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PERTURBATION OF INFINITESIMAL GENERATOR IN SEMIGROUP OF LINEAR OPERATOR

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Abstract: This paper consists of results on contraction semigroups thereby considering ω-order reversing partial contraction mapping (semigroup of linear operator) as the infinitesimal generator of the semigroup on perturbation by bounded linear operators which we will show that the addition of a bounded linear operator B to an infinitesimal generator A of a semigroup of linear opearator does not destroy the property of A.

Keywords and Phrases: ω-$ORCP_n$, C_0-Semigroup, Contraction Semigroup, Perturbation.

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1. Introduction

Let X be a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - ORCP_n$ be ω-order-reversing partial contraction mapping which is an example of C_0-semigroup, $\omega - ORCP_n \subseteq ORCP_n$ (Order Reversing Partial Contraction Mapping). Let $M_m(\mathbb{N} \cup 0)$ be a matrix, $L(X)$ the bounded linear operator in X, P_n, the partial transformation semigroup, $\rho(A)$ a resolvent of A, where A is the infinitesimal generator of a semigroup of linear operator and $F(x)$ be a duality

mapping on X. This paper consist of results on analytic and contraction in perturbation of bounded linear operator.
Balakrishnan [1], obtained an operator calculus for infinitesimal generators of semigroup. Beurling [3], deduced some analytic extension of semigroups of operator. Brezis and Cazenave [4], investigated linear semigroup of contractions, the Hille Yosida theory and some applications. Dafermos [5], established contraction semigroups and trends to equilibrium in continuum mechanics. Engel and Nagel [6], obtained one-parameter semigroup for linear evolution equations. Gutman [7], obtained some results on compact perturbation of m-accretive operators in general Banach spaces. Gustafson [8], proved some perturbation lemmas. Kato [9], introduced fractional powers of dissipative operators and also remarks on pseudo-resolvent and infinitesimal generators of semigroup (see [10]). Miyadera [11], established perturbation theory of semigroups. Pazy [12], introduced semigroup of linear operators and applications to partial differential equations. Rauf and Akinyele [13], obtained ω-order-preserving partial contraction mapping and established its properties, also in [14], Rauf *et.al.* established some results of stability and spectra properties on semigroup of linear operator. Vrabie [15], deduced some results of C_0-semigroup and its applications. Yosida [16], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [15]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-$ORCP_n$) [13]
A transformation $\alpha \in P_n$ is called ω-order-reversing partial contraction mapping if $\forall x, y \in Dom\alpha : x \le y \implies \alpha x \ge \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Analytic Semigroup) [15]
We say that a C_0-semigroup $\{T(t); t \ge 0\}$ is analytic if there exists $0 < \theta \le \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \ge 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each

$0 < \delta < \theta$, *the mapping* $z_1 \to S(z_1)$ *is bounded from* $\mathbb{C}_\delta$ *to* $L(X)$, *then the* C_0-*Semigroup* $\{T(t); t \geq 0\}$ *is called analytic and uniformly bounded.*

Definition 2.4 (Perturbation) [6]
Let $A : D(A) \subseteq X \to X$ *be the generator of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *and consider a second operator* $B : D(B) \subseteq X \to X$ *such that the sum* $A + B$ *generates a strongly continuous semigroup* $(S(t))_{t\geq 0}$. *We say that* A *is perturbed by operator* B *or that* B *is a perturbation of* A.

Definition 2.5 (C_0-semigroup of contraction)[15]
A C_0-*semigroup* $\{T(t); t \geq 0\}$ *is called of type* (M, ω) *with* $M \geq 1$ *and* $\omega \in \mathbb{R}$, *if for each* $t \geq 0$, *we have*
$\|T(t)\|_{L(X)} \leq Me^{t\omega}$.
A C_0-*semigroup* $\{T(t); t \geq 0\}$ *is called a* C_0-*semigroup of contraction or non expansive operator, if it is of type* $(1, 0)$, *that is, if for each* $t \geq 0$, *we have*
$\|T(t)\|_{L(X)} \leq 1$.

Definition 2.6 (Compact Semigroup) [6]
A C_0-*semigroup is compact if for each* $t > 0$, $T(t)$ *is a compact operator.*

Definition 2.7 (Dissipative)[15]
A linear operator $(A, D(A))$ *is dissipative if each* $x \in X$ *there exists* $x^* \in F(x)$ *such that* $Re(Ax, x^*) \leq 0$.

Definition 2.8 (m-dissipative)[12]
A dissipative operator A *for which* $R(I - A) = X$ *is called m-dissipative if* A *is dissipative so is* μA *for all* $\mu > 0$ *and therefore if* A *is m-dissipative then* $R(\lambda I - A) = X$ *for every* $\lambda > 0$.

Example 1. 3×3 matrix $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{3t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{t\lambda} \\ e^{3t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 2. Suppose $A : D(A) \subseteq X \to X$ is an unbounded generator of a strongly continuous semigroup and take an isomorphism $S \in L(X)$ such that

$D(A) \cap S(D(A)) = \{0\}$. Then $B = SAS^{-1}$ is a generator as well, but $A + B$ is defined only on $D(A+B) = D(A) \cap D(B) = D(A) \cap S(D(A)) = \{0\}$.
A concrete example for this situation is given on $X = C_0(\mathbb{R}_+)$ by $Af = f'$ with its canonical domain $D(A) = C_0'(\mathbb{R}_+)$ and $Sf = q.f$ for some continuous, positive function q such that q and q^{-1} are bounded and nowhere differentiable. Defining the operator B as $Bf = q.(q^1.f)'$ on $D(B) = \{f \in X : q^{-1}.f \in D(A)\}$, we obtain that the sum $A + B$ is defined only on $\{0\}$.

Theorem 2.1. (Hille-Yoshida) [15]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

Theorem 2.2. (Lumer-Phillips) [15]
Let $A : D(A) \subseteq X \to X$ be a densely defined operator. Then A generates a C_0-semigroup of contractions on X if and only if

i. A is dissipatives; and

ii. there exists $\lambda > 0$ such that $\lambda I - A$ is surjective.

Moreover, if A generates a C_0-semigroup of contractions, then $\lambda I - A$ is surjective for any $\lambda > 0$, and we have $Re(Ax, x^*) \leq 0$ for each $x \in D(A)$ and each $x^* \in F(x)$.

3. Main Results

In this section, perturbation of infinitesimal generators on contraction semigroups results on ω-$ORCP_n$ in semigroup of linear operator (C_0-semigroup) were considered:

Theorem 3.1. *Let X be a Banach space and A be the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$ on X such that $A \in \omega$-$ORCP_n$, satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is bounded linear operator on X, and $B \in \omega$-$ORCP_n$. Then:*

i. $A + B$ is the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$ on X, satisfying $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$; and *ii. there exists a unique family $U(t)$, $t \geq 0$ of bounded linear operators on X such that $t \to U(t)$ is continuous on $[0, \infty]$ for every $x \in X$ and $U(t)x = T(t)x + \int_0^t T(t-s)BU(s)xds$ for all $x \in X$.*

Proof. Assume there exists a norm $|.|$ on X such that $\|x\| \leq |x| \leq M\|x\|$ for every $x \in X$, $|T(t)| \leq e^{\omega t}$ and $|R(\lambda; A)| \leq (\lambda - \omega)^{-1}$ for real λ satisfying $\lambda > \omega$. Thus, for $\lambda > \omega + |B|$, the bounded linear operator $BR(\lambda; A)$ satisfies $|BR(\lambda; A)| < 1$ and therefore $I - BR(\lambda; A)$ is invertible for $\lambda > \omega + |B|$. Let

$$R = R(\lambda; A)(I - BR(\lambda; A))^{-1} = \sum_{k=0}^{\infty} R(\lambda; A)[BR(\lambda; A)]^k \tag{3.1}$$

then, $(\lambda I - A - B)R = (I - BR(\lambda; A))^{-1} - BR(\lambda; A)(I - BR(\lambda; A))^{-1} = I$ and

$$\begin{aligned} R(\lambda I - A - B)x &= R(\lambda; A)(\lambda I - A - B)x + \sum_{k=1}^{\infty} R(\lambda; A)[BR(\lambda; A)]^k(\lambda I - A - B)x \\ &= x - R(\lambda; A)Bx + \sum_{k=1}^{\infty}[R(\lambda; A)B]^k x - \sum_{k=1}^{\infty}[R(\lambda; A)B]^k x = x \end{aligned} \tag{3.2}$$

for every $x \in D(A)$ and $A \in \omega\text{-}ORCP_n$.Therefore, the resolvent of $A + B$ exists for $\lambda > \omega + |B|$ and it is given by the operator R. Moreover,

$$\begin{aligned} |(\lambda I - A - B)^{-1}| &= |\sum_{k=0}^{\infty} R(\lambda; A)[BR(\lambda; A)]^k| \\ &\leq (\lambda - \omega)^{-1}(I - |BR(\lambda; A)|)^{-1} \\ &\leq (\lambda - \omega - |B|)^{-1}. \end{aligned} \tag{3.3}$$

By definition 2.4, it follows that $A + B$ is the infinitesimal generator of a C_0-semigroup $\{T(t); t \leq 0\}$, satisfying $|S(t)| \leq e^{(\omega+|B|)t}$. Back to the original norm $\|.\|$ on X, we have

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t} \tag{3.4}$$

Now let's look at the relationship between semigroup $\{T(t); t \leq 0\}$ generated by A and the semigroup $\{T(t); t \leq 0\}$ generated by $A + B$ for all A,$B \in \omega\text{-}ORCP_n$. To this end, we consider the operator $H(s) = T(t-s)S(s)$. For $x \in D(A) = D(A+B)$, so $s \to H(s)x$ is differentiable and $H^{'}(s)x = T(t-s)BS(s)x$. Integrating $H^{'}(s)x$ from 0 to t yields

$$S(t)x = T(t) + \int_0^t T(t-s)BS(s)xds \tag{3.5}$$

for all $x \in D(A)$ and A,$B \in \omega$-$ORCP_n$. Since the operators on both sides of (3.5) are bounded, then (3.5) holds for every $x \in X$. The semigroup $\{T(t); t \leq 0\}$ is therefore the solution of the integral equations (3.5). For such integral equations, we have (ii) in the Theorem 3.1, and that proves (i).
To prove (ii), let

$$U_0(t)x = T(t) \tag{3.6}$$

and define $U_n(t)$ inductively by

$$V_{n+1}(t)x = \int_0^t T(t-s)BU_n(s)xds \tag{3.7}$$

for all $x \in X$, $n \geq 0$ and $B \in \omega$-$ORCP_n$. From this definition, it is obvious that $t \to V_n(t)x$ is continuous for all$x \in X$, $t \geq 0$ and every $n \geq 0$. Next we prove by induction that,

$$\|U_n(t)\| \leq Me^{\omega t}\frac{M^n\|B\|^n t^n}{n!}. \tag{3.8}$$

Indeed, for $n = 0$, (3.8) holds for by our assumption on $T(t)$ and the definition of $U_n(t)$. Assume (3.8) holds for n then by (3.7), we have

$$\begin{aligned}\|U_{n+1}(t)x\| &\leq \int_0^t Me^{\omega(t-s)}\|B\|Me^{\omega s}\frac{M^n\|B\|^n t^n}{n!}\\ &= Me^{\omega t}\frac{M^{n+1}\|B\|^{n+1}t^{n+1}}{n+1!}\end{aligned}$$

and thus (3.8) holds for $n > 0$. Defining

$$U(t) = \sum_{n=0}^{\infty} U_n(t), \tag{3.9}$$

then, it follows from (3.8) that (3.9) converges uniformly in the in the operator topology on bounded intervals. Therefore $t \to U(t)x$ is continuous for every $x \in X$ and moreso by (3.6) and (3.7), it follows that for every $x \in X$, $U(t)x$ satisfies the integral in (ii) of Theorem 3.1. This concludes the proof the existence statement. To prove the uniqueness, let $V(t)$, $t \geq 0$ be the family of bounded linear operators foe which $t \to V(t)x$ is continuous for every $x \in X$ and V(t)x =

$$V(t)x = T(t)x + \int_0^t T(t-s)BV(s)xds \tag{3.10}$$

for all $x \in X$, $n \geq 0$ and $B \in \omega\text{-}ORCP_n$. Subtracting (3.10) from (3.5) and estimating the difference yields

$$\|(U(t) - V(t))x\| \leq \int_0^t Me^{\omega(t-s)}\|B\|\|(U(s) - V(s))x\|ds. \tag{3.11}$$

But (ii) implies, for example by Gronwall's inequality, that $\|(U(t) - V(t))x\| = 0$ for every $t \geq 0$ and therefore $U(t) = V(t)$. From (ii) in Theorem 3.1, and the fact that the semigroup $\{S(t); t \geq 0\}$ generated by $A+B$ satisfies (3.5), we immediately obtained the following explicit representation of $S(t)$ in terms of $T(t)$, then

$$S(t) = \sum_{n=0}^{\infty} S_n(t) \tag{3.12}$$

where $S_0(t) = T(t)$, and

$$S_{n+1}(t)x = \int_0^t T(t-s)BS_n(s)xdx \tag{3.13}$$

for all $x \in X$, and the convergence in (3.12) is in the uniform operator topology. For the difference between $T(t)$ and $S(t)$, we have to prove a lemma, say

Lemma 3.2. *Suppose A is the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$ where $A \in \omega\text{-}ORCP_n$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Let B be a bounded linear operator and let $S(t)$ be the C_0-semigroup $\{S(t); t \geq 0\}$ generated by $A + B$ such that $A,B \in \omega\text{-}ORCP_n$. Then*
$\|S(t) - T(t)\| \leq Me^{\omega t}(e^{M\|B\|t} - 1)$.

Proof. From (3.5) and (i) of Theorem 3.1, we have

$$\begin{aligned}
\|S(t)x - T(t)x\| &\leq \int_0^t \|T(t-s)\|\|B\|\|S(s)\|\|x\|ds \\
&\leq M^2e^{\omega t}\|B\| \int_0^t e^{M\|B\|s}\|x\|ds \\
&= Me^{\omega t}(e^{M\|B\|t} - 1)\|x\|.
\end{aligned}$$

Hence the proof is complete.

Theorem 3.3. *Let A be the infinitesimal generator of an analytic semigroup where $A \in \omega\text{-}ORCP_n$ and let B be a closed operator satisfying $D(A) \supset D(A)$ and*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \tag{3.14}$$

for all $x \in D(A)$ and $A,B \in \omega$-$ORCP_n$, and there exists a positive number δ such that if $0 \leq a \leq \delta$, then $A+B$ is the infinitesimal generator of an analytic semigroup.

Proof. Suppose the semigroup generated by A is uniformly bounded. Then

$\rho(A) \supset \Sigma = \{\lambda; |arg\lambda| \leq \pi/2 + \omega\}$ for some $\omega > 0$ and in Σ, $\|R(\lambda; A)\| \leq M|\lambda|^{-1}$. Consider the bounded operator $BR(\lambda; A)$. From (3.14), it follows that for every $x \in X$ and $A \in \omega$-$ORCP_n$, we have

$$\begin{aligned} \|BR(\lambda; A)x\| &\leq a\|AR(\lambda; A)x\| + b\|R(\lambda; A)x\| \\ &\leq a(M+1)\|x\| + \frac{bM}{|\lambda|}\|x\|. \end{aligned} \tag{3.15}$$

Choosing $\delta = \frac{1}{2}(1+M)^{-1}$ and $|\lambda| > 2bM$, then, we have $\|BR(\lambda; A)\| < 1$ and therefore the operator $I - BR(\lambda; A)$ is invertible. A simple computation shows that

$$(\lambda I - (A+B))^{-1} = R(\lambda; A)(I - BR(\lambda; A))^{-1}. \tag{3.16}$$

Thus for $|\lambda| > 2bM$ and $|arg\lambda| \leq \pi/2 + \omega$, we obtained from (3.16) that

$$\|R(\lambda; A+B\| \leq M|\lambda|^{-1} \tag{3.17}$$

which implies that $A+B$ is the infinitesimal generator of an analytic semigroup. Suppose $T(t)$ is not uniformly bounded, then let $\|T(t)\| \leq Me^{\omega t}$. Let consider the semigroup $e^{-\omega t}T(t)$ generated by $A_0 = A - \omega I$. Then from (3.14), we have

$\|Bx\| \leq a\|Ax\| + (a\omega + b)\|x\|$ for all $x \in X$ and $A \in \omega$-$ORCP_n$. Therefore, by the first part of the proof, if $0 \leq a \leq \delta$, $A_0 + B = A + B - \omega I$ is an infinitesimal generator of an analytic semigroup which implies that $A+B$ is also the infinitesimal generator of an analytic semigroup, and this complete the proof.

Theorem 3.4. *Suppose A and B are linear operators in X, where $A,B \in \omega$-$ORCP_n$ such that $D(B) \supset D(A)$, then,*
(i) $A + tB$ is dissipative for $0 \leq t \leq 1$, if

$$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\| \tag{3.18}$$

for all $x \in D(A)$; and
(ii) B is dissipative and satisfy

$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|$ for all $x \in D(A)$, where $0 \leq \alpha < 1$, $\beta \geq 0$ and for some $t_0 \in [0,1]$, where $A + tB$ is dissipative, so that $A + tB$ is m-dissiptive for all $t \in [0,1]$. Then $A + B$ is the infinitesimal generator of a C_0-semigroup of contractions.

Proof. We need to show that there exists a $\delta > 0$ such that if $A + tB$ is m-dissipative, $A + tB$ is also m-dissipative for all $t \in [0, 1]$ and A,$B \in \omega$-$ORCP_n$ satisfying $|t - t_0| \leq \delta$. Since every point in $[0, 1]$ can be reached from every other point by a finite number of steps of length δ or less, this implies the result. Assume that for some $t_0 \in [0, 1]$ and $A + t_0B$ is m-dissipative. Then, $I - (A + t_0B)$ is invertible. Denoting $((I - (A + t_0B))^{-1}$ by $R(t_0)$, we have $\|R(t_0)\| \leq 1$. We need to show now that the operator $BR(t_0)$ is a bounded linear operator. From (3.18) and the triangle inequality, we have

$$\begin{aligned}\|Bx\| &\leq \alpha\|(A + t_0B)x\| + \alpha t_0\|Bx\| + \beta\|x\| \\ &\leq \alpha\|(A + t_0B)x\| + \alpha\|Bx\| + \beta\|x\|\end{aligned}$$

for all $x \in D(A)$, A,$B \in \omega$-$ORCP_n$ and therefore

$$\|Bx\| \leq \frac{\alpha}{1-\alpha}\|(A + t_0B)x\| + \frac{\beta}{1-\alpha}\|x\|. \tag{3.19}$$

Since $R(t_0) : X \to D(A)$ and $(A + t_0B)R(t_0) = R(t_0) - I$, it follows from (3.19) that

$$\begin{aligned}\|BR(t_0)x\| &\leq \frac{\alpha}{1-\alpha}\|(R(t_0) - I)x\| + \frac{\beta}{1-\alpha}\|R(t_0)x\| \\ &\leq \frac{2\alpha+\beta}{1-\alpha}\|x\|\end{aligned} \tag{3.20}$$

for all $x \in X$ and so $BR(t_0)$ is bounded. To show that $A + tB$ is m-dissipative, we will need to show that $I - (A + tB)$ is invertible and thus its range is all of X, then we have

$$\begin{aligned}I - (A + tB) &= I - (A + t_0B) + (t_0 - t)B \\ &= (I + (t_0 - t)BR(t_0))(I - (A + tB)).\end{aligned} \tag{3.21}$$

Therefore $I - (A + tB)$ is invertible if and only if $I + (t_0 - t)BR(t_0)$ is invertible. But $I + (t_0 - t)BR(t_0)$ is invertible for all t satisfying

$|t - t_0| < (1 - \alpha)(2\alpha + \beta)^{-1} \leq \|BR(t_0)\|^{-1}$ and we therefore choose $\delta = (1 - \alpha)(4\alpha + 2\beta)^{-1}$, and that complete the proof of (i).

To prove (ii), let us consider Theorem 2.2 such that $\overline{D(A)} = X$ and A is m-dissipative. Therefore $A + tB$ is dissipative for every $0 \leq t \leq 1$. This follows from the fact that if A is m-dissipative $Re < Ax, x^* > \leq 0$ for every $x^* \in F(x)$ by definition (2.7). Indeed, if B is dissipative with $D(B) \supset D(A)$, then for every $x \in D(A)$ and A,$B \in \omega$-$ORCP_n$, there is a $x^* \in F(x)$ such that $Re < Bx, x^* > \leq 0$ and for this same x^*, $Re < Ax + tBx, x^* > \leq 0$. From (i) of Theorem 3.4, it follows that $A+tB$ is m-dissipative for all $t \in [0, 1]$ and in particular $A+B$ is m-dissipative since $D(A+B) = D(A)$ is dense in X, then, $A+B$ is the infinitesimal generator of a C_0-semigroup by Theorem (2.2). Hence the proof is complete.

Theorem 3.5. *Assume A is the infinitesimal generator of a C_0-semigroup of contractions where $A \in \omega$-$ORCP_n$. Let B be dissipative, $D(A) \subset D(B)$ such that $B \in \omega$-$ORCP_n$ and*

$$\|Bx\| \leq \|Ax\| + \beta\|x\| \tag{3.22}$$

where $\beta \geq 0$ is a contsant. If B^* is the adjoint of B and densely defined, then the closure $\overline{A+B}$ of $A+B$ is the infinitesimal generator of a C_0-semigroup of contractions.

Proof. $A+B$ is dissipative and densely defined since A is m-dissipative and B is dissipative with $D(B) \supset D(A)$. Therefore $A+B$ is closable and its closure $\overline{A+B}$ is dissipative. To prove that $\overline{A+B}$ is the infinitesimal generator of a C_0-semigroup of contractions, it is therefore sufficient to show that $R(I-(\overline{A+B})) = X$. Since $\overline{A+B}$ is dissipative and closed, its follows that $R(I-(\overline{A+B}))$ is closed and therefore it suffices to show that $R(I-(\overline{A+B}))$ is dense in X. Let $y^* \in X^*$ be "orthogonal" to the range of $I-(A+B)$, that is $< y^*, z >= 0$ for every $z \in R(I-(\overline{A+B}))$. Let $y \in X$ be such that $\|y^*\| \leq < y^*, y >$. From (ii) of Theorem 3.4, it follows that $A+tB$ is m-dissipative for $0 \leq t < 1$ and therefore the equation

$$x_t - Ax_t - tBx_t = y \tag{3.23}$$

has a unique solution x_t for every $0 \leq t < 1$. Since $A+tB$ is dissipative, then $\|x_t\| \leq \|y\|$. From (3.22), it follows that

$$\begin{aligned} \|Bx_t\| \leq \|Ax_t\| + \beta\|x_t\| &\leq \|(A+tB)x_t\| + t\|Bx_t\| + \beta\|x_t\| \\ &\leq \|y - x_t\| + t\|Bx_t\| + \beta\|x_t\| \end{aligned} \tag{3.24}$$

and therefore

$$(1-t)\|Bx_t\| \leq \|y - x_t\| + \beta\|x_t\| \leq (2+\beta)\|y\|. \tag{3.25}$$

Let $z^* \in D(B*)$, then,

$$\begin{aligned} |< z^*, (1-t)Bx_t >| &= (1-t)|< B^*z^*, x_t >| \\ &\leq (1-t)\|B^*z^*\|\|y\| \to 0 \ as \ t \to 1. \end{aligned} \tag{3.26}$$

Since $D(B^*)$ is dense in X^* and since by (3.25), $(1-t)Bx_t$ is uniformly bounded and it follows from (3.26) that $(1-t)Bx_t$ tends weakly to zero as $t \to 1$. In particular by the choice of y^*, we have

$$\begin{aligned} \|y^*\| \ \leq \ < y^*, y > \ &= \ < y^*, x_t - Ax_t - tBx_t > \\ &= \ < y^*, (1-t)Bx_t > \ \ \to 0 \ as \ t \to 1 \end{aligned}$$

which implies $y^* = 0$ and the range of $I - \overline{(A+B)}$ is dense in X. To complete the proof, let X be reflexive Banach space and let T be a closable densely defined operator in X. Then it is well known that T^* is closed and $D(T^*)$ is dense in X^* and therefore since X is a reflexive Banach space and $A \in \omega\text{-}ORCP_n$ is the infinitesimal generator of a C_0-semigroup of contractions in X, then B is dissipative such that $D(B) \supset D(A)$ and

$\|Bx\| \leq \|Ax\| + \beta\|x\|$, for all $x \in D(A)$ and A,$B \in \omega\text{-}ORCP_n$ where $\beta \geq 0$. Then $A + B$ is the infinitesimal generator of a C_0-semigroup of contractions in X and this achieved the proof.

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