_2, p_3, \dots; \dots; z \\ d_1, d_2, d_3, \dots; e_1, e_2, e_3, \dots; f_1, f_2, f_3, \dots; \dots \end{matrix} \right] \\ &= \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, a_3, \dots; p_1]_n [b_1, b_2, b_3, \dots; p_2]_n [c_1, c_2, c_3, \dots; p_3]_n \dots z^n}{[p_1]_n [d_1, d_2, d_3, \dots; p_1]_n [e_1, e_2, e_3, \dots; p_2]_n [f_1, f_2, f_3, \dots; p_3]_n \dots} \end{aligned} \quad (1.6)$$

valid under suitable convergence conditions.

2. Introduction:

In 1858 Cayley [1] published without proof the following remarkable theorem:
If

$$(1 - z)^{a+b-c} {}_2F_1 \left[\begin{matrix} 2a, 2b; z \\ 2c \end{matrix} \right] = \sum_{n=0}^{\infty} a_n z^n \quad (2.1)$$

then

$${}_2F_1 \left[\begin{matrix} a, b; z \\ c + 1/2 \end{matrix} \right] {}_2F_1 \left[\begin{matrix} c - a, c - b; z \\ 1/2 + c \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[c]_n a_n z^n}{[c + 1/2]_n} \quad (2.2)$$

It was not until forty years later that a proof of it was published by Orr [2] who discussed the differential equation satisfied by the product of two hypergeometric series and obtained some additional theorems of a similar nature. Other mathematicians such as Edwardes [3], Watson [8], Burchnall and Chandy [4], Whipple [9,10] also dealt with similar identities employing certain methods, not necessarily used by Orr.

Denis [5] in 1969 established certain very interesting identities of Cayley-Orr type involving basic bilateral hypergeometric series which are of number theoretic importance. Due to lack of suitable transformations, this work could not go further. Motivated by the above work of Denis and availability of a large number of transformations of bilateral poly-basic hypergeometric series due to Denis, et.al [6,7] we attempt to establish certain Cayley-Orr identities of special type, by using the above mentioned known transformations of poly-basic hypergeometric functions. However, our attempt will be to use only a few of these transformations as illustration and the rest can be dealt with similarly.

3. Results to be used:

In this section we shall use the following known transformations of poly-basic hypergeometric functions

$$\begin{aligned} & {}_4\psi_4 \left[\begin{matrix} aq_1^m, yq_1^m : q^{-n}, q^{-n}/\alpha\beta; q_1, q; q_1 \\ q_1^{1+m}, ayq_1^{1+m} : q^{-n}/\alpha, q^{-n}/\beta \end{matrix} \right] \\ &= \frac{[\alpha, \beta; q]_m [q_1, ayq_1; q_1]_m [aq_1, yq_1; q_1]_n [q, \alpha\beta q; q]_n q_1^m}{[q, \alpha\beta q; q]_m [a, y; q_1]_m [q_1, ayq_1; q_1]_n [\alpha q, \beta q; q]_n q_1^m} \\ & \quad \times {}_4\psi_4 \left[\begin{matrix} \alpha q^m, \beta q^m : q_1^{-n}, q_1^{-n}/ay; q, q_1; q \\ q^{1+m}, \alpha\beta q^{1+m} : q_1^{-n}/a, q_1^{-n}/y \end{matrix} \right] \end{aligned} \quad (3.1)$$

[Denis, et.al[6;3.7]]

$$\begin{aligned}
& {}_4\psi_4 \left[\begin{matrix} \alpha q_1^m, \beta q_1^m : q^{-n}, bp^{-n}/a; q_1, q, p; q_1/b \\ q_1^{1+m}, \alpha\beta q_1^{m+1} : q^{-n}/b, p^{-n}/a \end{matrix} \right] \\
&= \frac{[q_1, \alpha\beta q_1; q_1]_m [q; q]_n [ap/b; p]_n [apq; pq]_m}{[\alpha, \beta; q_1]_m [ap; p]_n [bq; q]_n [a; pq]_m [q; q]_m} \\
&\quad \times \frac{[a; p]_m [b; q]_m [\alpha q_1, \beta q_1; q_1]_n b^n}{[ap/b; p]_m [q_1, \alpha\beta q_1; q_1]_n (bq_1)^m} \\
&\times {}_5\psi_5 \left[\begin{matrix} ap^{1+m} q^{1+m} : ap^m : bq^m : q_1^{-n}, q_1^{-n}/\alpha\beta; pq, p, q, q_1; 1/b \\ ap^m q^m : ap^{1+m}/b : q^{1+m} : q_1^{-n}/\alpha, q_1^{-n}/\beta \end{matrix} \right] \quad (3.2)
\end{aligned}$$

[Denis et. al.[6;3.18]]

$$\begin{aligned}
& {}_{10}\psi_{10} \left[\begin{matrix} ap^{m+1} q^{m+1} : bp^{m+1}/q^{m+1} : ap^m, bp^m : cq^m, aq^m/bc : \\ a(pq)^m : b(p/q)^m : ap^{m+1}/c, bcp^{m+1} : q^{m+1}, \\ q_1^{-n}, eq_1^{-n}, fq_1^{-n}/d, gq_1^{-n}/d; pq, p/q, p, q, q_1; q \\ aq^{m+1}/b : q_1^{-n}/d, q_1^{-n}/e, q_1^{-n}/f, q_1^{-n}/g \end{matrix} \right] \\
&= \frac{[a; pq]_m [b; p/q]_m [q, aq/b]_m [ap/c, bcp; p]_m [q_1, dq_1/e, dq_1/f, dq_1/g; q_1]_n}{[apq; pq]_m [bp/q; p/q]_m [a, b; p]_m [c, a/bc; q]_m [dq_1, eq_1, fq_1, gq_1; q_1]_n} \\
&\times \frac{[d, q_1\sqrt{d}, -q_1\sqrt{d}, e, f, g; q_1]_m [ap, bp; p]_n [cq, aq/bc; q]_n q_1^m}{[q_1, \sqrt{d}, -\sqrt{d}, dq_1/e, dq_1/f, dq_1/g; q_1]_m [q, aq/b; q]_n [ap/c, bcp; p]_n q^m} \\
&\times {}_{10}\psi_{10} \left[\begin{matrix} dq_1^m, q_1^{1+m}\sqrt{d}, -q_1^{1+m}\sqrt{d}, eq_1^m, fq_1^m, gq_1^m : q^{-n}, \\ q_1^{m+1}, q_1^m\sqrt{d}, -q_1^m\sqrt{d}, dq_1^{m+1}/e, \\ bq^{-n}/a : cp^{-n}/a, p^{-n}/bc; q_1, q, p; q_1 \\ dq_1^{m+1}/f, dq_1^{m+1}/g : q^{-n}/c, bcq^{-n}/a : p^{-n}/a, p^{-n}/b \end{matrix} \right] \quad (3.3)
\end{aligned}$$

[Denis et. al.[6;3.11]]

where d=efg.

4. Results to be established:

In this section we shall establish the following identities:

If,

$${}_2\psi_2 \left[\begin{matrix} a, y; zq \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q, \beta q; z \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

then,

$$\begin{aligned} & {}_2\psi_2 \left[\begin{matrix} \alpha, \beta; zq \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} aq, yq; z \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] \\ &= \frac{[q/a, q/y, 1/ay, 1/\alpha, 1/\beta; q]_m}{[1/a, 1/y, 1/\alpha\beta, q/\alpha, q/\beta; q]_m} \sum_{n=-\infty}^{\infty} \frac{[ayq, \alpha\beta q^{1-m}; q]_n [q^{-n}/\alpha\beta]_m}{[\alpha\beta q, ayq^{1-m}; q]_n [q^{-n}/ay; q]_m} A_n z^n \quad (4.1.a) \end{aligned}$$

If,

$${}_2\psi_2 \left[\begin{matrix} aq^m, yq^m; zq \\ q^{1+m}, ayq^{1+m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q, \beta q; z \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

then,

$$\begin{aligned} & {}_2\psi_2 \left[\begin{matrix} \alpha, \beta; zq \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} aq^{1+m}, yq^{1+m}; z \\ q^{1+m}, ayq^{1+m} \end{matrix} \right] \\ &= \frac{[a, y, 1/\alpha, 1/\beta; q]_m q^{2m}}{[q/\alpha, q/\beta, aq, yq, 1/\alpha\beta; q]_m} \sum_{n=-\infty}^{\infty} \frac{[\alpha\beta q^{1-m}; q]_n [q^{-n}/\alpha\beta]_m}{[\alpha\beta q; q]_n} A_n (q^m z)^n \quad (4.1.b) \end{aligned}$$

If,

$${}_2\psi_2 \left[\begin{matrix} a, y; zq \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q^{1+m}, \beta q^{1+m}; z \\ q^{1+m}, \alpha\beta q^{1+m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

then,

$$\begin{aligned} & {}_2\psi_2 \left[\begin{matrix} aq, yq; z \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q^m, \beta q^m; zq \\ q^{1+m}, \alpha\beta q^{1+m} \end{matrix} \right] \\ &= \frac{[q/a, q/y, 1/ay, \alpha q, \beta q; q]_m}{[1/a, 1/y, \alpha, \beta; q]_m} \sum_{n=-\infty}^{\infty} \frac{[ayq; q]_n}{[ayq^{1-m}; q]_n [q^{-n}/ay; q]_m} A_n z^n \quad (4.1.c) \end{aligned}$$

If,

$${}_4\psi_4 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-m}, b; zq^m/b \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q, \beta q; z \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

then,

$${}_2\psi_2 \left[\begin{matrix} \alpha, \beta; zbq^{1-m} \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} bq, aq^{1-m}; z \\ q^{1+m}, aq/b \end{matrix} \right]$$

$$\begin{aligned}
&= \frac{[1/\sqrt{a}, -1/\sqrt{a}, 1/\alpha, q/b; q]_m [q/a; q]_{2m} a^{2m} (-1)^m q^m}{[1/b, q/a, q/\sqrt{a}, -q/\sqrt{a}, q/\alpha, q/\beta, 1/\alpha\beta, \beta q]_m \beta^m q^{2m^2}} \\
&\times \sum_{n=-\infty}^{\infty} \frac{[\alpha\beta q^{1-m}; q]_n [aq^{1-m}; q]_n [q^{-n}/\alpha\beta, q^{-n}/a; q]_m}{[\alpha\beta q; q]_n [\beta q^{1+m}; q]_n} A_n (bq^m z)^n \quad (4.2)
\end{aligned}$$

and if,

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-m}, b, c, aq^m/bc; zq \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq^{1-m} \end{matrix} \right] {}_4\psi_4 \left[\begin{matrix} dq^{1-m}, eq, fq, gq^{1+m}; z \\ q^{1+m}, dq/e, dq/f, dq^{1-m}/g \end{matrix} \right] \\
&= \sum_{n=-\infty}^{\infty} A_n z^n
\end{aligned}$$

then,

$$\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} dq^{-m}, q\sqrt{d}, -q\sqrt{d}, e, f, gq^m; zq \\ q^{1+m}, \sqrt{d}, -\sqrt{d}, dq/e, dq/f, dq^{1-m}/g \end{matrix} \right] {}_4\psi_4 \left[\begin{matrix} aq^{1-m}, bq, cq, aq^{1+m}/bc; z \\ q^{1+m}, aq/b, aq/c, bcq^{1-m} \end{matrix} \right] \\
&= \frac{[1/\sqrt{a}, -1/\sqrt{a}, q/b, q/c, 1/bc; q]_m}{[q/a, q/\sqrt{a}, -q/\sqrt{a}; q]_m} \\
&\times \frac{[a/bc, gq, 1/e, 1/f, q/d, q/\sqrt{d}, -q/\sqrt{d}; q]_m [q/a, 1/d; q]_{2m}}{[1/b, 1/c, 1/\sqrt{d}, -1/\sqrt{d}, q/e, q/f, g, g/d, aq/bc; q]_m [1/a, q/d; q]_{2m}} \\
&\times \sum_{n=-\infty}^{\infty} \frac{[aq^{1-m}]_n [bcq]_n [dq]_n [dq^{1-m}/g]_n [gq^{-n}/d]_m [q^{-n}/a]_m A_n z^n}{[aq]_n [bcq^{1-m}]_n [dq^{1-m}]_n [dq^{1-m}/g]_n [q^{-n}/bc]_m [q^{-n}/d]_m} \quad (4.3)
\end{aligned}$$

5. Proof:

In this section we shall discuss the outline of proof of our results. We start with (4.1(a)). To do so we consider the transformation (3.1) with $q_1 = q$ and $m = 0$ and then replace n, a, y, α and β by $2m + n, aq^{-m}, yq^{-m}, \alpha q^{-m}$ and βq^{-m} , respectively in it to have,

$$\begin{aligned}
&\sum_{r=0}^{2m+n} \frac{[aq^{-m}, yq^{-m}, \alpha q^{-m}, q^{-2m-n}, q^{-n}/\alpha\beta]_r q^r}{[q, ayq^{1-2m}, q^{-m-n}/\alpha, q^{-m-n}/\beta]_r} = \frac{[aq^{1-m}, yq^{1-m}, \alpha^{1-2m}]_{2m+n}}{[ayq^{1-2m}, \alpha q^{1-m}, \beta q^{1-m}]_{2m+n}} \\
&\times \sum_{r=0}^{2m+n} \frac{[\alpha q^{-m}, \beta q^{-m}, q^{-2m-n}, q^{-n}/ay]_r q^r}{[q, \alpha\beta q^{1-2m}, q^{-m-n}/a, q^{-m-n}/y]_r}
\end{aligned}$$

Now replacing r by $m + s$ in above, we get

$$\begin{aligned}
& [q/a, q/y, q^{-n}/\alpha\beta, 1/ay]_n [\alpha q, \beta q]_n \times {}_4\psi_4 \left[\begin{matrix} a, y, q^{-m-n}, q^{-n+m}/\alpha\beta; q \\ q^{1+m}, ayq^{1-m}, q^{-n}/\alpha, q^{-n}/\beta \end{matrix} \right] \\
&= \frac{[1/a, 1/y]_m [\alpha\beta q, aq, yq]_n [q/\alpha, q/\beta, q^{-n}/ay, 1/\alpha\beta]_m}{[1/\alpha, 1/\beta]_m [ayq]_n} \\
&\quad \times {}_4\psi_4 \left[\begin{matrix} \alpha, \beta, q^{-m-n}, q^{-n+m}/ay; q \\ q^{1+m}, \alpha\beta q^{1-m}, q^{-n}/a, q^{-n}/y \end{matrix} \right] \tag{5.1}
\end{aligned}$$

Let us consider

$${}_2\psi_2 \left[\begin{matrix} a, y; zq \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q, \beta q; z \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

It is obvious from the above that

$$A_n = \frac{[\alpha q, \beta q]_n}{[q^{1+m}, \alpha\beta q^{1-m}]_n} \times {}_4\psi_4 \left[\begin{matrix} a, y, q^{-m-n}, q^{-n+m}/\alpha\beta; q \\ q^{1+m}, ayq^{1-m}, q^{-n}/\alpha, q^{-n}/\beta \end{matrix} \right] \tag{5.2}$$

Also

$${}_2\psi_2 \left[\begin{matrix} \alpha, \beta; zq \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} aq, yq; z \\ q^{1+m}, ayq^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} B_n z^n$$

implies that

$$B_n = \frac{[aq, yq]_n}{[q^{1+m}, ayq^{1-m}]_n} {}_4\psi_4 \left[\begin{matrix} \alpha, \beta, q^{-m-n}, q^{-n+m}/ay; q \\ q^{1+m}, \alpha\beta q^{1-m}, q^{-n}/a, q^{-n}/y \end{matrix} \right] \tag{5.3}$$

Now the equations (5.1), (5.2) and (5.3) lead to the proof of (4.1(a)). Next we shall prove (4.1(b)). Let us consider the transformation (3.1) with $q_1 = q$ and $m = 0$ with n, α and β replaced by $2m + n, \alpha q^{-m}$ and βq^{-m} respectively. The proof follows exactly on the same steps as those of the proof of (4.1(a)).

Similarly to prove (4.1(c)), we replace n, a and y by $2m + n, aq^{-m}$ and yq^{-m} , respectively in (3.1) with $q_1 = q$ and $m = 0$. The proof, now, follows exactly on the line of the above results.

Next we shall prove (4.2). To do so we use the transformation (3.2) with $q_1 = p = q$ and $m = 0$ and replace n, a, b, α and β by $2m + n, aq^{-2m}, bq^{-m}, \alpha q^{-m}$ and βq^{-m} , respectively, we get,

$$\sum_{r=0}^{2m+n} \frac{[\alpha q^{-m}, \beta q^{-m}, q^{-2m-n}, bq^{-m-n}/a]_r (q^{1+m}/b)^r}{[q, \alpha\beta q^{1-2m}, q^{-m-n}/b, q^{-n}/a]_r} = \frac{[aq^{1-m}/b, \alpha q^{1-m}, \beta q^{1-m}]_{2m+n}}{[aq^{1-m}, bq^{1-m}, \alpha\beta q^{1-2m}]_{2m+n}}$$

$$\times (bq^{-m})^{2m+n} \sum_{r=0}^{2m+n} \frac{[q^{1-m}\sqrt{a}, -q^{1-m}\sqrt{a}, aq^{-2m}, bq^{-m}, q^{-2m-n}, q^{-n}/\alpha\beta]_r (q^m/b)^r}{[q, q^{-m}\sqrt{a}, -q^{-m}\sqrt{a}, aq^{1-m}/b, q^{-m-n}/\alpha, q^{-m-n}/\beta]_r}$$

Now replacing r by $m + s$ in above relation, we get

$$\begin{aligned} & \frac{[q/\alpha, q/\beta, 1/\alpha\beta]_m [bq]_n}{[q^{-n}/a]_m [aq/b]_n} {}_4\psi_4 \left[\begin{matrix} \alpha, \beta, q^{-m-n}, bq^{-n}/a; q^{1+m}/b \\ q^{1+m}, \alpha\beta q^{1-m}, q^{-n}/b, q^{m-n}/a \end{matrix} \right] \\ &= \frac{[1/\sqrt{a}, -1/\sqrt{a}, 1/\alpha, 1/\beta, q/b, q^{-n}/\alpha\beta]_m [q/a]_{2m} [\alpha q, \beta q]_n q^{-m(n-1)b^n}}{[1/b, q/a, q/\sqrt{a}, -q/\sqrt{a}]_m [1/a]_{2m} [aq, \alpha\beta q]_n} \\ & \quad \times {}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-m}, b, q^{-m-n}, q^{m-n}/\alpha\beta; q^m/b \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b, q^{-n}/\alpha, q^{-n}/\beta \end{matrix} \right] \end{aligned} \quad (5.4)$$

Now let us consider

$${}_4\psi_4 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^m, b; zq^m/b \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} \alpha q, \beta q; z \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] = \sum_{n=-\infty}^{\infty} A_n z^n$$

It is obvious that

$$\begin{aligned} A_n &= \frac{[\alpha q, \beta q]_n}{[q^{1+m}, \alpha\beta q^{1-m}]_n} \times \\ & {}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-m}, b, q^{-m-n}, q^{m-n}/\alpha\beta; q^m/b \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b, q^{-n}/\alpha, q^{-n}/\beta \end{matrix} \right] \end{aligned} \quad (5.5)$$

Similarly

$${}_2\psi_2 \left[\begin{matrix} \alpha, \beta; z bq^{1-m} \\ q^{1+m}, \alpha\beta q^{1-m} \end{matrix} \right] {}_2\psi_2 \left[\begin{matrix} bq, aq^{1-m}; z \\ q^{1+m}, aq/b \end{matrix} \right] = \sum_{n=-\infty}^{\infty} B_n z^n$$

leads to

$$B_n = \frac{[bq, aq^{1-m}]_n}{[q^{1+m}, aq/b]_n} \times {}_4\psi_4 \left[\begin{matrix} \alpha, \beta, q^{-m-n}, bq^{-n}/a; q^{1+m}/b \\ q^{1+m}, \alpha\beta q^{1-m}, q^{-n}/b, q^{m-n}/a \end{matrix} \right] \quad (5.6)$$

Now, (5.4), (5.5), and (5.6) lead to the proof of (4.2).

Next, in order to prove (4.3) we use the transformation (3.3) with $p = q_1 = q$, $m = 0$ and n, a, b, c, d, e and f replaced by $2m + n, aq^{-2m}, bq^{-m}, cq^{-m}, dq^{-2m}, eq^{-m}$ and $f q^{-m}$, respectively, to have

$$\sum_{r=0}^{2m+n} \frac{[q^{1-m}\sqrt{a}, -q^{1-m}\sqrt{a}, aq^{-2m}, bq^{-m}, cq^{-m}]_r}{[q, q^{-m}\sqrt{a}, -q^{-m}\sqrt{a}, aq^{1-m}/c, bcq^{1-2m}]_r}$$

$$\begin{aligned}
& \times \frac{[a/bc, q^{-2m-n}, eq^{-m-n}/d, fq^{-m-n}/d, gq^{-n}/d]_r q^r}{[aq^{1-m}/b, q^{-n}/d, q^{-m-n}/e, q^{-m-n}/f, q^{-2m-n}/g]_r} \\
& = \frac{[bq^{-m}, cq^{1-m}, dq^{1-m}/e, dq^{1-m}/f, aq^{1-2m}, dq^{1-2m}/g, aq/bc]_{2m+n}}{[eq^{1-m}, fq^{1-m}, aq^{1-m}/b, aq^{1-m}/c, dq^{1-2m}, bcq^{1-2m}, gq]_{2m+n}} \\
& \quad \times \sum_{r=0}^{2m+n} \frac{[dq^{-2m}, q^{1-m}\sqrt{d}, -q^{1-m}\sqrt{d}, eq^{-m}, fq^{-m}]_r}{[q, q^{-m}\sqrt{d}, -q^{-m}\sqrt{d}, dq^{1-m}/e, dq^{1-m}/f]_r} \\
& \quad \times \frac{[g, q^{-2m-n}, bq^{-m-n}/a, cq^{-m-n}/a, q^{-n}/bc]_r q^r}{[dq^{1-2m}/g, q^{-m-n}/c, bcq^{-2m-n}/a, q^{-n}/a, q^{-m-n}/b]_r}
\end{aligned}$$

Now, replacing r by $m+s$ in the above relation, we get, after some simplification

$$\begin{aligned}
& \frac{[1/\sqrt{a}, -1/\sqrt{a}, q/b, q/c, 1/bc, gq^{-n}/d, gq]_m [q/a]_{2m} [gq^{1+m}, eq, fq]_n}{[q/\sqrt{a}, -q/\sqrt{a}, q/a, q^{-n}/d]_m [dq/e, dq/f]_n} \times \\
& {}_{10}\psi_{10} \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-n}, b, c, aq^m/bc, q^{-m-n}, eq^{-n}/d, fq^{-n}/d, gq^{m-n}/d; q \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/c, bcq^{1-m}, aq/b, q^{m-n}/d, q^{-n}/e, q^{-n}/f, q^{m-n}/g \end{matrix} \right] \\
& = \frac{[1/b, 1/ca, q/bc, 1/\sqrt{d}, -1/\sqrt{d}, q/e, q/f, g, g/d, q^{-n}/bc]_m}{[1/e, 1/f, q/d, q/\sqrt{d}, -q/\sqrt{d}, q^{-n}/a]_m} \\
& \quad \times \frac{[1/a, q/d]_{2m} [aq, dq/g, cq, bq, aq^{1+m}/bc]_n}{[1/d]_{2m} [dq, bcq, aq/b, aq/c]_n} \\
& \quad \times {}_{10}\psi_{10} \left[\begin{matrix} dq^{-m}, q\sqrt{d}, -q\sqrt{d}, e, f, gq^m, \\ q^{1+m}, \sqrt{d}, -\sqrt{d}, dq/e, dq/f, \\ q^{-m-n}, bq^{-n}/a, cq^{-n}/a, q^{-n+m}/bc; q \\ dq^{1-m}/g, q^{-n}/c, bcq^{-n-m}/a, q^{-n+m}/a, q^{-n}/b \end{matrix} \right] \quad (5.7)
\end{aligned}$$

Now, let us consider,

$$\begin{aligned}
& {}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, aq^{-m}, b, c, aq^m/bc; zq \\ q^{1+m}, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq^{1-m} \end{matrix} \right] {}_4\psi_4 \left[\begin{matrix} dq^{1-m}, eq, fq, gq^{1+m}; z \\ q^{1+m}, dq/e, dq/f, dq^{1-m}/g \end{matrix} \right] \\
& = \sum_{n=-\infty}^{\infty} A_n z^n
\end{aligned}$$

and

$${}_6\psi_6 \left[\begin{matrix} dq^{-m}, q\sqrt{d}, -q\sqrt{d}, e, f, gq^m; zq \\ q^{1+m}, \sqrt{d}, -\sqrt{d}, dq/e, dq/f, dq^{1-m}/g \end{matrix} \right] {}_4\psi_4 \left[\begin{matrix} aq^{1-m}, bq, cq, aq^{1+m}/bc; z \\ q^{1+m}, aq/b, aq/c, bcq^{1-m} \end{matrix} \right]$$

$$= \sum_{n=-\infty}^{\infty} B_n z^n$$

Now, comparing the coefficient of z^n in the above two relations and making use of (5.7) we get (4.3).

The other transformations of Denis et.al. [9,10] will also lead to interesting Cayley-Orr type of identities.

Taking $m = 0$ in (4.3), we get,

If

$${}_6\phi_5 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, a, b, c, a/bc; zq \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{matrix} \right] {}_4\phi_3 \left[\begin{matrix} d, eq, fq, g; z \\ dq/e, dq/f, dq/g \end{matrix} \right] = \sum_{n=0}^{\infty} A_n z^n$$

Then,

$$\begin{aligned} & {}_6\phi_5 \left[\begin{matrix} d, q\sqrt{d}, -q\sqrt{d}, e, f, g; zq \\ \sqrt{d}, -\sqrt{d}, dq/e, dq/f, dq/g \end{matrix} \right] {}_4\phi_3 \left[\begin{matrix} aq, bq, cq, aq/bc; z \\ aq/b, aq/c, bcq \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{[aq]_n [bcq]_n [dq]_n [dq/g]_n A_n z^n}{[aq]_n [bcq]_n [dq]_n [dq/g]_n} \end{aligned} \quad (5.8)$$

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