

SOME RESULTS ON COMPOSITION OF THREE ENTIRE FUNCTIONS

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Abstract: The purpose of this paper is to study some growth properties of composition of three entire functions of finite $[p, q]$ order.

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1. Introduction and Definitions

Let f and g be two transcendental entire functions. It is well known by a result of Clunie [5] that $\lim_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty$ and $\lim_{r \rightarrow \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty$. Many authors [3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 18] have investigated the composition of entire functions with finite order and obtained many interesting results. In 2009, Tu et.al [16] investigated the growth of two composite entire functions of finite iterated order and proved several results connecting $T(r, fog)$, $T(r, f)$, $T(r, g)$. After this in 2015, using the idea of generalised iteration of two entire functions Banerjee and Mandal [1, 2] extended the results of Jin tu et.al [16] for generalised iterated entire functions with finite iterated order. In this paper, we investigate the growth of composition of three entire functions to generalise the results of Xu et.al [17]. Here we use some basic definitions and standard notations of Nevanlinna theory [7]. First we introduce some basic definitions [11, 16, 17].

Definition 1.1. Let $f(z)$ be an entire function. Then the order and lower order

of $f(z)$ are respectively defined by

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

Definition 1.2. The iterated i order $\rho_i(f)$ of an entire function f is defined by

$$\rho_i(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[i+1]} M(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log^{[i]} T(r, f)}{\log r}, \quad (i \in \mathbb{N}).$$

Similarly, the iterated i lower order $\lambda_i(f)$ of an entire function f is defined by

$$\lambda_i(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[i+1]} M(r, f)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log^{[i]} T(r, f)}{\log r}, \quad (i \in \mathbb{N}),$$

where

$$\log^{[1]}(r) = \log r, \quad \log^{[i+1]}(r) = \log(\log^{[i]}(r)) \quad (i \in \mathbb{N}), \quad \text{for all sufficiently large } r.$$

Remark 1.1. We define $\exp^{[1]}(r) = e^r$, $\exp^{[i+1]}(r) = \exp(\exp^{[i]}(r))$ ($i \in \mathbb{N}$), $\exp^{[0]}r = r = \log^{[0]} r$ and $\exp^{[-1]}r = \log r$, for all $r \in (0, \infty)$.

Remark 1.2. Clearly $\rho_1(f) = \rho(f)$ and $\lambda_1(f) = \lambda(f)$.

Definition 1.3. If $f(z)$ is a transcendental entire function, the $[p, q]$ -order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r} = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r}.$$

Similarly, the lower $[p, q]$ -order of $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r} = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r},$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Definition 1.4. A transcendental entire function $f(z)$ is said to have index-pair $[p, q]$, if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1, q-1]}(f)$ is not a nonzero finite number.

For an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the maximum term $\mu(r, f)$ of f on $|z| = r$ is defined by

$$\mu(r, f) = \max_{n \geq 0} \{|a_n| r^n\}.$$

Since for $0 < r < R$ we have

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r} \mu(R, f),$$

an equivalent form for $\rho_{[p,q]}(f)$ and $\lambda_{[p,q]}(f)$ are

$$\rho_{[p,q]}(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} r}$$

and

$$\lambda_{[p,q]}(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} r},$$

where p, q are positive integers satisfying $p \geq q \geq 1$.

Let f, g, h be three entire functions and having index-pair $[p_1, q_1], [p_2, q_2], [p_3, q_3]$ respectively. Set

$$A = \rho_{[p_1, q_1]}(f), A_l = \lambda_{[p_1, q_1]}(f), B = \rho_{[p_2, q_2]}(g), B_l = \lambda_{[p_2, q_2]}(g), C = \rho_{[p_3, q_3]}(h) \text{ and } C_l = \lambda_{[p_3, q_3]}(h).$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [12] *Let f and g be two entire functions. If $M(r, g) > \frac{2+\epsilon}{\epsilon} |g(0)|$ for any $\epsilon > 0$, then*

$$T(r, f \circ g) < (1 + \epsilon) T(M(r, g), f).$$

In particular if $g(0) = 0$, then

$$T(r, f \circ g) \leq T(M(r, g), f)$$

for all $r > 0$.

Lemma 2.2. [5] *Let f and g be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$*

$$M(M(r, g), f) \geq M(r, f \circ g) \geq M(c(\alpha)M(\alpha r, g), f).$$

Furthermore if $\alpha = \frac{1}{2}$, for sufficiently large r

$$M(r, f \circ g) \geq M\left(\frac{1}{8}M\left(\frac{1}{2}r, g\right), f\right).$$

Lemma 2.3. [15] Let f and g be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$. Then

$$\mu(r, f \circ g) \geq (1 - \delta)\mu(c(\alpha)\mu(\alpha\delta r, g), f).$$

And if g is any entire function with $\alpha = \delta = \frac{1}{2}$, for sufficiently large r

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right).$$

Lemma 2.4. [13] Let f and g be two entire functions. Then for all large values of r

$$T(r, f \circ g) \geq \frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right).$$

3. Main Theorems

In this section we present the main results of the paper.

Theorem 3.1. Let f, g, h be three entire functions having index pair $[p_1, q_1]$, $[p_2, q_2]$, $[p_3, q_3]$ respectively and $0 < A_l \leq A \leq \infty$, $B_l > 0$. We have

- (i) if $(p_2 + p_3 + 2 - q_1 - q_2) > 0$ then $\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) = C$;
- (ii) if $(p_2 + p_3 + 2 - q_1 - q_2) = 0$ then $A_l C \leq \rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \leq AC$;
- (iii) if $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ then $A_l \leq \rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) \leq A$.

Proof. We have

$$\begin{aligned} A &= \limsup_{r \rightarrow \infty} \frac{\log^{[p_1]} T(r, f)}{\log^{[q_1]} r}, B = \limsup_{r \rightarrow \infty} \frac{\log^{[p_2+1]} M(r, g)}{\log^{[q_2]} r} \quad \text{and} \\ C &= \limsup_{r \rightarrow \infty} \frac{\log^{[p_3+1]} M(r, h)}{\log^{[q_3]} r}. \end{aligned} \quad (3.1)$$

For arbitrary $\epsilon > 0$ and sufficiently large r , we have

$$\begin{aligned} T(r, f) &\leq \exp^{[p_1]} \left\{ (A + \epsilon) \log^{[q_1]}(r) \right\}, \quad M(r, g) \leq \exp^{[p_2+1]} \left\{ (B + \epsilon) \log^{[q_2]}(r) \right\} \\ \text{and } M(r, h) &\leq \exp^{[p_3+1]} \left\{ (C + \epsilon) \log^{[q_3]}(r) \right\}. \end{aligned} \quad (3.2)$$

From (3.2) and using Lemma 2.1, we have for sufficiently large values of r

$$\begin{aligned}
T(r, f \circ g \circ h) &\leq (1 + o(1))T(M(r, h), f \circ g) \\
&\leq (1 + o(1))T(M(M(r, h), g), f) \\
&\leq (1 + o(1))\exp^{[p_1]} \left\{ (A + \epsilon) \log^{[q_1]} M(M(r, h), g) \right\} \\
&\leq (1 + o(1))\exp^{[p_1]} \left[(A + \epsilon) \log^{[q_1]} \left\{ \exp^{[p_2+1]} ((B + \epsilon) \log^{[q_2]} M(r, h)) \right\} \right] \\
&\leq (1 + o(1)) \\
&\exp^{[p_1]} \left[(A + \epsilon) \log^{[q_1]} \left\{ \exp^{[p_2+1]} \left((B + \epsilon) \log^{[q_2]} \left(\exp^{[p_3+1]} \left((C + \epsilon) \log^{[q_3]}(r) \right) \right) \right) \right\} \right] \\
&= (1 + o(1))\exp^{[p_1]} \left[(A + \epsilon) \exp^{[p_2+1-q_1]} \left\{ (B + \epsilon) \exp^{[p_3+1-q_2]} ((C + \epsilon) \log^{[q_3]}(r)) \right\} \right] \\
&\leq (1 + o(1))\exp^{[p_1]} \left[(A + \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C + 2\epsilon) \log^{[q_3]}(r) \right\} \right]. \tag{3.3}
\end{aligned}$$

Now

(i) if $(p_2 + p_3 + 2 - q_1 - q_2) > 0$ then we have from (3.3)

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+2-q_1-q_2]} T(r, f \circ g \circ h)}{\log^{[q_3]} r} \leq C,$$

since we always have $(p_1 + p_2 + p_3 + 2 - q_1 - q_2) \geq q_3$.

i.e.,

$$\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \leq C. \tag{3.4}$$

(ii) If $(p_2 + p_3 + 2 - q_1 - q_2) = 0$ then since $p_1 \geq q_3$ we have from (3.3)

$$\rho_{[p_1, q_3]}(f \circ g \circ h) \leq AC.$$

(iii) If $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ then $(q_1 + q_2 + q_3 - p_2 - p_3 - 2) \geq 1$ and we have from (3.3)

$$\rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) \leq A,$$

as $p_1 \geq (q_1 + q_2 + q_3 - p_2 - p_3 - 2)$.

Since $C > 0$ then there exists a sequence $\{r_n\} \rightarrow \infty$ such that for any ϵ ($0 < \epsilon < C$) and sufficiently large r_n , we have

$$M(r_n, h) \geq \exp^{[p_3+1]} \left\{ (C - \epsilon) \log^{[q_3]}(r_n) \right\}. \tag{3.5}$$

Now by using Lemma 2.2 and Lemma 2.4 for sufficiently large r_n , we have from

(3.5)

$$\begin{aligned}
T(r_n, f \circ g \circ h) &\geq \frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{r_n}{4}, g \circ h\right), f\right) \\
&\geq \frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{1}{8}M\left(\frac{r_n}{8}, h\right), g\right), f\right) \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ (A_l - \epsilon) \log^{[q_1]} \left(\frac{1}{9}M\left(\frac{1}{8}M\left(\frac{r_n}{8}, h\right), g\right) \right) \right\} \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ (A_l - 2\epsilon) \log^{[q_1]} \exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} \left(\frac{1}{8}M\left(\frac{r_n}{8}, h\right) \right) \right\} \right\} \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ (A_l - 2\epsilon) \log^{[q_1]} \exp^{[p_2+1]} \left\{ (B_l - 2\epsilon) \log^{[q_2]} \right. \right. \\
&\quad \left. \exp^{[p_3+1]} \left\{ (C - \epsilon) (\log^{[q_3]}(r_n) + O(1)) \right\} \right\} \left. \right\} \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ (A_l - 2\epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C - 2\epsilon) (\log^{[q_3]}(r_n) + O(1)) \right\} \right\}.
\end{aligned} \tag{3.6}$$

Now (i) if $(p_2 + p_3 + 2 - q_1 - q_2) > 0$ then from (3.7) we have

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+2-q_1-q_2]} T(r, f \circ g \circ h)}{\log^{[q_3]} r} \geq C$$

i.e.,

$$\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \geq C. \tag{3.8}$$

From (3.4) and (3.8) we have

$$\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) = C.$$

(ii) If $(p_2 + p_3 + 2 - q_1 - q_2) = 0$ then from (3.7) we have

$$\rho_{[p_1, q_3]}(f \circ g \circ h) \geq A_l C.$$

(iii) If $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ then from (3.7) and since $(q_1 + q_2 + q_3 - p_2 - p_3 - 2) \geq 1$, we have

$$\rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) \geq A_l.$$

This completes the proof of Theorem 3.1.

Theorem 3.2. Let f, g, h be three entire functions having index pair $[p_1, q_1]$, $[p_2, q_2]$, $[p_3, q_3]$ respectively and $0 < A < \infty$, $0 < B_l < \infty$, $0 < C_l \leq C < \infty$. We

have

- (i) if $(p_2 + p_3 + 2 - q_1 - q_2) > 0$ then $C_l \leq \rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \leq C$;
- (ii) if $(p_2 + p_3 + 2 - q_1 - q_2) = 0$ then $AC_l \leq \rho_{[p_1, q_3]}(f \circ g \circ h) \leq AC$;
- (iii) if $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ then

$$\rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) = A.$$

Proof. Since $A > 0$ then there exists a sequence $\{R_n\} \rightarrow \infty$ such that for any ϵ ($0 < \epsilon < A$) and sufficiently large $\{R_n\}$, we have

$$M(R_n, f) \geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \log^{[q_1]}(R_n) \right\}.$$

Since $M(r, h)$ is an increasing continuous function, then there exists a sequence $\{r_n\} \rightarrow \infty$ satisfying

$$R_n = \frac{1}{8} M\left(\frac{r_n}{4}, h\right) \geq \frac{1}{3} \exp^{[p_3+1]} \left\{ (C_l - \epsilon) \log^{[q_3]} \left(\frac{r_n}{4}\right) \right\}.$$

From Lemma 2.2, for sufficiently large $\{r_n\}$

$$\begin{aligned} M(r_n, f \circ g \circ h) &\geq M\left(\frac{1}{8} M\left(\frac{r_n}{2}, g \circ h\right), f\right) \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \log^{[q_1]} M\left(\frac{r_n}{2}, g \circ h\right) \right\} + O(1) \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \log^{[q_1]} M\left(\frac{1}{8} M\left(\frac{r_n}{4}, h\right), g\right) \right\} + O(1) \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} R_n + O(1) \right\} \right) \right\} \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} \exp^{[p_3+1]} \right. \right. \right. \\ &\quad \left. \left. \left\{ (C_l - \epsilon) \log^{[q_3]}(r_n) + O(1) \right\} \right\} \right\} \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \exp^{[p_2+1-q_1]} \left\{ (B_l - \epsilon) \exp^{[p_3+1-q_2]} \right. \right. \\ &\quad \left. \left. \left\{ (C_l - \epsilon) \log^{[q_3]}(r_n) + O(1) \right\} \right\} \right\} \\ &\geq \exp^{[p_1+1]} \left\{ (A - \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C_l - 2\epsilon) \log^{[q_3]}(r_n) + O(1) \right\} \right\}. \end{aligned} \tag{3.9}$$

Now,

- (i) if $(p_2 + p_3 + 2 - q_1 - q_2) > 0$, then from (3.9) we have

$$\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \geq C_l. \tag{3.10}$$

Again from Lemma 2.2, we have

$$\begin{aligned}
M(r, f \circ g \circ h) &\leq M(M(r, g \circ h), f) \\
&\leq M(M(M(r, h), g), f) \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} M(M(r, h), g) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B + \epsilon) \log^{[q_2]} M(r, h) \right\} \right) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} \left\{ \exp^{[p_2+1]} \left\{ (B + \epsilon) \log^{[q_2]} \left(\exp^{[p_3+1]} \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left\{ (C + \epsilon) \log^{[q_3]}(r) \right\} \right\} \right\} \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \left(\exp^{[p_2+1-q_1]} \left\{ (B + \epsilon) \exp^{[p_3+1-q_2]} \right. \right. \right. \\
&\quad \left. \left. \left. \left\{ (C + \epsilon) \log^{[q_3]}(r) \right\} \right\} \right) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C + 2\epsilon) \log^{[q_3]}(r) \right\} \right\}.
\end{aligned} \tag{3.11}$$

So for $(p_2 + p_3 + 2 - q_1 - q_2) > 0$, we have from (3.11)

$$\rho_{[p_1+p_2+p_3+2-q_1-q_2, q_3]}(f \circ g \circ h) \leq C. \tag{3.12}$$

So (i) follows from the (3.10) and (3.12).

(ii) If $(p_2 + p_3 + 2 - q_1 - q_2) = 0$, then from (3.9), we have

$$\rho_{[p_1, q_3]}(f \circ g \circ h) \geq AC_l;$$

and from (3.11) we get

$$\rho_{[p_1, q_3]}(f \circ g \circ h) \leq AC.$$

So, we have

$$AC_l \leq \rho_{[p_1, q_3]}(f \circ g \circ h) \leq AC.$$

This completes the proof of (ii).

(iii) Now if $(p_2 + p_3 + 2 - q_1 - q_2) < 0$, then from (3.9) we have

$$\rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) \geq A.$$

and from (3.11) we have

$$\rho_{[p_1, q_1+q_2+q_3-p_2-p_3-2]}(f \circ g \circ h) \leq A.$$

So,

$$\rho_{[p_1, q_1 + q_2 + q_3 - p_2 - p_3 - 2]}(f \circ g \circ h) = A.$$

This completes the proof of (iii).

Thus the proof of the Theorem 3.2 is completed.

Theorem 3.3 *Let f, g, h be three entire functions having index pair $[p_1, q_1], [p_2, q_2], [p_3, q_3]$ respectively.*

(1) *If $(p_2 + p_3 + 2 - q_1 - q_2) > 0$*

and (i) $q_1 > q_3$ and $C_l > 0$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1 + p_2 + p_3 + 3 - q_1 - q_2]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} = \infty;$$

(ii) $q_1 = q_3$ and $0 < A, A_l, C_l, C < \infty$, then we have

$$\frac{C_l}{A} \leq \lim_{r \rightarrow \infty} \frac{\log^{[p_1 + p_2 + p_3 + 3 - q_1 - q_2]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} \leq \frac{C}{A_l};$$

(iii) $q_1 < q_3$ and $A_l > 0$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1 + p_2 + p_3 + 3 - q_1 - q_2]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} = 0.$$

(2) *If $(p_2 + p_3 + 2 - q_1 - q_2) = 0$*

and (i) $q_1 > q_3$ and $A_l, C_l > 0$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1 + 1]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} = \infty;$$

(ii) $q_1 = q_3$ and $0 < A, A_l, C_l, C < \infty$, then we have

$$\frac{C_l A_l}{A} \leq \lim_{r \rightarrow \infty} \frac{\log^{[p_1 + 1]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} \leq \frac{AC}{A_l};$$

(iii) $q_1 < q_3$ and $0 < A, A_l, C_l, C < \infty$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1 + 1]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} = 0.$$

(3) *If $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ and $0 < A, A_l < \infty$, then we have*

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1 + 1]} M(r, f \circ g \circ h)}{\log^{[p_1 + 1]} M(r, f)} = \infty.$$

Proof. From Lemma 2.2 and for $\epsilon > 0$ there exists a positive number r_0 such that for all $r > r_0$ we have

$$\begin{aligned}
M(r, f \circ g \circ h) &\geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g \circ h\right), f\right) \\
&\geq M\left(\frac{1}{8}M\left(\frac{1}{8}M\left(\frac{r}{4}, h\right), g\right), f\right) \\
&\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} \left(\frac{1}{8}M\left(\frac{1}{8}M\left(\frac{r}{4}, h\right), g\right) \right) \right\} \\
&\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} \left(\frac{1}{8}M\left(\frac{r}{4}, h\right) \right) \right\} \right) \right\} + O(1) \\
&\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} \left(\exp^{[p_3+1]} \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left\{ (C_l - \epsilon) \log^{[q_3]}(r) \right\} \right\} \right) \right\} + O(1) \\
&\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C_l - 2\epsilon) \log^{[q_3]}(r) \right\} \right\} + O(1) \tag{3.13}
\end{aligned}$$

and

$$\begin{aligned}
M(r, f \circ g \circ h) &\leq M(M(r, g \circ h), f) \\
&\leq M(M(M(r, h), g), f) \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} M(M(r, h), g) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} \left(\exp^{[p_2+1]} \left\{ (B + \epsilon) \log^{[q_2]} M(r, h) \right\} \right) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} \left\{ \exp^{[p_2+1]} \left\{ (B + \epsilon) \log^{[q_2]} \left(\exp^{[p_3+1]} \left\{ (C + \epsilon) \log^{[q_3]}(r) \right\} \right) \right\} \right\} \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \left(\exp^{[p_2+1-q_1]} \left\{ (B + \epsilon) \exp^{[p_3+1-q_2]} \left\{ (C + \epsilon) \log^{[q_3]}(r) \right\} \right\} \right) \right\} \\
&\leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C + 2\epsilon) \log^{[q_3]}(r) \right\} \right\}. \tag{3.14}
\end{aligned}$$

Again

$$\exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} r \right\} \leq M(r, f) \leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} r \right\} \tag{3.15}$$

Now we consider the following three cases.

Case I. $(p_2 + p_3 + 2 - q_1 - q_2) > 0$. Then from (3.13) and (3.15), we have

$$\begin{aligned}
\frac{(C + 2\epsilon) \log^{[q_3]}(r)}{(A_l - \epsilon) \log^{[q_1]} r} &\geq \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} \\
&\geq \frac{(C_l - 2\epsilon) \log^{[q_3]}(r) + O(1)}{(A + \epsilon) \log^{[q_1]} r}. \tag{3.16}
\end{aligned}$$

(i) When $q_1 > q_3$. Since $C_l > 0$, from (3.16) we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} = \infty.$$

(ii) When $q_1 = q_3$. Since $0 < A, A_l, C_l, C < \infty$, from (3.16) we have

$$\frac{C_l}{A} \leq \lim_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} \leq \frac{C}{A_l}.$$

(iii) When $q_1 < q_3$. Since $A_l > 0$, from (3.16) we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} = 0.$$

Case II. $(p_2 + p_3 + 2 - q_1 - q_2) = 0$. Then from (3.13) and (3.15), we have

$$\begin{aligned} \frac{(A + \epsilon)(C + 2\epsilon) \log^{[q_3]}(r)}{(A_l - \epsilon) \log^{[q_1]} r} &\geq \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} \\ &\geq \frac{(A_l - \epsilon)(C_l - 2\epsilon) \log^{[q_3]}(r) + O(1)}{(A + \epsilon) \log^{[q_1]} r}. \end{aligned} \quad (3.17)$$

(i) When $q_1 > q_3$. Since $A_l, C_l > 0$, from (3.17) we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} = \infty.$$

(ii) When $q_1 = q_3$. Since $0 < A, A_l, C_l, C < \infty$, from (3.17) we have

$$\frac{C_l A_l}{A} \leq \lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} \leq \frac{AC}{A_l}.$$

(iii) When $q_1 < q_3$. Since $0 < A, A_l, C_l, C < \infty$, from (3.17) we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} = 0.$$

Case III. $(p_2 + p_3 + 2 - q_1 - q_2) < 0$. Since $p_3 \geq q_3 \geq 1$, we have from (3.13) and (3.15) and $0 < A, A_l < \infty$, we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} M(r, f \circ g \circ h)}{\log^{[p_1+1]} M(r, f)} \geq \frac{(A_l - \epsilon) \log^{[q_1+q_2+q_3-p_2-p_3-2]}(r)}{(A + \epsilon) \log^{[q_1]} r} \rightarrow \infty. \quad (3.18)$$

This completes the proof of the Theorem 3.3.

Remark 3.1. In Theorem 3.3 the conditions $q_1 > q_3$ and $C_l > 0$ are necessary. For this we consider the following example.

Example 3.1. We consider the functions $f(z) = \exp^{[3]}(z)$, $g(z) = \exp^{[2]}(z)$ and $h(z) = \exp(z)$.

Then $(f \circ g \circ h)(z) = \exp^{[6]}(z)$ and so $M(r, f \circ g \circ h) = \exp^{[6]}(r)$.

Choose $p_1 = q_1 = 2$, $p_2 = 2$, $q_2 = 1$, $p_3 = 2$, $q_3 = 1$, so that $q_1 = 2 > 1 = q_3$ and $p_2 + p_3 + 2 - q_1 - q_2 = 3 > 0$.

Now

$$C_l = \liminf_{r \rightarrow \infty} \frac{\log^{[3]} M(r, h)}{\log r} = 0$$

and

$$\liminf_{r \rightarrow \infty} \frac{\log^{[5]} M(r, f \circ g \circ h)}{\log^{[3]} M(r, f)} = 1.$$

Remark 3.2. The conditions $q_1 = q_3$ and $0 < A, A_l, C, C_l > 0$ are necessary. For this we consider the following example.

Example 3.2. We take the following functions $f(z) = \exp(z)$, $g(z) = \exp^{[2]}(z)$ and $h(z) = \exp(z)$.

Then $(f \circ g \circ h)(z) = \exp^{[4]}(z)$ and so $M(r, f \circ g \circ h) = \exp^{[4]}(r)$.

Choose $p_1 = q_1 = 1$; $p_2 = 1$, $q_2 = 1$; $p_3 = 2$, $q_3 = 1$ so that $q_1 = 1 = q_3$ and $p_2 + p_3 + 2 - q_1 - q_2 = 3 > 0$.

Now $C_l = C = 0$ and

$$\liminf_{r \rightarrow \infty} \frac{\log^{[5]} M(r, f \circ g \circ h)}{\log^{[2]} M(r, f)} = 1.$$

Remark 3.3. The conditions $q_1 < q_3$ and $A_l > 0$ are necessary. For this we consider the following example.

Example 3.3. We take the following functions $f(z) = \exp(z)$, $g(z) = \exp^{[2]}(z)$ and $h(z) = \exp^{[4]}(z)$.

Then $(f \circ g \circ h)(z) = \exp^{[7]}(z)$ and so $M(r, f \circ g \circ h) = \exp^{[7]}(r)$.

Choose $p_1 = 2$, $q_1 = 1$; $p_2 = 2$, $q_2 = 2$; $p_3 = 2$, $q_3 = 2$ so that $q_1 = 1 = q_3$ and $p_2 + p_3 + 2 - q_1 - q_2 = 3 > 0$.

Now $A_l = 0$ and

$$\liminf_{r \rightarrow \infty} \frac{\log^{[6]} M(r, f \circ g \circ h)}{\log^{[3]} M(r, f)} = \infty.$$

In terms of $\mu(r, f)$ instead of $M(r, f)$ Theorem 3.3 can be stated as follows:

Theorem 3.4. Let f, g, h be three entire functions having index pair $[p_1, q_1]$,

$[p_2, q_2], [p_3, q_3]$ respectively. Then

(1) If $(p_2 + p_3 + 2 - q_1 - q_2) > 0$

and (i) $q_1 > q_3$ and $C_l > 0$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} \mu(r, f \circ g \circ h)}{\log^{[p_1+1]} \mu(r, f)} = \infty;$$

(ii) $q_1 = q_3$ and $0 < A, C_l < \infty$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+p_2+p_3+3-q_1-q_2]} \mu(r, f \circ g \circ h)}{\log^{[p_1+1]} \mu(r, f)} \geq \frac{C_l}{A}.$$

(2) If $(p_2 + p_3 + 2 - q_1 - q_2) = 0$

and (i) $q_1 > q_3$ and $A_l, C_l > 0$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} \mu(r, f \circ g \circ h)}{\log^{[p_1+1]} \mu(r, f)} = \infty;$$

(ii) $q_1 = q_3$ and $0 < A, A_l, C_l, C < \infty$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} \mu(r, f \circ g \circ h)}{\log^{[p_1+1]} \mu(r, f)} \leq \frac{A_l C_l}{A}.$$

(3) If $(p_2 + p_3 + 2 - q_1 - q_2) < 0$ and $0 < A_l, A < \infty$, then we have

$$\lim_{r \rightarrow \infty} \frac{\log^{[p_1+1]} \mu(r, f \circ g \circ h)}{\log^{[p_1+1]} \mu(r, f)} = \infty.$$

Proof. From Lemma 2.3 for sufficiently large r we have

$$\mu(r, f \circ g \circ h) \geq \mu\left(\frac{1}{8}\mu\left(\frac{r}{2}, g \circ h\right), f\right).$$

Then from the assumptions of Theorem 3.3, for any $\epsilon > 0$ there exists a positive number r_0 and for all $r > r_0$ we have

$$\begin{aligned} \mu(r, f \circ g \circ h) &\geq \left(\frac{1}{8}\mu\left(\frac{r}{2}, g \circ h\right), f\right) \\ &\geq \mu\left(\frac{1}{8}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, h\right), g\right), f\right) \\ &\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} \left(\frac{1}{8}\mu\left(\frac{r}{8}, h\right), g\right) \right\} \\ &\geq \exp^{[p_1+1]} \left\{ (A_l - \epsilon) \exp^{[p_2+p_3+2-q_1-q_2]} \left\{ (C_l - 2\epsilon) \log^{[q_3]}(r) + O(1) \right\} \right\}. \end{aligned}$$

and

$$\exp^{[p_1+1]} \left\{ (A_l - \epsilon) \log^{[q_1]} r \right\} \leq \mu(r, f) \leq \exp^{[p_1+1]} \left\{ (A + \epsilon) \log^{[q_1]} r \right\}.$$

Using the same arguments as in Theorem (3.3), we can get the conclusions of the Theorem 3.4.

Theorem 3.5. *Let f, g, h be three entire functions and g, h have index pair $[p_2, q_2], [p_3, q_3]$ respectively satisfying $B_l, C_l > 0$. If $(f \circ g \circ h)$ have index-pair $[p_1, q_1]$ and $0 < \lambda_{[p_1, q_1]}(f \circ g \circ h) = \chi < \infty$, then $\lambda_{[p_1, q_1]}(f \circ g) = 0$ and $\lambda_{[p_1, q_1]}(f) = 0$.*

Proof. By definition and Lemma 2.4 there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n , we have

$$\frac{1}{3} \log M\left(\frac{1}{9} M\left(\frac{r_n}{4}, h\right), fog\right) \leq T(r_n, f \circ g \circ h) \leq \exp^{[p_1]} \left\{ (\chi + \epsilon) \log^{[q_1]} r_n \right\}. \quad (3.19)$$

Since $\lambda_{[p_3, q_3]}(h) = C_l > 0$ so for given ϵ ($0 < \epsilon < C_l$) and sufficiently large r_n , we have

$$\begin{aligned} \frac{1}{9} M\left(\frac{r_n}{4}, h\right) &\geq \exp^{[p_3+1]} \left\{ \left(C_l - \frac{\epsilon}{2}\right) \log^{[q_3]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p_3+1]} \left\{ (C_l - \epsilon) \log^{[q_3]} r_n \right\}. \end{aligned}$$

Set $R_n = \frac{1}{9} M\left(\frac{r_n}{4}, h\right)$. Then

$$r_n \leq \exp^{[q_3]} \left\{ \frac{1}{(C_l - \epsilon)} \log^{[p_3+1]} R_n \right\}. \quad (3.20)$$

Then from (3.19) and (3.20), we get

$$\log M(R_n, fog) \leq 3 \exp^{[p_1]} \left\{ (\chi + \epsilon) \log^{[q_1]} \left(\exp^{[q_3]} \left\{ \frac{1}{(C_l - \epsilon)} \log^{[p_3+1]} R_n \right\} \right) \right\}.$$

Since $(p_3 + 1 - q_3) > 0$, we have

$$\frac{\log^{[p_1+1]} M(R_n, fog)}{\log^{[q_1]} R_n} \leq \frac{(\chi + \epsilon) \exp^{[q_3 - q_1]} \left\{ \frac{1}{(C_l - \epsilon)} \log^{[p_3+1]} R_n \right\}}{\log^{[q_1]} R_n} \rightarrow 0.$$

Thus

$$\lambda_{[p_1, q_1]}(f \circ g) = 0.$$

Again for sufficiently large r_n , we get

$$\frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{r_n}{4}, g\right), f\right) \leq T(r_n, f \circ g) \leq \exp^{[p_1]} \left\{ \epsilon \log^{[q_1]} r_n \right\}. \quad (3.21)$$

Since $\lambda_{[p_2, q_2]}(g) = B_l > 0$ so for given ϵ ($0 < \epsilon < B_l$) and sufficiently large r_n , we have

$$\begin{aligned} \frac{1}{9}M\left(\frac{r_n}{4}, g\right) &\geq \exp^{[p_2+1]} \left\{ \left(B_l - \frac{\epsilon}{2}\right) \log^{[q_2]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p_2+1]} \left\{ (B_l - \epsilon) \log^{[q_2]} r_n \right\}. \end{aligned}$$

Set $R'_n = \frac{1}{9}M\left(\frac{r_n}{4}, g\right)$. Then

$$r_n \leq \exp^{[q_2]} \left\{ \frac{1}{(B_l - \epsilon)} \log^{[p_2+1]} R'_n \right\}. \quad (3.22)$$

Then from (3.21) and (3.22), we get

$$\log M(R'_n, f) \leq 3 \exp^{[p_1]} \left\{ \epsilon \log^{[q_1]} \left(\exp^{[q_2]} \left\{ \frac{1}{(B_l - \epsilon)} \log^{[p_2+1]} R'_n \right\} \right) \right\}.$$

Since $(p_2 + 1 - q_2) > 0$ we have

$$\frac{\log^{[p_1+1]} M(R'_n, f)}{\log^{[q_1]} R'_n} \leq \frac{(\epsilon \cdot \exp^{[q_2-q_1]} \left\{ \frac{1}{(B_l - \epsilon)} \log^{[p_2+1]} R'_n \right\})}{\log^{[q_1]} R'_n} \rightarrow 0.$$

Thus,

$$\lambda_{[p_1, q_1]}(f) = 0.$$

This completes the proof.

Theorem 3.6. *Let f, g, h be three entire functions and p, q be two positive integers satisfying $p \geq q \geq 1$. If $\lambda_{[p, q]}(f \circ g \circ h) = \tau < \lambda_{[p, q]}(h) = \sigma < \infty$ then $\lambda_{[p, q]}(f \circ g) = 0$ and $\lambda_{[p, q]}(f) = 0$.*

Proof. By definition and Lemma 2.4 there exists a sequence $\{r_n\}$ tending to infinity such that for sufficiently large r_n , we have

$$\frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{r_n}{4}, h\right), f \circ g\right) \leq T(r_n, f \circ g \circ h) \leq \exp^{[p]} \left\{ (\tau + \epsilon) \log^{[q]} r_n \right\}. \quad (3.23)$$

Now for given ϵ ($0 < \epsilon < \sigma - \tau$) and sufficiently large r_n , we have

$$\begin{aligned} \frac{1}{9}M\left(\frac{r_n}{4}, h\right) &\geq \exp^{[p+1]} \left\{ \left(\sigma - \frac{\epsilon}{2}\right) \log^{[q]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p+1]} \left\{ (\sigma - \epsilon) \log^{[q]} r_n \right\}. \end{aligned}$$

Set $R_n = \frac{1}{9}M\left(\frac{r_n}{4}, h\right)$. Then

$$\log^{[q]} r_n \leq \frac{1}{(\sigma - \epsilon)} \log^{[p+1]} R_n. \quad (3.24)$$

Then from (3.23) and (3.24), we get

$$\log M(R_n, fog) \leq 3\exp^{[p]} \left\{ \frac{\tau + \epsilon}{\sigma - \epsilon} \log^{[p+1]} R_n \right\}.$$

Since $p \geq q \geq 1$, so we have

$$\frac{\log^{[p+1]} M(R_n, fog)}{\log^{[q]} R_n} \leq \frac{\frac{\tau + \epsilon}{\sigma - \epsilon} \log^{[p+1]} R_n}{\log^{[q]} R_n} \rightarrow 0.$$

Thus

$$\lambda_{[p,q]}(f \circ g) = 0.$$

Again for sufficiently large r_n , we get

$$\frac{1}{3} \log M\left(\frac{1}{9}M\left(\frac{r_n}{4}, g\right), f\right) \leq T(r_n, fog) \leq \exp^{[p]} \left\{ \epsilon \log^{[q]} r \right\}. \quad (3.25)$$

Since $\lambda_{[p,q]}(g) = B_l > 0$, then for given ϵ ($0 < \epsilon < B_l$) and sufficiently large r_n , we have

$$\begin{aligned} \frac{1}{9}M\left(\frac{r_n}{4}, g\right) &\geq \exp^{[p+1]} \left\{ \left(B_l - \frac{\epsilon}{2}\right) \log^{[q]} \frac{r_n}{4} \right\} \\ &\geq \exp^{[p+1]} \left\{ (B_l - \epsilon) \log^{[q]} r_n \right\}. \end{aligned}$$

Set $R'_n = \frac{1}{9}M\left(\frac{r_n}{4}, g\right)$. Then

$$r_n \leq \exp^{[q]} \left\{ \frac{1}{(B_l - \epsilon)} \log^{[p+1]} R'_n \right\}. \quad (3.26)$$

Then from (3.25) and (3.26), we get

$$\begin{aligned}\log M(R'_n, f) &\leq 3\exp^{[p]} \left\{ \epsilon \log^{[q]} \left(\exp^{[q]} \left\{ \frac{1}{(B_l - \epsilon)} \log^{[p+1]} R'_n \right\} \right) \right\} \\ &\leq 3\exp^{[p]} \left\{ \epsilon \frac{1}{(B_l - \epsilon)} \log^{[p+1]} R'_n \right\}.\end{aligned}$$

So

$$\frac{\log^{[2]} M(R_n, f)}{\log R_n} \leq \frac{\exp^{[p-1]} \left\{ \frac{\epsilon}{(B_l - \epsilon)} \log^{[p+1]} R'_n \right\}}{\log^{[q]} R'_n} \rightarrow 0.$$

Thus

$$\lambda_{[p,q]}(f) = 0.$$

This completes the proof.

4. Conclusion and Future Scope

The main aim of this paper is to extend the notion of growth of two composite entire functions of finite iterated order to growth of three composite entire functions. But still there are some problems to be extended for composition of n entire functions of finite iterated order. These type of studies can be regarded as open problem for further workers in this branch.

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