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WEIGHTED SHARING OF SETS ON THE CARDINALITY OF THE UNIQUE RANGE SETS FOR MEROMORPHIC AND ENTIRE FUNCTIONS

Harina P. Waghamore and Vijaylaxmi S.B.

Department of Mathematics,
Jnanabharathi Campus, Bangalore University,
Bengaluru, Karnataka - 560056, INDIA

E-mail : harinapw@gmail.com, harina@bub.ernet.in, vijaylaxmib@gmail.com

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Abstract: The present paper deals with the study on cardinality of unique range sets of meromorphic(entire) functions. By using the concept of weighted sharing of sets, we prove one result which greatly improves the result stated in [3].

Keywords and Phrases: Meromorphic (Entire) function, URSM(URSE), Weighted sharing of sets.

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1. Introduction and Main Results

In the literature, by a meromorphic function we mean that the function has singularities as its poles only, in the whole complex plane $\mathbb{C}$. For the standard notations and results in Nevanlinna theory, the reader can refer the book by W. K. Hayman (see [11]).

We now discuss the necessary definitions and notations used in the paper.

Definition 1.1. ([18]) *For a non-constant meromorphic function f and any set $S \subset \mathbb{C} \cup \{\infty\}$, we define*

$$E_f(S) = \bigcup_{a \in S} \{(z,p) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \{(z,1) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p \text{ or } 1\}.$$

If $E_f(S) = E_g(S)$ *(resp.* $\overline{E}_f(S) = \overline{E}_g(S)$*), then it is said that* f *and* g *share the set* S *counting multiplicities (CM) (resp. ignoring multiplicities (IM)). Also, if set* S *has only one element, then it coincides with the usual definition of value sharing.*

F. Gross and C. C. Yang ([10]) introduced the study on unique range set for entire function(URSE). Later the analogous definition for meromorphic function was also introduced.

Definition 1.2. ([18]) *Let* $S \subset \mathbb{C} \cup \{\infty\}$ *and* f *and* g *be two non-constant meromorphic (resp. entire) functions. If* $E_f(S) = E_g(S)$ *implies* $f \equiv g$*, then* S *is called a unique range set for meromorphic (resp. entire) functions or we write URSM(resp. URSE).*

Also, H. X. Yi ([20]) gave the analogous definition for reduced unique range sets.

Definition 1.3. ([20]) *A set* $S \subset \mathbb{C} \cup \{\infty\}$ *is said to be a unique range set for meromorphic (resp. entire) functions in ignoring multiplicity, we write URSM-IM (resp. URSE-IM) or a reduced unique range set for meromorphic (resp. entire) functions, we also write RURSM (resp. RURSE) if* $\overline{E}_f(S) = \overline{E}_g(S)$ *implies* $f \equiv g$ *for any pair of non-constant meromorphic (resp. entire) functions.*

We also need the following notations:

$$\lambda_M = inf\{\sharp(S) | S \text{ is an } URSM\} \text{ and}$$
$$\lambda_E = inf\{\sharp(S) | S \text{ is an } URSE\},$$

where $\sharp(S)$ *is the cardinality of the set* S*.*

Definition 1.4. ([15]) *Suppose* $a \in \mathbb{C} \cup \{\infty\}$ *and* $m \in \mathbb{N}$*. We denote by* $N(r, a; f \mid= 1)$*, the counting function of simple a-points of* f*, by* $N(r, a; f \mid\leq m)$ *(resp.* $N(r, a; f \mid\geq m)$*), we denote the counting function of those a-points of* f *whose multiplicities are not greater (resp. less) than* m *where each a-point is counted according to its multiplicity.*

Similarly, one can define $\overline{N}(r, a; f \mid\leq m)$ and $\overline{N}(r, a; f \mid\geq m)$ as the reduced counting function of $N(r, a; f \mid\leq m)$ and $N(r, a; f \mid\geq m)$ respectively. Analogously, $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined.

Definition 1.5. ([12]) *Suppose that* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(a, 0)$*. Further suppose that* z_0 *be an a-point of* f *with multiplicity* p*, an a-point of* g *with multiplicity* q*. We denote the following :*

(i) by $\overline{N}_L(r, a; f)$, the reduced counting function of those a-points of f and g where $p > q$,

(ii) by $N_E^{1)}(r,a;f)$, the counting function of those a-points of f and g where $p=q=1$,

(iii) by $\overline{N}_E^{(2}(r,a;f)$, the reduced counting function of those a-points of f and g where $p=q\geq 2$.

Similarly, we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$. *Also when* f *and* g *share* (a,m), $m\geq 1$, *then* $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 1.6. *We denote by* $\overline{N}(r,a;f\mid=k)$, *the reduced counting function of those* a*-points of* f *whose multiplicities is exactly* k, *where* $k\geq 2$ *is an integer.*

Definition 1.7. ([12]) *Let* f, g *share a value* a *IM. We denote by* $\overline{N}_*(r,a;f,g)$, *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g. *Clearly,* $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

In the recent research study, there are many authors (see [2], [7], [8], [9], [13], [14]) who have been working on reducing the cardinality of unique range sets.

L. W. Liao and C. C. Yang ([15]) introduced the following notation:

Definition 1.8. ([15]) *Let* f *be a non-constant meromorphic function and* $S\subset\mathbb{C}\cup\{\infty\}$. *We define*

$$E_{1)}(S,f)=\bigcup_{a\in S}E_{1)}(a,f),$$

where $E_{1)}(a,f)$ *is the set of all simple* a*-points of* f.

For positive integers $n\geq 3$ and $c\neq 0,1$, we shall denote by $P(z)$, the Frank-Reinders polynomial ([6]) as:

$$P(z)=\frac{(n-1)(n-2)}{2}z^n-n(n-2)z^{n-1}+\frac{n(n-1)}{2}z^{n-2}-c. \tag{1.1}$$

Clearly the restrictions on c implies that $P(z)$ has only simple zeros. Using the methods of Frank-Reinders ([6]), L. W. Liao and C. C. Yang ([15]) proved following Theorem:

Theorem A. ([15]) *Suppose that* $n(\geq 1)$ *be a positive integer. Further suppose that* $S=\{z:P(z)=0\}$ *where the polynomial* $P(z)$ *of degree* n *defined by (1.1). Let* f *and* g *be two non-constant meromorphic functions satisfying* $E_{1)}(S,f)=E_{1)}(S,g)$. *If* $n\geq 15$, *then* $f\equiv g$.

B. Chakraborthy and S. Chakraborthy ([3]) improved the above result by using the definition of weighted sharing of a set with weight 1 as follows:

Theorem B. ([3]) *Suppose that $n(\geq 1)$ be a positive integer. Further suppose that $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ of degree n defined by (1.1). Let f and g be two non-constant meromorphic functions satisfying $E_f(S, 1) = E_g(S, 1)$. If $n \geq 13$, then $f \equiv g$.*

We now give the definition of weighted sharing of a set as follows:

Definition 1.9. ([12]) *For a non-constant meromorphic function f and any set $S \subset \mathbb{C} \cup \{\infty\}$, $l \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, we define*

$$E_f(S, l) = \bigcup_{a \in S} \{(z, t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},$$

where $t = p$ if $p \leq l$ and $t = p + 1$ if $p > l$.

Two meromorphic functions f and g are said to share the set S with weight l, if $E_f(S, l) = E_g(S, l)$. Clearly, $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Our aim in the paper is to further reduce the cardinality of the unique range sets using Frank-Rienders techniques and taking into consideration of weighted sharing of sets, we obtain a result which improves Theorem B.

The following theorem is our main result:

Theorem 1. *Suppose that $n(\geq 1)$ be a positive integer. Further, suppose that $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ of degree n is defined by (1.1). Let f and g be two non-constant meromorphic functions satisfying $E_f(S, l) = E_g(S, l)$ and if one of the following conditions holds:*

$$(i)\ \text{when } l \geq 2, n \geq 9 \tag{1.2}$$

$$(ii)\ \text{when } l = 1, n \geq 10 \tag{1.3}$$

$$(iii)\ \text{when } l = 0, n \geq 15 \tag{1.4}$$

then $f \equiv g$.

Corollary 1. *Suppose that $n(\geq 1)$ be a positive integer. Further, suppose that $S = \{z : P(z) = 0\}$ where the polynomial $P(z)$ of degree n is defined by (1.1). Let f and g be two non-constant entire functions satisfying $E_f(S, l) = E_g(S, l)$ and if one of the following conditions holds:*

(i) when $l \geq 2$, $n \geq 5$

(ii) when $l = 1$, $n \geq 5$

$$(iii)\ \text{when } l = 0,\ n \geq 8$$

then $f \equiv g$.

We give an example which shows that the conditions obtained in the above result are necessary for $f \equiv g$, but not sufficient.

Example 1. Let $f(z) = e^z$, $g(z) = e^{-z}$ and let $S = \{z : P(z) = 0\}$, where $P(z)$ is as defined in (1.1). Since f and g share the set S with weight l and one of the conditions in (1.2), (1.3) and (1.4) is satisfied. But, $f \not\equiv g$.

We define for any two non-constant meromorphic functions f and g

$$Q(z) = \frac{P(z) + c}{c},\ F = Q(f),\ G = Q(g).$$

Now, we denote H by using F and G as follows

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

2. Some Lemmas

Lemma 2.1. ([17]) *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

Lemma 2.2. ([18]) *For a non-constant meromorphic function f,*

$$T(r, \frac{1}{f}) = T(r, f) + O(1),$$

where $O(1)$ is a bounded quantity depending on a.

Lemma 2.3. ([18]) *For a non-constant meromorphic function f and for a complex number $a \in \mathbb{C} \cup \{\infty\}$*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1),$$

where $O(1)$ is a bounded quantity depending on a.

Lemma 2.4. ([18]) *Suppose that f is a non-constant meromorphic function in the complex plane and $a_1, a_2, ..., a_q$ are $q(\geq 3)$ distinct values in $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} N(r,a_j;f) + S(r,f)$$

where $S(r,f)$ is a quantity such that $\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to +\infty$ out side of a set E in $(0,\infty)$ with finite linear measure.

A polynomial $P(z)$ over $\mathbb{C}$, is called a uniqueness polynomial for meromorphic (resp. entire) functions, if for any two non-constant meromorphic (resp. entire) functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$.

In 2000, H. Fujimoto ([7]) first discovered a special property of a polynomial, which was later termed as critical injection property. A polynomial $P(z)$ is said to satisfy critical injection property if $P(\alpha) \neq P(\beta)$ for any two distinct zeros α, β of the derivative $P^{'}(z)$.

Let $P(z)$ be a monic polynomial without multiple zero whose derivatives has mutually distinct k-zeros given by $d_1, d_2, ..., d_k$ with multiplicities $q_1, q_2, ..., q_k$ respectively. The following theorem of Fujimoto helps us to find many uniqueness polynomials taken here as a lemma.

Lemma 2.5. ([8]) *Suppose that $P(z)$ satisfy critical injection property. Then $P(z)$ will be a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

Lemma 2.6. ([16]) *Let F, G be two non-constant meromorphic functions sharing $(1, 2)$, $(\infty, 0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & + \overline{N}_*(r,\infty,F;G) + S(r,F) + S(r,G). \end{aligned}$$

Similar expression holds for G.

Lemma 2.7. ([16]) *Let F, G be two non-constant meromorphic functions sharing $(1,1)$, $(\infty,0)$ and $H \not\equiv 0$. Then*

$$T(r,F) \le N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r;0;F) + \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G).$$

Similar expression holds for G.

Lemma 2.8. ([16]) *Let F, G be two non-constant meromorphic functions sharing $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \le & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + \\ & 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G). \end{aligned}$$

Similar expression holds for G.

3. Proof of Theorem 1.

By the assumption, it is clear that F and G share $(1,l)$. Now we consider two cases as follows:

Case 1. Let $H \not\equiv 0$.

Subcase 1.1. By using Lemma 2.6 when $l \ge 2$, we have

$$\begin{aligned} T(r,F) \le & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \\ & \overline{N}_*(r,\infty,F;G) + S(r,F) + S(r,G). \end{aligned}$$

Similarly,

$$\begin{aligned} T(r,G) \le & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \\ & \overline{N}_*(r,\infty,G;F) + S(r,F) + S(r,G). \end{aligned}$$

Here,

$$\overline{N}_*(r,1,F;G) = \overline{N}_*(r,\infty,G;F).$$

By adding these, we get

$$\begin{aligned} T(r,F) + T(r,G) \le & \, 2N_2(r,0;F) + 2N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + \\ & 2\overline{N}_*(r,\infty,F;G) + S(r,F) + S(r,G) \\ n(T(r,f) + T(r,g)) \le & \, 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \\ & 2\overline{N}_*(r,\infty,f;g) + S(r,f) + S(r,g) \\ \le & \, 8\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Then $n \leq 8$, which contradicts (1.2).

Subcase 1.2. By using Lemma 2.7 when $l = 1$, we have

$$\begin{aligned} T(r,F) \leq & N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r;0;F) + \\ & \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G). \end{aligned}$$

Similarly,

$$\begin{aligned} T(r,G) \leq & N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + \frac{1}{2}\overline{N}(r;0;G) + \\ & \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G). \end{aligned}$$

By adding these, we get

$$\begin{aligned} T(r,F) + T(r,G) \leq & 2N_2(r,0;F) + 2N_2(r,0;G) + \frac{3}{2}\{\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\} + \\ & \{\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\} + \frac{1}{2}\{\overline{N}(r,0;F) + \overline{N}(r,0;G)\} + \\ & 2\overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G) \\ n(T(r,f) + T(r,g)) \leq & 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + \frac{5}{2}\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} + \\ & \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + 2\overline{N}_*(r,\infty;f;g) + S(r,f) + S(r,g) \\ \leq & 9\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Then, $n \leq 9$, which contradicts (1.3)

Subcase 1.3. By using Lemma 2.8 when $l = 0$, we have

$$\begin{aligned} T(r,F) \leq & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) + \\ & \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G). \end{aligned}$$

Similarly,

$$\begin{aligned} T(r,G) \leq & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;G) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,0;G) + \\ & \overline{N}(r,0;F) + \overline{N}_*(r,\infty;F;G) + S(r,F) + S(r,G). \end{aligned}$$

By adding these, we get

$$\begin{aligned}T(r,F)+T(r,G) \leq\; & 2N_2(r,0;F)+2N_2(r,0;G)+5\{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}+\\ & 3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+2\overline{N}_*(r,\infty;F;G)+\\ & S(r,F)+S(r,G)\\ n(T(r,f)+T(r,g)) \leq\; & 4\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+5\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\\ & 3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+2\overline{N}_*(r,\infty;f;g)+\\ & S(r,f)+S(r,g)\\ \leq\; & 14\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

Then, $n \leq 14$, which contradicts (1.4).

Case 2. Let $H \equiv 0$. On integrating twice, we get

$$F = \frac{AG+B}{CG+D},$$

where A, B, C, D are constants satisfying $AD-BC \neq 0$. By applying Lemma 2.1 to the above equation, we get

$$T(r,f) = T(r,g)+O(1).$$

Subcase 2.1. First let us assume that $AC \neq 0$. Then we can rewrite F as

$$F-\frac{A}{C} = \frac{BC-AD}{C(CG+D)}.$$

By second fundamental theorem, we get

$$\begin{aligned}nT(r,f)+O(1) &= T(r,F)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,\frac{A}{C};F)+S(r,F)\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)\\ &\leq 3T(r,f)+S(r,f).\end{aligned}$$

Thus, $n \leq 3$, which contradicts (1.2), (1.3), (1.4).

Subcase 2.2. Here, we consider $AC = 0$. Since $AD-BC \neq 0$, $A = C = 0$ will never occur and hence we need to discuss the two obvious subcases:

Subcase 2.2.1. Proceeding as in proof of Subcase 2.2.1 in Proof of Theorem 2.1

(See [3]), we arrive at contradiction to (1.2), (1.3), (1.4).

Subcase 2.2.2. Proceeding as in proof of Subcase 2.2.2 in Proof of Theorem 2.1 (See [3]), we arrive at contradiction to (1.2), (1.3), (1.4).

Now By using Lemma 2.5, $P(z)$ will be a uniqueness polynomial.
Hence, $f \equiv g$.
Hence the proof of theorem 1.

Proof of Corollary 1: By taking $\overline{N}(r, \infty; F) = S(r, F)$ in proof of theorem 1, we get proof of the corollary.

Open problem : Can the result in Theorem 1 be proved for any set, other than only the set by considering Frank-Rienders polynomial?

Future Scope of research: Relying on the Nevanlinna's methods, it will be interesting to further reduce the cardinality of the unique range sets for meromorphic and entire functions, by considering different sets. Also, the results obtained here can be extended to difference polynomials and difference operators.

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