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ON MULTISET RELATIONS AND FACTOR MULTIGROUPS

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Abstract: Crisp congruence relations on groups are very well known. This paper attempts to define factor multigroups by using the proposed multiset relations in this study and prove some basic properties.

Keywords and Phrases: Multiset, multiset relation, multiset congruence, factor multigroup.

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1. Introduction

The notion of multisets was formulated by N. G. Bruijn in a private communication to Knuth [19] as a mathematical structure that violates basic set conditions. It is a collection of objects in which repetition is significant. Since then, multisets have been applied to various fields of mathematics and computer science. Nazmul et al. [20] introduced the concept of multigroups with multiset settings different from the earlier definitions given by [3] and [21] and obtained equivalents of some basic results in group theory. This has been investigated further by [1],[2],[4],[5],[6],[7],[8],[9],[10],[11],[16],[17] and [18]. In [22], a new fashion of multigroups was developed with some properties considered. The concept of multiset relations on set was defined by Girish and John [12]. They have also generalized the concept by considering multiset relations on multisets and developed multiset topologies (see [13]). In [14], multiset relations are applied to consider rough multiset relations.

Since multigroups are a generalization of groups and there exist close relationships between the normal subgroups and the relation of congruency in a group, we have been motivated to define factor multigroups using the proposed multiset relations in this paper and derive some basic results.

2. Preliminaries

In this section, we review briefly some definitions and results that are required in this paper.

Definition 2.1. [15] *A multiset $\mathcal{A}$ is a countable set ξ together with a function $\mathcal{M}_{\mathcal{A}} : \xi \longrightarrow \mathbb{N} \cup \{0\}$ that defines the multiplicity of the elements of ξ in $\mathcal{A}$.*
The usual set operations can be carried over to multisets. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be multisets over ξ . Then

(i) $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if $\mathcal{M}_{\mathcal{A}_1}(a) \leq \mathcal{M}_{\mathcal{A}_2}(b)$.

(ii) Let $\mathcal{A} = \mathcal{A}_1 \bigcap \mathcal{A}_2$ such that $\forall\, a \in \xi,\ \mathcal{M}_{\mathcal{A}}(a) = \mathcal{M}_{\mathcal{A}_1}(a) \bigwedge \mathcal{M}_{\mathcal{A}_2}(a)$.

(ii) Let $\mathcal{A} = \mathcal{A}_1 \bigcup \mathcal{A}_2$ such that $\forall\, a \in \xi,\ \mathcal{M}_{\mathcal{A}}(a) = \mathcal{M}_{\mathcal{A}_1}(a) \bigvee \mathcal{M}_{\mathcal{A}_2}(a)$.

Definition 2.2. [12] *A submultiset R of $\mathcal{A} \times \mathcal{A}$ is said to be a multiset relation on $\mathcal{A}$ if every member of R has a multiplicity $\mathcal{M}_1(a, b) \cdot \mathcal{M}_2(a, b)$.*

Definition 2.3. [20] *Let G be any group. By a multigroup we mean a multiset $\mathcal{A}$ over G satisfying the following conditions:*

(i) $\mathcal{M}_{\mathcal{A}}(ab) \geq \mathcal{M}_{\mathcal{A}}(a) \bigwedge \mathcal{M}_{\mathcal{A}}(b) = min\{\mathcal{M}_{\mathcal{A}}(a), \mathcal{M}_{\mathcal{A}}(b)\},\ \forall\, a, b \in G,$

(ii) $\mathcal{M}_{\mathcal{A}}(a^{-1}) \geq \mathcal{M}_{\mathcal{A}}(a),\ \forall\, a \in G.$

It is immediate from the definition that $\mathcal{M}_{\mathcal{A}}(e) \geq \mathcal{M}_{\mathcal{A}}(a)$ and $\mathcal{M}_{\mathcal{A}}(a^{-1}) = \mathcal{M}_{\mathcal{A}}(a)$, where e is the identity element of G.

Definition 2.4. [20] *A multigroup H over G is called a normal multigroup if $\mathcal{M}_H(ab) = \mathcal{M}_H(ba),\ \forall\, a, b \in G$.*

Definition 2.5. [20] *If a multiset is a multigroup over G/H, then it is called factor multigroup. Analogously, if it is a normal multigroup over G/H, then it is called normal factor multigroup.*

3. Multiset relation and multiset congruence

In this section, we define the proposed multiset relations and outline some results.

Definition 3.1. *Let G be a group and H be a normal multigroup over G. A* *multiset relation* η *can be defined on* G *by*

$$\mathcal{M}_\eta(a,b) = \begin{cases} \geq \mathcal{M}_H(e), & if \quad a=b=e, \\ \mathcal{M}_H(e), & if \quad a=b, \\ \mathcal{M}_H(a) \bigwedge \mathcal{M}_H(b), & if \quad a \neq b, \ \forall \ a,b \in G. \end{cases}$$

Definition 3.2. *Let* G *be a group and* η_1 *and* η_2 *be multiset relations on* G. *Then the composition* $\eta_1 \circ \eta_2$ *is defined as follows:*

$$\mathcal{M}_{\eta_1 \circ \eta_2}(a,c) = \bigvee_{b \in G} \{\mathcal{M}_{\eta_1}(a,b) \bigwedge \mathcal{M}_{\eta_2}(b,c)\}$$

Definition 3.3. *Let* G *be a group. A binary function* $\mathcal{M}_\eta : G \times G \longrightarrow \mathbb{N} \cup \{0\}$ *is called a multiset congruence on* G *if the following conditions are satisfied for all* $a,b,c,d \in G$:

(C1) $\mathcal{M}_\eta(e,e) \geq \mathcal{M}_\eta(a,a)$

(C2) $\mathcal{M}_\eta(a,b) = \mathcal{M}_\eta(b,a)$

(C3) $\mathcal{M}_{\eta \circ \eta}(a,c) \leq \mathcal{M}_\eta(a,c)$

(C4) $\mathcal{M}_\eta(ac,bd) \geq \mathcal{M}_\eta(a,b) \bigwedge \mathcal{M}_\eta(c,d)$

Proposition 3.1. *Let* G *be a group and* H *be a normal multigroup over* G. *Then the multiset relation* η *defined on* G *is a multiset congruence.*

Proof. Let $a,b,c,d \in G$.

(C1): $\mathcal{M}_\eta(e,e) \geq \mathcal{M}_H(e) = \mathcal{M}_\eta(a,a)$.

(C2):

$$\begin{aligned} \mathcal{M}_\eta(a,b) &= \mathcal{M}_H(a) \bigwedge \mathcal{M}_H(b) \\ &= \mathcal{M}_H(b) \bigwedge \mathcal{M}_H(a) \\ &= \mathcal{M}_\eta(b,a) \end{aligned}$$

(C3):

$$\begin{aligned}
\mathcal{M}_{\eta\circ\eta}(a,c) &= \bigvee_{b\in G}\{\mathcal{M}_\eta(a,b)\bigwedge\mathcal{M}_\eta(b,c)\}\\
&= \bigvee_{b\in G}\{\Big(\mathcal{M}_H(a)\bigwedge\mathcal{M}_H(b)\Big)\bigwedge\Big(\mathcal{M}_H(b)\bigwedge\mathcal{M}_H(c)\Big)\}\\
&\leq \Bigg(\bigvee_{b\in G}\{\mathcal{M}_H(a)\bigwedge\mathcal{M}_H(b)\}\Bigg)\bigwedge\Bigg(\bigvee_{b\in G}\{\mathcal{M}_H(b)\bigwedge\mathcal{M}_H(c)\}\Bigg)\\
&\leq \bigvee_{b\in G}\{\mathcal{M}_H(a)\}\bigwedge\bigvee_{b\in G}\{\mathcal{M}_H(c)\}\\
&= \mathcal{M}_H(a)\wedge\mathcal{M}_H(c)\\
&= \mathcal{M}_\eta(a,c)
\end{aligned}$$

(C4):

$$\begin{aligned}
\mathcal{M}_\eta(ac,bd) &= \mathcal{M}_H(ac)\bigwedge\mathcal{M}_H(bd)\\
&\geq \Big(\mathcal{M}_H(a)\bigwedge\mathcal{M}_H(c)\Big)\bigwedge\Big(\mathcal{M}_H(b)\bigwedge\mathcal{M}_H(d)\Big)\\
&= \Big(\mathcal{M}_H(a)\bigwedge\mathcal{M}_H(b)\Big)\bigwedge\Big(\mathcal{M}_H(c)\bigwedge\mathcal{M}_H(d)\Big)\\
&= \mathcal{M}_\eta(a,b)\bigwedge\mathcal{M}_\eta(c,d)
\end{aligned}$$

Therefore η is a multiset congruence on G.

Proposition 3.2. *Let G be a group and H be a normal multigroup over G, then the function $\mathcal{M}_E : G/H \longrightarrow \mathbb{N}\cup\{0\}$ defined by $\mathcal{M}_E(aH) = \mathcal{M}_\eta(a,h)\ \forall\ h \in H$ such that $\mathcal{M}_H(h) > \mathcal{M}_H(a)\ \forall\ a \neq e \in G$ is a factor multigroup over G/H.*

Proof. For every $aH, bH \in G/H$, we have

$$\begin{aligned}
\mathcal{M}_E(aHbH) &= \mathcal{M}_\eta(ab,h)\\
&= \mathcal{M}_H(ab)\bigwedge\mathcal{M}_H(h)\\
&= \mathcal{M}_H(ab)\\
&\geq \mathcal{M}_H(a)\bigwedge\mathcal{M}_H(b)\\
&= \Big(\mathcal{M}_H(a)\bigwedge\mathcal{M}_H(h)\Big)\bigwedge\Big(\mathcal{M}_H(b)\bigwedge\mathcal{M}_H(h)\Big)\\
&= \mathcal{M}_\eta(a,h)\bigwedge\mathcal{M}_\eta(b,h)\\
&= \mathcal{M}_E(aH)\bigwedge\mathcal{M}_E(bH)
\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_E(a^{-1}H) &= \mathcal{M}_\eta(a^{-1}, h) \\ &= \mathcal{M}_H(a^{-1}) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_H(a^{-1}) \\ &\geq \mathcal{M}_H(a) \\ &= \mathcal{M}_H(a) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_\eta(a, h) = \mathcal{M}_E(aH)\end{aligned}$$

Thus E is a factor multigroup over G/H.

Proposition 3.3. *The function* $\mathcal{M}_E : G/H \longrightarrow \mathbb{N} \cup \{0\}$ *defined by* $\mathcal{M}_E(aH) = \mathcal{M}_\eta(a, h)$ $\forall$ $h \in H$ *with* $\mathcal{M}_H(h) > \mathcal{M}_H(a)$ $\forall$ $a \neq e \in G$ *is a normal factor multigroup over* G/H.
Proof. Since H is a multigroup over G, then

$$\begin{aligned}\mathcal{M}_E(aH, bH) &= \mathcal{M}_\eta(ab, h) \\ &= \mathcal{M}_H(ab) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_H(ab) = \mathcal{M}_H(ba) \\ &= \mathcal{M}_H(ba) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_\eta(ba, h) \\ &= \mathcal{M}_E(bH, aH)\end{aligned}$$

Hence E is a normal factor multigroup over G/H.

Proposition 3.4. *If* E *is a factor multigroup over* G/H *and* $\mathcal{M}_H(h) > \mathcal{M}_H(a)$ $\forall$ $a \neq e \in G$*, then* $\mathcal{M}_E(a^n H) \geq \mathcal{M}_E(aH)$.
Proof. Let $aH \in G/H$. Then

$$\begin{aligned}\mathcal{M}_E(a^n H) &= \mathcal{M}_\eta(a^n, h) \\ &= \mathcal{M}_H(a^{n-1}a) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_H(a^{n-1}a) \\ &\geq \mathcal{M}_H(a^{n-1}) \bigwedge \mathcal{M}_H(a) \\ &\geq \mathcal{M}_H(a) \bigwedge \mathcal{M}_H(a) \bigwedge ... \bigwedge \mathcal{M}_H(a) \qquad \textit{(by recursion)}\end{aligned}$$

$$\begin{aligned} &= \mathcal{M}_H(a) \\ &= \mathcal{M}_H(a) \bigwedge \mathcal{M}_H(h) \\ &= \mathcal{M}_\eta(a,h) = \mathcal{M}_E(aH) \end{aligned}$$

Hence the proof.

Proposition 3.5. *Let E be a factor multigroup over G/H and let $aH \in G/H$ with $\mathcal{M}_H(h) > \mathcal{M}_H(a)\ \forall\ a \neq e \in G$. Then $\mathcal{M}_E(aHbH) = \mathcal{M}_E(bH)\ \forall\ bH \in G/H$ if and only if $\mathcal{M}_E(aH) = \mathcal{M}_E(H)$.*

Proof. If $\mathcal{M}_E(aHbH) = \mathcal{M}_E(bH)\ \forall\ bH \in G/H$, then $bH = H$.
Conversely, assume $\mathcal{M}_E(aH) = \mathcal{M}_E(H)$. Since E is a multigroup over G/H, then

$$\begin{aligned} \mathcal{M}_E(aHbH) &\geq \mathcal{M}_E(aH) \bigwedge \mathcal{M}_E(bH) \\ &= \mathcal{M}_E(H) \bigwedge \mathcal{M}_E(bH) \\ &= \mathcal{M}_\eta(e,h) \bigwedge \mathcal{M}_\eta(b,h) \\ &= \mathcal{M}_H(h) \bigwedge \Big(\mathcal{M}_H(b) \bigwedge \mathcal{M}_H(h)\Big) \\ &= \mathcal{M}_H(h) \bigwedge \mathcal{M}_H(b) \\ &= \mathcal{M}_\eta(b,h) \\ &= \mathcal{M}_E(bH) \end{aligned}$$

and, on the other hand, $\mathcal{M}_E(bH) \geq \mathcal{M}_E(H) \bigwedge \mathcal{M}_E(bH) = \mathcal{M}_E(aHbH)$. This completes the proof.

Proposition 3.6. *Let E and F be two normal factor multigroups over G/H. Then $E \bigcap F$ is a normal factor multigroup over G/H.*

Proof. For every $aH, bH \in G/H$,

$$\begin{aligned} \mathcal{M}_{E\bigcap F}(aHbH) &= \mathcal{M}_E(aHbH) \bigwedge \mathcal{M}_F(aHbH) \\ &\geq \Big(\mathcal{M}_E(aH) \bigwedge \mathcal{M}_E(bH)\Big) \bigwedge \Big(\mathcal{M}_F(aH) \bigwedge \mathcal{M}_F(bH)\Big) \\ &= \Big(\mathcal{M}_E(aH) \bigwedge \mathcal{M}_F(aH)\Big) \bigwedge \Big(\mathcal{M}_E(bH) \bigwedge \mathcal{M}_F(bH)\Big) \\ &= \mathcal{M}_{E\bigcap F}(aH) \bigwedge \mathcal{M}_{E\bigcap F}(bH) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{E\bigcap F}(a^{-1}H) &= \mathcal{M}_E(a^{-1}H) \bigwedge \mathcal{M}_F(a^{-1}H) \\ &\geq \mathcal{M}_E(aH) \bigwedge \mathcal{M}_F(aH) = \mathcal{M}_{E\bigcap F}(aH) \end{aligned}$$

Hence $E\bigcap F$ is a multigroup over G/H.

$$\begin{aligned}\mathcal{M}_{E\bigcap F}(aHbH) &= \mathcal{M}_E(aHbH)\bigwedge \mathcal{M}_F(aHbH)\\ &= \mathcal{M}_E(bHaH)\bigwedge \mathcal{M}_F(bHaH) = \mathcal{M}_{E\bigcap F}(bHaH)\end{aligned}$$

Thus $E\bigcap F$ is a normal multigroup over G/H.

Proposition 3.7. *Let E be a normal factor multigroup over G/H such that $\mathcal{M}_H(h) > \mathcal{M}_H(a)\ \forall\ a \neq e \in G$. Then the function $\mathcal{M}_\delta : G/H \times G/H$ defined by $\mathcal{M}_\delta(aH, bH) = \mathcal{M}_E(aHb^{-1}H)$ is a multiset congruence on G/H.*

Proof. Let $aH, bH \in G/H$. Then

(C1): $\mathcal{M}_\delta(eH, eH) \geq \mathcal{M}_E(H) = \mathcal{M}_\delta(aH, aH)$

(C2):

$$\begin{aligned}\mathcal{M}_\delta(aH, bH) &= \mathcal{M}_E(aHb^{-1}H)\\ &= \mathcal{M}_E((ba^{-1})^{-1}H)\\ &= \mathcal{M}_E(ba^{-1}H)\\ &= \mathcal{M}_E(bHa^{-1}H)\\ &= \mathcal{M}_\delta(bH, aH)\end{aligned}$$

(C3):

$$\begin{aligned}\mathcal{M}_{\delta\circ\delta}(aH, bH) &= \bigvee_{b\in G/H} \{\mathcal{M}_\delta(aH, bH)\bigwedge \mathcal{M}_\delta(bH, cH)\}\\ &= \bigvee_{b\in G/H} \{\mathcal{M}_E(aHb^{-1}H)\bigwedge \mathcal{M}_E(bHc^{-1}H)\}\\ &= \bigvee_{b\in G/H} \{\mathcal{M}_E(ab^{-1}H)\bigwedge \mathcal{M}_E(bc^{-1}H)\}\\ &= \bigvee_{b\in G} \{\mathcal{M}_\eta(ab^{-1}, h)\bigwedge \mathcal{M}_\eta(bc^{-1}, h)\}\\ &= \bigvee_{b\in G} \{\Big(\mathcal{M}_H(ab^{-1})\bigwedge \mathcal{M}_H(h)\Big)\bigwedge\Big(\mathcal{M}_H(bc^{-1})\bigwedge \mathcal{M}_H(h)\Big)\}\\ &= \bigvee_{b\in G} \{\Big(\mathcal{M}_H(ab^{-1})\bigwedge \mathcal{M}_H(bc^{-1})\Big)\bigwedge \mathcal{M}_H(h)\}\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{b\in G}\{\Big(\mathcal{M}_H(ab^{-1})\bigwedge \mathcal{M}_H(bc^{-1})\Big)\\
&\leq \bigvee_{b\in G}\{\mathcal{M}_H(ac^{-1})\}\\
&= \mathcal{M}_H(ac^{-1})\\
&= \mathcal{M}_H(ac^{-1})\bigwedge \mathcal{M}_H(h)\\
&= \mathcal{M}_\eta(ac^{-1},h)\\
&= \mathcal{M}_E(aHc^{-1}H)=\mathcal{M}_\delta(aH,cH)
\end{aligned}
$$

(C4):

$$
\begin{aligned}
\mathcal{M}_\delta(acH,bdH) &= \mathcal{M}_E(ac(bd)^{-1}H)\\
&= \mathcal{M}_\eta(ac(bd)^{-1},h)\\
&= \mathcal{M}_H(ac(bd)^{-1})\bigwedge \mathcal{M}_H(h)\\
&= \mathcal{M}_H(acd^{-1}b^{-1})\\
&= \mathcal{M}_H(b^{-1}acd^{-1})\\
&\geq \mathcal{M}_H(b^{-1}a)\bigwedge \mathcal{M}_H(cd^{-1})\\
&= \mathcal{M}_H(ab^{-1})\bigwedge \mathcal{M}_H(cd^{-1})\\
&= \Big(\mathcal{M}_H(ab^{-1})\bigwedge \mathcal{M}_H(h)\Big)\bigwedge\Big(\mathcal{M}_H(cd^{-1})\bigwedge \mathcal{M}_H(h)\Big)\\
&= \mathcal{M}_\eta(ab^{-1},h)\bigwedge \mathcal{M}_\eta(cd^{-1},h)\\
&= \mathcal{M}_E(ab^{-1}H)\bigwedge \mathcal{M}_E(cd^{-1}H)\\
&= \mathcal{M}_E(aHb^{-1}H)\bigwedge \mathcal{M}_E(cHd^{-1}H)\\
&= \mathcal{M}_\delta(aH,bH)\bigwedge \mathcal{M}_\delta(cH,dH)
\end{aligned}
$$

Thus δ is a multiset congruence on G/H.

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