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On Transformation Formulae for q-Series

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Abstract: In this paper, we have established two identities by making use of Bailey transform and bilateral Bailey transform. Using these identities and some known summation formulae, certain transformation formulae for basic hypergeometric series as well as for poly-basic hypergeometric series have been established.
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1. Introduction, Notations and Definitions

Let q be a fixed complex parameter with $|q| < 1$. The q-shifted factorial for any complex parameter 'a' is defined by

$$[a, q]_k = \begin{cases} 1 & \text{for } k = 0 \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{k-1}), & \text{k=1,2,3,...,} \end{cases}$$

$$[a; q]_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

Also as usual, we write

$$[a_1, a_2, ..., a_r]_k = [a_1; q]_k[a_2; q]_k...[a_r; q]_k.$$

A basic hypergeometric series is defined by

$$_r\Phi_s\left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_k z^k}{[q, b_1, b_2, ..., b_s; q]_k} \left[(-)^k q^{k(k-1)/2}\right]^{1+s-r}, \qquad (1.1)$$

which is convergent in the whole complex plane if $r \leq s$ and it converges in unit disc $|z| < 1$ if $r = s + 1$.

A truncated basic hypergeometric series is represented by

$$ {}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix} \right]_N = \sum_{k=0}^{N} \frac{[a_1, a_2, ..., a_r; q]_k z^k}{[q, b_1, b_2, ..., b_s; q]_k} \left[(-)^k q^{k(k-1)/2} \right]^{1+s-r}. \quad (1.2) $$

A bilateral basic hypergeometric series is defined as,

$$ {}_r\Psi_r \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_r \end{matrix} \right] = \sum_{k=-\infty}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_k z^k}{[b_1, b_2, ..., b_r; q]_k}, \quad (1.3) $$

convergent in the region $\left| \frac{b_1 b_2 ... b_r}{a_1 a_2 ... a_r} \right| < |z| < 1$.
A poly-basic hypergeometric series is represented as,

$$ \Phi \left[\begin{matrix} a_1, a_2, .., a_r : c_{1,1}, ..., c_{1,r_1}; ...; c_{m,1}, ..., c_{m,r_m}; q, q_1, ..., q_m; z \\ b_1, b_2, .., b_s : d_{1,1}, ..., d_{1,s_1}; ...; d_{m,1}, ..., d_{m,s_m} \end{matrix} \right] $$

$$ = \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n}{[q, b_1, b_2, ..., b_s; q]_n} \left[(-)^n q^{n(n-1)/2} \right]^{1+s-r} \times $$

$$ \times \prod_{j=1}^{\infty} \frac{[c_{j,1}, ..., c_{j,r_j}; q_j]_n}{[d_{j,1}, ..., d_{j,s_j}; q_j]_n} \left[(-)^n q^{n(n-1)/2} \right]^{s_j - r_j}. \quad (1.4) $$

We shall be in need of following well known summations

$$ {}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right]_n = \frac{[\alpha q, \beta q; q]_n}{[q, \alpha\beta q; q]_n}. \quad (1.5) $$

$$ {}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right]_n = \frac{[aq, bq; q]_n}{[q, aq/b; q]_n b^n}. \quad (1.6) $$

$$ {}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{matrix} \right]_n = \frac{[aq, bq, cq, aq/bc; q]_n}{[q, aq/b, aq/c, bcq; q]_n}. \quad (1.7) $$

$$ \sum_{k=-m}^{n} \frac{[adpq; pq]_k \left[\frac{bp}{dq}; \frac{p}{q} \right]_k [a, b; p]_k \left[c, \frac{ad^2}{bc}; q \right]_k q^k}{[ad; pq]_k \left[\frac{b}{d}; \frac{p}{q} \right]_k \left[\frac{adp}{c}, \frac{bcp}{d}; p \right]_k \left[dq, \frac{adq}{b}; q \right]_k} $$

$$= \frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)} \times$$

$$\times \left[\frac{[ap,bp;p]_n \left[cq, \frac{ad^2q}{bc};q\right]_n}{\left[\frac{adp}{c}, \frac{bcp}{d};p\right]_n \left[dq, \frac{adq}{b};q\right]_n} - \frac{\left[\frac{c}{ad}, \frac{d}{bc};p\right]_{m+1} \left[\frac{1}{d}, \frac{b}{ad};q\right]_{m+1}}{\left[\frac{1}{a}, \frac{1}{b};p\right]_{m+1} \left[\frac{1}{c}, \frac{bc}{ad^2};q\right]_{m+1}} \right] \tag{1.8}$$

[Gasper & Rahman 2; App. II (II 36)]

$$\sum_{k=-m}^{n} \frac{[\beta;p]_k[c;q]_k[y;P]_k \left[\frac{\beta yc}{d^2};\frac{pP}{q}\right]_k \left[\frac{yP}{dq};\frac{P}{q}\right]_k \left[\frac{\beta p}{dq};\frac{p}{q}\right]_k \left[\frac{\beta yc}{d}pP;pP\right]_k q^k}{[dq;q]_k \left[\frac{\beta cp}{d};p\right]_k \left[\frac{cyP}{d};P\right]_k \left[\frac{\beta ypP}{dq};\frac{pP}{q}\right]_k \left[\frac{y}{d};\frac{P}{q}\right]_k \left[\frac{\beta}{d};\frac{p}{q}\right]_k \left[\frac{\beta cy}{d};pP\right]_k}$$

$$= \frac{d(1-\beta)(1-c)(1-y)(d^2-\beta cy)}{(d-\beta cy)(d-y)(d-\beta)(c-d)} \times$$

$$\times \left[\frac{\left[\frac{d}{\beta c};p\right]_{m+1} \left[\frac{1}{d};q\right]_{m+1} \left[\frac{d}{cy};P\right]_{m+1} \left[\frac{d}{\beta y};q\right]_{m+1}}{\left[\frac{1}{\beta};p\right]_{m+1} \left[\frac{1}{c};q\right]_{m+1} \left[\frac{1}{y};P\right]_{m+1} \left[\frac{d^2}{\beta cy};\frac{pP}{q}\right]_{m+1}} - \right.$$

$$\left. - \frac{[\beta p;p]_n[cq;q]_n[yP;P]_n \left[\frac{\beta cypP}{d^2q};\frac{pP}{q}\right]_n}{\left[\frac{\beta cp}{d};p\right]_n [dq;q]_n \left[\frac{cyP}{d};P\right]_n \left[\frac{\beta ypP}{dq};\frac{pP}{q}\right]_n} \right]. \tag{1.9}$$

[Verma 3; (12A) p.86]

$$\sum_{k=-m}^{n} \frac{[adpqPQ;pqPQ]_k \left[\frac{dPQ}{cpq};\frac{PQ}{pq}\right]_k \left[\frac{bpP}{dqQ};\frac{pP}{qQ}\right]_k \left[\frac{adPQ}{bcqp};\frac{pQ}{Pq}\right]_k}{[ad;pqPQ]_k \left[\frac{d}{c};\frac{PQ}{pq}\right]_k \left[\frac{b}{d};\frac{pP}{qQ}\right]_k \left[\frac{ad}{bc};\frac{pQ}{Pq}\right]_k} \times$$

$$\times \frac{[a;p^2]_k[c;q^2]_k[b;P^2]_k \left[\frac{ad^2}{bc};Q^2\right]_k q^{2k}}{\left[d\frac{qPQ}{p};\frac{qPQ}{p}\right]_k \left[\frac{ad}{c}\frac{pPQ}{q};\frac{pPQ}{q}\right]_k \left[\frac{adpqQ}{bP};\frac{pqQ}{P}\right]_k \left[\frac{bcpqP}{dQ};\frac{pqP}{Q}\right]_k}$$

$$= \frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(c-d)(bc-ad)} \times$$

$$\times \left[\frac{[ap^2;p^2]_n[cq^2;q^2]_n[bP^2;P^2]_n \left[\frac{ad^2Q^2}{bc};Q^2\right]_k}{\left[\frac{dqPQ}{p};\frac{qPQ}{p}\right]_n \left[\frac{adpPQ}{cq},\frac{pPQ}{q}\right]_n \left[\frac{adpqQ}{bP};\frac{pqQ}{P}\right]_n \left[\frac{bcpqP}{dQ};\frac{pqP}{Q}\right]_n} \right.$$

$$\left. - \frac{\left[\frac{c}{ad};\frac{pPQ}{q}\right]_{m+1} \left[\frac{1}{d};\frac{qPQ}{p}\right]_{m+1} \left[\frac{b}{ad};\frac{pqQ}{P}\right]_{m+1} \left[\frac{d}{bc};\frac{pqP}{Q}\right]_{m+1}}{\left[\frac{1}{a};p^2\right]_{m+1} \left[\frac{1}{c};q^2\right]_{m+1} \left[\frac{1}{b};P^2\right]_{m+1} \left[\frac{bc}{ad^2};Q^2\right]_{m+1}} \right] \tag{1.10}$$

[Verma 3; (18) p.89]

2. Identities

In this section establish two identities which will be used in next section

(i) In 1947, Bailey showed that,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.1}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.2}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.3}$$

where u_r, v_r, α_r and δ_r are sequences of r alone and all infinite series involved in (2.2) and (2.3) are convergent.
Taking $u_r, v_r = 1$, we have

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r,$$

which on simplification gives the identity,

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r + \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r - \sum_{n=0}^{\infty} \alpha_n \delta_n, \tag{2.4}$$

where α_r and δ_r are any arbitrary sequences of r alone such that all infinite series involved in (2.4) are convergent.

(ii) Andrews and Warnaar in 2007 [1] gave bilateral version of Bailey's transform. Symmetric bilateral Bailey transform is given by,

If

$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.5}$$

and

$$\gamma_n = \sum_{r=|n|} \delta_r u_{r-n} v_{r+n} \tag{2.6}$$

then

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.7}$$

[Andrews and Warnaar 1; lemma (2.1)]

where u_r, v_r, α_r and δ_r are arbitrary sequences of r alone such that all infinite series involved in (2.6) and (2.7) are convergent.

Taking $u_r, v_r = 1$ we can write (2.7) as

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r,$$

which on simplification gives the identity,

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r = \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r + \sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=-\infty}^{\infty} \alpha_n \delta_n, \tag{2.8}$$

where α_r and δ_r are any arbitrary sequences of r alone chosen such that all infinite series involved in (2.8) are convergent.

3. Main Results

In this section we shall establish main transformation formulae.

(i) Choosing $\alpha_n = \dfrac{[a, q\sqrt{a}, -q\sqrt{a}, e; q]_n}{[q, \sqrt{a}, -\sqrt{a}, aq/e; q]_n e^n}$ and $\delta_n = \dfrac{[\alpha, \beta; q]_n q^n}{[q, \alpha\beta q; q]_n}$ in (2.4) and using (1.5) and (1.6) we find,

$${}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e, \alpha, \beta; q; q/e \\ \sqrt{a}, -\sqrt{a}, aq/e, q, \alpha\beta q \end{matrix}\right] - {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e, \alpha q, \beta q; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e, q, \alpha\beta q \end{matrix}\right]$$

$$= {}_4\Phi_3 \left[\begin{matrix} \alpha, \beta, aq, eq; q; q/e \\ q, \alpha\beta q, aq/e \end{matrix} \right], \tag{3.1}$$

where $|1/e| < 1$.

(ii) Choosing

$\alpha_n = \dfrac{[a, q\sqrt{a}, -q\sqrt{a}, e; q]_n}{[q, \sqrt{a}, -\sqrt{a}, aq/e; q]_n e^n}$ and $\delta_n = \dfrac{[\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \alpha/\beta\gamma; q]_n q^n}{[q, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \beta\gamma q; q]_n}$ in (2.4) and using (1.6) and (1.7) we find,

$${}_{10}\Phi_9 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e, \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \alpha/\beta\gamma; q; q/e \\ \sqrt{a}, -\sqrt{a}, aq/e, q, \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \beta\gamma q \end{matrix} \right]$$

$$= {}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, e, \alpha q, \beta q, \gamma q, \alpha q/\beta\gamma; q; 1/e \\ \sqrt{a}, -\sqrt{a}, aq/e, q, \alpha q/\beta, \alpha q/\gamma, \beta\gamma q \end{matrix} \right]$$

$$+ {}_8\Phi_7 \left[\begin{matrix} \alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, \beta, \gamma, \alpha/\beta\gamma, aq, eq; q; q/e \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha q/\beta, \alpha q/\gamma, \beta\gamma q, q, aq/e \end{matrix} \right], \tag{3.2}$$

provided $|1/e| < 1$.

(iii) Taking $\alpha_n = \dfrac{[\alpha, \beta; q]_n q^n}{[q, \alpha\beta q; q]_n}$ and $\delta_n = \dfrac{[a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q]_n q^n}{[q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq; q]_n}$ in (2.4) and using (1.5) and (1.7) we get,

$$\frac{[\alpha q, \beta q, aq, bq, cq, aq/bc; q]_\infty}{[q, \alpha\beta q, q, aq/b, aq/c, bcq; q]_\infty} + {}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc, \alpha, \beta; q; q^2 \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq, q, \alpha\beta q \end{matrix} \right]$$

$$= {}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc, \alpha q, \beta q; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq, q, \alpha\beta q \end{matrix} \right]$$

$$+ {}_6\Phi_5 \left[\begin{matrix} \alpha, \beta, aq, bq, cq, aq/bc; q; q \\ q, \alpha\beta q, aq/b, aq/c, bcq \end{matrix} \right]. \tag{3.3}$$

(iv) Choosing

$\alpha_n = \dfrac{[adpq; pq]_n \left[\frac{bp}{dq}; \frac{p}{q}\right]_n [a, b; p]_n \left[c, \frac{ad^2}{bc}; q\right]_n q^n}{[ad; pq]_n \left[\frac{b}{d}; \frac{p}{q}\right]_n \left[\frac{adp}{c}, \frac{bcp}{d}; p\right]_n \left[dq, \frac{adq}{b}; q\right]_n}$ and $\delta_n = \dfrac{[\alpha, \beta; q_1]_n q_1^n}{[q_1, \alpha\beta q_1; q_1]_n}$ in (2.4) and using (1.5) and (1.8) after taking m=0 in it we find,

$$\frac{d(1-a)(1-b)(1-c)(bc-ad^2)[ap, bp; p]_\infty \left[cq, \frac{ad^2q}{bc}; q\right]_\infty [\alpha q_1, \beta q_1; q]_\infty}{(1-ad)(d-b)(d-c)(bc-ad) \left[\frac{adp}{c}, \frac{bcp}{d}; p\right]_\infty \left[dq, \frac{adq}{b}; q\right]_\infty [q_1, \alpha\beta q_1; q_1]_\infty}$$

$$+\frac{(ad-c)(bc-d)(1-d)(b-ad)[\alpha q_1,\beta q_1;q_1]_\infty}{(1-ad)(d-b)(d-c)(bc-ad)[q_1,\alpha\beta q_1;q_1]_\infty}$$

$$+\sum_{n=0}^{\infty}\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_n[\alpha,\beta;q_1]_n(qq_1)^n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n[q_1,\alpha\beta q_1;q_1]_n}$$

$$=\sum_{n=0}^{\infty}\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_n[\alpha q_1,\beta q_1;q_1]_n q^n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n[q_1,\alpha\beta q_1;q_1]_\infty}$$

$$+\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[\alpha,\beta;q_1]_n[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n q_1^n}{[q_1,\alpha\beta q_1;q_1]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}, \tag{3.4}$$

where max. $(|p|,|q|,|q_1|)<1$.

(v) Choosing

$$\alpha_r=\frac{[\beta;p_1]_r[\alpha;q_1]_r[y;P]_r\left[\frac{\alpha\beta y}{d_1^2};\frac{p_1P}{q_1}\right]_r\left[\frac{yP}{d_1q_1};\frac{P}{q_1}\right]_r}{[d_1q_1;q_1]_r\left[\frac{\alpha\beta p_1}{d_1};p_1\right]_r\left[\frac{\alpha yP}{d_1};P\right]_r\left[\frac{\beta yp_1P}{d_1q_1};\frac{p_1P}{q_1}\right]_r}\times$$

$$\times\frac{\left[\frac{\beta p_1}{d_1q_1};\frac{p_1}{q_1}\right]_r\left[\frac{\alpha\beta yp_1P}{d_1};p_1P\right]_r q_1^r}{\left[\frac{y}{d_1};\frac{P}{q_1}\right]_r\left[\frac{\beta}{d_1};\frac{p_1}{q_1}\right]_r\left[\frac{\alpha\beta y}{d_1};p_1P\right]_r}$$

and

$$\delta_r=\frac{[adpq;pq]_r\left[\frac{bp}{dq};\frac{p}{q}\right]_r[a,b;p]_r\left[c,\frac{ad^2}{bc};q\right]_r q^r}{[ad;pq]_r\left[\frac{b}{d};\frac{p}{q}\right]_r\left[\frac{adp}{c},\frac{bcp}{d};p\right]_r\left[dq,\frac{adq}{b};q\right]_r}$$

in (2.8) and using m=0 case of (1.8) and m=n case of (1.9) we have

$$\frac{d_1(1-\alpha)(1-\beta)(1-y)(d_1^2-\alpha\beta y)}{(d_1-\alpha\beta y)(d_1-y)(d_1-\beta)(\alpha-d_1)}\times\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}$$

$$\times\left[\frac{\left[\frac{d_1}{\alpha\beta};p_1\right]_\infty\left[\frac{1}{d_1};q_1\right]_\infty\left[\frac{d_1}{\alpha y};P\right]_\infty\left[\frac{d_1}{\beta y};\frac{p_1P}{q_1}\right]_\infty}{\left[\frac{1}{\beta};p_1\right]_\infty\left[\frac{1}{\alpha};q_1\right]_\infty\left[\frac{1}{y};P\right]_\infty\left[\frac{d_1^2}{\alpha\beta y};\frac{p_1P}{q_1}\right]_\infty}\right.$$

$$\left.-\frac{[\beta p_1;p_1]_\infty[\alpha q_1;q_1]_\infty[yP;P]_\infty\left[\frac{\alpha\beta y p_1P}{d_1^2q_1};\frac{p_1P}{q_1}\right]_\infty}{\left[\frac{\alpha\beta p_1}{d_1};p_1\right]_\infty[d_1q_1;q_1]_\infty\left[\frac{\alpha yP}{d_1};P\right]_\infty\left[\frac{\beta y p_1P}{d_1q_1};\frac{p_1P}{q_1}\right]_\infty}\right]\times$$

$$\left[\frac{[ap,bp;p]_\infty\left[cq,\frac{ad^2q}{bc};q\right]_\infty}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_\infty\left[dq,\frac{adq}{b};q\right]_\infty}-\frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right]$$

$$+\sum_{n=-\infty}^{\infty}\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}\times$$

$$\times\frac{[\beta;p_1]_n[\alpha;q_1]_n[y;P]_n\left[\frac{\alpha\beta y}{d_1^2};\frac{p_1P}{q_1}\right]_n\left[\frac{yP}{d_1q_1};\frac{P}{q_1}\right]_n}{[d_1q_1;q_1]_n\left[\frac{\alpha\beta p_1}{d_1};p_1\right]_n\left[\frac{\alpha yP}{d_1};P\right]_n\left[\frac{\beta y p_1P}{d_1q_1};\frac{p_1P}{q_1}\right]_n}\times$$

$$\times\frac{\left[\frac{\beta p_1}{d_1q_1};\frac{p_1}{q_1}\right]_n\left[\frac{\alpha\beta y p_1P}{d_1};p_1P\right]_n(qq_1)^n}{\left[\frac{y}{d_1};\frac{P}{q_1}\right]_n\left[\frac{\beta}{d_1};\frac{p_1}{q_1}\right]_n\left[\frac{\alpha\beta y}{d_1};p_1P\right]_n}$$

$$=\frac{d_1(1-\alpha)(1-\beta)(1-y)(d_1^2-\alpha\beta y)}{(d_1-\alpha\beta y)(d_1-y)(d_1-\beta)(\alpha-d_1)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_nq^n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}\times$$

$$\times\left[\frac{\left[\frac{d_1}{\alpha\beta};p_1\right]_{n+1}\left[\frac{1}{d_1};q_1\right]_{n+1}\left[\frac{d_1}{\alpha y};P\right]_{n+1}\left[\frac{d_1}{\beta y};q_1\right]_{n+1}}{\left[\frac{1}{\beta};p_1\right]_{n+1}\left[\frac{1}{\alpha};q_1\right]_{n+1}\left[\frac{1}{y};P\right]_{n+1}\left[\frac{d_1^2}{\alpha\beta y};\frac{p_1P}{q_1}\right]_{n+1}}\right.$$

$$\left.-\frac{[\beta p_1;p_1]_n[\alpha q_1;q_1]_n[yP;P]_n\left[\frac{\alpha\beta yp_1P}{d_1^2q_1};\frac{p_1P}{q_1}\right]_n}{\left[\frac{\alpha\beta p_1}{d_1};p_1\right]_n[d_1q_1;q_1]_n\left[\frac{\alpha yP}{d_1};P\right]_n\left[\frac{\beta yp_1P}{d_1q_1};\frac{p_1P}{q_1}\right]_n}\right]$$

$$+\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}$$

$$\times\sum_{n=-\infty}^{\infty}\frac{[\beta;p_1]_n[\alpha;q_1]_n[y;P]_n\left[\frac{\alpha\beta y}{d_1^2};\frac{p_1P}{q_1}\right]_n\left[\frac{yP}{d_1q_1};\frac{P}{q_1}\right]_n}{[d_1q_1;q_1]_n\left[\frac{\alpha\beta p_1}{d_1};p_1\right]_n\left[\frac{\alpha yP}{d_1};P\right]_n\left[\frac{\beta yp_1P}{d_1q_1};\frac{p_1P}{q_1}\right]_n}\times$$

$$\times\frac{\left[\frac{\beta p_1}{d_1q_1};\frac{p_1}{q_1}\right]_n\left[\frac{\alpha\beta yp_1P}{d_1};p_1P\right]_n q_1^n}{\left[\frac{y}{d_1};\frac{P}{q_1}\right]_n\left[\frac{\beta}{d_1};\frac{p_1}{q_1}\right]_n\left[\frac{\alpha\beta y}{d_1};p_1P\right]_n}$$

$$\times\left[\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}-\frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right]. \qquad (3.5)$$

(vi) Choosing

$$\alpha_n=\frac{[a_1d_1p_1q_1PQ;p_1q_1PQ]_n\left[\frac{d_1PQ}{c_1p_1q_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1p_1P}{d_1q_1Q};\frac{p_1P}{q_1Q}\right]_n}{[a_1d_1;p_1q_1PQ]_n\left[\frac{d_1}{c_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1}{d_1};\frac{p_1P}{q_1Q}\right]_n\left[\frac{a_1d_1}{b_1c_1};\frac{p_1Q}{q_1P}\right]_n}\times$$

$$\frac{\left[\frac{a_1d_1p_1Q}{b_1c_1q_1P};\frac{p_1Q}{q_1P}\right]_n[a_1;p_1^2]_n[c_1;q_1^2]_n[b_1;P^2]_n\left[\frac{a_1d_1^2}{b_1c_1};Q^2\right]_n q_1^{2n}}{\left[\frac{d_1q_1PQ}{p_1};\frac{q_1PQ}{p_1}\right]_n\left[\frac{a_1d_1p_1PQ}{c_1q_1};\frac{p_1PQ}{q_1}\right]_n\left[\frac{a_1d_1p_1q_1Q}{b_1P};\frac{p_1q_1Q}{P}\right]_n\left[\frac{b_1c_1p_1q_1P}{d_1Q};\frac{p_1q_1P}{Q}\right]_n}$$

and

$$\delta_n = \frac{[adpq;pq]_n \left[\frac{bp}{dq};\frac{p}{q}\right]_n [a,b;p]_n \left[c,\frac{ad^2}{bc};q\right]_n q^n}{[ad;pq]_n \left[\frac{b}{d};\frac{p}{q}\right]_n \left[\frac{adp}{c},\frac{bcp}{d};p\right]_n \left[dq,\frac{adq}{b};q\right]_n}$$

in (2.8) and using m=0 case of (1.8) and m=n case of (1.10) we find,

$$\frac{d_1(1-a_1)(1-b_1)(1-c_1)(b_1c_1-a_1d_1^2)}{(1-a_1d_1)(d_1-b_1)(c_1-d_1)(b_1c_1-a_1d_1)}\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}\times$$

$$\left[\frac{[a_1p_1^2;p_1^2]_\infty[c_1q_1^2;q_1^2]_\infty[b_1P^2;P^2]_\infty\left[\frac{a_1d_1^2Q^2}{b_1c_1};Q^2\right]_\infty}{\left[\frac{d_1q_1PQ}{p_1};\frac{q_1PQ}{p_1}\right]_\infty\left[\frac{a_1d_1p_1PQ}{c_1q_1};\frac{p_1PQ}{q_1}\right]_\infty\left[\frac{a_1d_1p_1q_1Q}{b_1P};\frac{p_1q_1Q}{P}\right]_\infty\left[\frac{b_1c_1p_1q_1P}{d_1Q};\frac{p_1q_1P}{Q}\right]_\infty}\right.$$

$$\left.-\frac{\left[\frac{c_1}{a_1d_1};\frac{p_1PQ}{q_1}\right]_\infty\left[\frac{1}{d_1};\frac{q_1PQ}{p_1}\right]_\infty\left[\frac{b_1}{a_1d_1};\frac{p_1q_1Q}{P}\right]_\infty\left[\frac{d_1}{b_1c_1};\frac{p_1q_1P}{Q}\right]_\infty}{\left[\frac{1}{a_1};p_1^2\right]_\infty\left[\frac{1}{c_1^2};q_1^2\right]_\infty\left[\frac{1}{b_1^2};P^2\right]_\infty\left[\frac{b_1c_1}{a_1d_1^2};Q^2\right]_\infty}\right]\times$$

$$\times\left[\frac{[ap,bp;p]_\infty\left[cq,\frac{ad^2q}{bc};q\right]_\infty}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_\infty\left[dq,\frac{adq}{b};q\right]_\infty}-\frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right]$$

$$+\sum_{n=-\infty}^{\infty}\frac{[a_1d_1p_1q_1PQ;p_1q_1PQ]_n\left[\frac{d_1PQ}{c_1p_1q_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1p_1P}{d_1q_1Q};\frac{p_1P}{q_1Q}\right]_n}{[a_1d_1;p_1q_1PQ]_n\left[\frac{d_1}{c_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1}{d_1};\frac{p_1P}{q_1Q}\right]_n\left[\frac{a_1d_1}{b_1c_1};\frac{p_1Q}{q_1P}\right]_n}\times$$

$$\frac{\left[\frac{a_1d_1p_1Q}{b_1c_1q_1P};\frac{p_1Q}{q_1P}\right]_n[a_1;p_1^2]_n[c_1;q_1^2]_n[b_1;P^2]_n\left[\frac{a_1d_1^2}{b_1c_1};Q^2\right]_n}{\left[\frac{d_1q_1PQ}{p_1};\frac{q_1PQ}{p_1}\right]_n\left[\frac{a_1d_1p_1PQ}{c_1q_1};\frac{p_1PQ}{q_1}\right]_n\left[\frac{a_1d_1p_1q_1Q}{b_1P};\frac{p_1q_1Q}{P}\right]_n\left[\frac{b_1c_1p_1q_1P}{d_1Q};\frac{p_1q_1P}{Q}\right]_n}$$

$$\times\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_n(qq_1^2)^n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}$$

$$= \frac{d_1(1-a_1)(1-b_1)(1-c_1)(b_1c_1-a_1d_1^2)}{(1-a_1d_1)(d_1-b_1)(c_1-d_1)(b_1c_1-a_1d_1)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{[adpq;pq]_n\left[\frac{bp}{dq};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{ad^2}{bc};q\right]_n q^n}{[ad;pq]_n\left[\frac{b}{d};\frac{p}{q}\right]_n\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}$$

$$\left[\frac{[a_1p_1^2;p_1^2]_n[c_1q_1^2;q_1^2]_n[b_1P^2;P^2]_n\left[\frac{a_1d_1^2Q^2}{b_1c_1};Q^2\right]_n}{\left[\frac{d_1q_1PQ}{p_1};\frac{q_1PQ}{p_1}\right]_n\left[\frac{a_1d_1p_1PQ}{c_1q_1};\frac{p_1PQ}{q_1}\right]_n\left[\frac{a_1d_1p_1q_1Q}{b_1P};\frac{p_1q_1Q}{P}\right]_n\left[\frac{b_1c_1p_1q_1P}{d_1Q};\frac{p_1q_1P}{Q}\right]_n}\right.$$

$$\left.-\frac{\left[\frac{c_1}{a_1d_1};\frac{p_1PQ}{q_1}\right]_{n+1}\left[\frac{1}{d_1};\frac{q_1PQ}{p_1}\right]_{n+1}\left[\frac{b_1}{a_1d_1};\frac{p_1q_1Q}{P}\right]_{n+1}\left[\frac{d_1}{b_1c_1};\frac{p_1q_1P}{Q}\right]_{n+1}}{\left[\frac{1}{a_1};p_1^2\right]_{n+1}\left[\frac{1}{c_1^2};q_1^2\right]_{n+1}\left[\frac{1}{b_1^2};P^2\right]_{n+1}\left[\frac{b_1c_1}{a_1d_1^2};Q^2\right]_{n+1}}\right]$$

$$+\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}\times$$

$$\times\sum_{n=-\infty}^{\infty}\frac{[a_1d_1p_1q_1PQ;p_1q_1PQ]_n\left[\frac{d_1PQ}{c_1p_1q_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1p_1P}{d_1q_1Q};\frac{p_1P}{q_1Q}\right]_n}{[a_1d_1;p_1q_1PQ]_n\left[\frac{d_1}{c_1};\frac{PQ}{p_1q_1}\right]_n\left[\frac{b_1}{d_1};\frac{p_1P}{q_1Q}\right]_n\left[\frac{a_1d_1}{b_1c_1};\frac{p_1Q}{q_1P}\right]_n}\times$$

$$\frac{\left[\frac{a_1d_1p_1Q}{b_1c_1q_1P};\frac{p_1Q}{q_1P}\right]_n[a_1;p_1^2]_n[c_1;q_1^2]_n[b_1;P^2]_n\left[\frac{a_1d_1^2}{b_1c_1};Q^2\right]_n q_1^{2n}}{\left[\frac{d_1q_1PQ}{p_1};\frac{q_1PQ}{p_1}\right]_n\left[\frac{a_1d_1p_1PQ}{c_1q_1};\frac{p_1PQ}{q_1}\right]_n\left[\frac{a_1d_1p_1q_1Q}{b_1P};\frac{p_1q_1Q}{P}\right]_n\left[\frac{b_1c_1p_1q_1P}{d_1Q};\frac{p_1q_1P}{Q}\right]_n}$$

$$\times\left[\frac{[ap,bp;p]_n\left[cq,\frac{ad^2q}{bc};q\right]_n}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n\left[dq,\frac{adq}{b};q\right]_n}-\frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)}\right]. \tag{3.6}$$

Similar other transformation can also be established by proper choice α_n and δ_n

4. Special Cases

In this section we shall deduce certain interesting transformations as special cases of the results established in previous section.

(i) Taking $q_1 = q, d = 1$ in (3.4) we find,

$$\frac{[ap, bp; p]_\infty [cq, aq/bc; q]_\infty [\alpha q, \beta q; q]_\infty}{[ap/c, bcp; p]_\infty [q, aq/b; q]_\infty [q, \alpha\beta q; q]_\infty}$$

$$+\sum_{n=0}^{\infty} \frac{[apq; pq]_n \left[\frac{bp}{q}; \frac{p}{q}\right]_n [a, b; p]_n \left[c, \frac{a}{bc}; q\right]_n [\alpha, \beta; q]_n (q)^{2n}}{[a; pq]_n \left[b; \frac{p}{q}\right]_n \left[\frac{ap}{c}, bcp; p\right]_n \left[q, \frac{aq}{b}; q\right]_n [q, \alpha\beta q; q]_n}$$

$$= \sum_{n=0}^{\infty} \frac{[apq; pq]_n \left[\frac{bp}{q}; \frac{p}{q}\right]_n [a, b; p]_n \left[c, \frac{a}{bc}; q\right]_n [\alpha q, \beta q; q]_n q^n}{[a; pq]_n \left[b; \frac{p}{q}\right]_n \left[\frac{ap}{c}, bcp; p\right]_n \left[q, \frac{aq}{b}; q\right]_n [q, \alpha\beta q; q]_n}$$

$$+\sum_{n=0}^{\infty} \frac{[\alpha, \beta; q]_n [ap, bp; p]_n \left[cq, \frac{aq}{bc}; q\right]_n q^n}{[q, \alpha\beta q; q]_n \left[\frac{ap}{c}, bcp; p\right]_n \left[q, \frac{aq}{b}; q\right]_n}, \tag{4.1}$$

(ii) For c=1, (4.1) yields the summation formula

$${}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; q; q \\ \alpha\beta q \end{matrix} \right] = \frac{[\alpha q, \beta q; q]_\infty}{[q, \alpha\beta q; q]_\infty}, \tag{4.2}$$

which is basic analogue of Gauss summation formula.
(iii) Taking $\beta = 1$ in (4.1) we get the summation formula

$$\sum_{n=0}^{\infty} \frac{[apq; pq]_n \left[\frac{bp}{q}; \frac{p}{q}\right]_n [a, b; p]_n \left[c, \frac{a}{bc}; q\right]_n q^n}{[a; pq]_n \left[b; \frac{p}{q}\right]_n \left[\frac{ap}{c}, bcp; p\right]_n \left[q, \frac{aq}{b}; q\right]_n}$$

$$= \frac{[ap, bp; p]_\infty [cq, aq/bc; q]_\infty}{[ap/c, bcp; p]_\infty [q, aq/b; q]_\infty}, \qquad |q| < 1. \tag{4.3}$$

(iv) For p=q, (4.1) yields (3.3).
(v) For p=q, (4.3) yields

$${}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{matrix} \right] = \frac{[aq, bq, cq, aq/bc; q]_\infty}{[q, aq/b, aq/c, bcq; q]_\infty} \tag{4.4}$$

which can be deduced from [Gasper & Rahman 2; App.II (II.20)] by replaced d by a/bc.

(vi) Taking $d_1 = d, p_1 = p$ and $q_1 = q$ in (3.5) we get

$$\frac{d^2(1-\alpha)(1-\beta)(1-y)(d^2-\alpha\beta y)}{(d-\alpha\beta y)(d-y)(d-\beta)(\alpha-d)} \times \frac{(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)} \times$$

$$\times \left[\frac{\left[\frac{d}{\alpha\beta};p\right]_\infty \left[\frac{1}{d};q\right]_\infty \left[\frac{d}{\alpha y};P\right]_\infty \left[\frac{d}{\beta y};\frac{pP}{q}\right]_\infty}{\left[\frac{1}{\beta};p\right]_\infty \left[\frac{1}{\alpha};q\right]_\infty \left[\frac{1}{y};P\right]_\infty \left[\frac{d^2}{\alpha\beta y};\frac{pP}{q}\right]_\infty} \right.$$

$$\left. - \frac{[\beta p;p]_\infty [\alpha q;q]_\infty [yP;P]_\infty \left[\frac{\alpha\beta y pP}{d^2 q};\frac{pP}{q}\right]_\infty}{\left[\frac{\alpha\beta p}{d};p\right]_\infty [dq;q]_\infty \left[\frac{\alpha y P}{d};P\right]_\infty \left[\frac{\beta y pP}{dq};\frac{pP}{q}\right]_\infty} \right] \times$$

$$\left[\frac{[ap,bp;p]_\infty \left[cq,\frac{ad^2q}{bc};q\right]_\infty}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_\infty \left[dq,\frac{adq}{b};q\right]_\infty} - \frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)} \right]$$

$$+ \sum_{n=-\infty}^{\infty} \frac{[adpq;pq]_n \left[\frac{bp}{dq};\frac{p}{q}\right]_n [a,b;p]_n \left[c,\frac{ad^2}{bc};q\right]_n}{[ad;pq]_n \left[\frac{b}{d};\frac{p}{q}\right]_n \left[\frac{adp}{c},\frac{bcp}{d};p\right]_n \left[dq,\frac{adq}{b};q\right]_n} \times$$

$$\times \frac{[\beta;p]_n [\alpha;q]_n [y;P]_n \left[\frac{\alpha\beta y}{d^2};\frac{pP}{q}\right]_n \left[\frac{yP}{dq};\frac{P}{q}\right]_n}{[dq;q]_n \left[\frac{\alpha\beta p}{d};p\right]_n \left[\frac{\alpha y P}{d};P\right]_n \left[\frac{\beta y pP}{dq};\frac{pP}{q}\right]_n} \times$$

$$\times \frac{\left[\frac{\beta p}{dq};\frac{p}{q}\right]_n \left[\frac{\alpha\beta y pP}{d};pP\right]_n q^{2n}}{\left[\frac{y}{d};\frac{P}{q}\right]_n \left[\frac{\beta}{d};\frac{p}{q}\right]_n \left[\frac{\alpha\beta y}{d};pP\right]_n}$$

$$= \frac{d(1-\alpha)(1-\beta)(1-y)(d^2-\alpha\beta y)}{(d-\alpha\beta y)(d-y)(d-\beta)(\alpha-d)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{[adpq;pq]_n \left[\frac{bp}{dq};\frac{p}{q}\right]_n [a,b;p]_n \left[c,\frac{ad^2}{bc};q\right]_n q^n}{[ad;pq]_n \left[\frac{b}{d};\frac{p}{q}\right]_n \left[\frac{adp}{c},\frac{bcp}{d};p\right]_n \left[dq,\frac{adq}{b};q\right]_n} \times$$

$$\times \left[\frac{\left[\frac{d}{\alpha\beta};p\right]_{n+1} \left[\frac{1}{d};q\right]_{n+1} \left[\frac{d}{\alpha y};P\right]_{n+1} \left[\frac{d}{\beta y};q\right]_{n+1}}{\left[\frac{1}{\beta};p\right]_{n+1} \left[\frac{1}{\alpha};q\right]_{n+1} \left[\frac{1}{y};P\right]_{n+1} \left[\frac{d^2}{\alpha\beta y};\frac{pP}{q}\right]_{n+1}} \right.$$

$$\left. - \frac{[\beta p;p]_n[\alpha q;q]_n[yP;P]_n \left[\frac{\alpha\beta ypP}{d^2q};\frac{pP}{q}\right]_n}{\left[\frac{\alpha\beta p}{d};p\right]_n [dq;q]_n \left[\frac{\alpha yP}{d};P\right]_n \left[\frac{\beta ypP}{dq};\frac{pP}{q}\right]_n} \right]$$

$$+\frac{d(1-a)(1-b)(1-c)(bc-ad^2)}{(1-ad)(d-b)(d-c)(bc-ad)}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{[\beta;p]_n[\alpha;q]_n[y;P]_n \left[\frac{\alpha\beta y}{d^2};\frac{pP}{q}\right]_n \left[\frac{yP}{dq};\frac{P}{q}\right]_n}{[dq;q]_n \left[\frac{\alpha\beta p}{d};p\right]_n \left[\frac{\alpha yP}{d};P\right]_n \left[\frac{\beta ypP}{dq};\frac{pP}{q}\right]_n} \times$$

$$\times \frac{\left[\frac{\beta p}{dq};\frac{p}{q}\right]_n \left[\frac{\alpha\beta ypP}{d};pP\right]_n q^n}{\left[\frac{y}{d};\frac{P}{q}\right]_n \left[\frac{\beta}{d};\frac{p}{q}\right]_n \left[\frac{\alpha\beta y}{d};pP\right]_n}$$

$$\left[\frac{[ap,bp;p]_n \left[cq,\frac{ad^2q}{bc};q\right]_n}{\left[\frac{adp}{c},\frac{bcp}{d};p\right]_n \left[dq,\frac{adq}{b};q\right]_n} - \frac{(ad-c)(bc-d)(1-d)(b-ad)}{d(1-a)(1-b)(1-c)(bc-ad^2)} \right]. \qquad (4.5)$$

(vii) For d=1, (4.5) yields

$$\frac{[\beta p;p]_\infty[\alpha q;q]_\infty[yP;P]_\infty \left[\frac{\alpha\beta ypP}{q};\frac{pP}{q}\right]_\infty [ap,bp;p]_\infty \left[cq,\frac{aq}{bc};q\right]_\infty}{[\alpha\beta p;p]_\infty [q;q]_\infty [\alpha yP;P]_\infty \left[\frac{\beta ypP}{q};\frac{pP}{q}\right]_\infty \left[\frac{ap}{c},bcp;p\right]_\infty \left[q,\frac{aq}{b};q\right]_\infty}$$

$$+\sum_{n=0}^{\infty}\frac{[apq;pq]_n\left[\frac{bp}{q};\frac{p}{q}\right]_n\left[\frac{\beta p}{q};\frac{p}{q}\right]_n[a,b,\beta;p]_n\left[c,\frac{a}{bc},\alpha;q\right]_n}{[a;pq]_n\left[b,\beta;\frac{p}{q}\right]_n\left[\frac{ap}{c},bcp,\alpha\beta p;p\right]_n\left[q,\frac{aq}{b},q;q\right]_n}$$

$$\times\frac{[y;P]_n\left[\alpha\beta y;\frac{pP}{q}\right]_n\left[\frac{yP}{q};\frac{P}{q}\right]_n[\alpha\beta ypP;pP]_n}{[\alpha yP;P]_n\left[\frac{\beta ypP}{q};\frac{pP}{q}\right]_n\left[y;\frac{P}{q}\right]_n[\alpha\beta y;pP]_n}$$

$$=\sum_{n=0}^{\infty}\frac{[apq;pq]_n\left[\frac{bp}{q};\frac{p}{q}\right]_n[a,b;p]_n\left[c,\frac{a}{bc};q\right]_n q^n}{[a;pq]_n\left[b;\frac{p}{q}\right]_n\left[\frac{ap}{c},bcp;p\right]_n\left[q,\frac{aq}{b};q\right]_n}\times$$

$$\times\frac{[\beta p;p]_n[\alpha q;q]_n[yP;P]_n\left[\frac{\alpha\beta ypP}{q};\frac{pP}{q}\right]_n}{[\alpha\beta p;p]_n\,[q;q]_n\,[\alpha yP;P]_n\left[\frac{\beta ypP}{q};\frac{pP}{q}\right]_n}$$

$$+\sum_{n=0}^{\infty}\frac{[\beta;p]_n[\alpha;q]_n[y;P]_n\left[\alpha\beta y;\frac{pP}{q}\right]_n\left[\frac{yP}{q};\frac{P}{q}\right]_n}{[q;q]_n\,[\alpha\beta p;p]_n\,[\alpha yP;P]_n\left[\frac{\beta ypP}{q};\frac{pP}{q}\right]_n}\times$$

$$\times\frac{\left[\frac{\beta p}{q};\frac{p}{q}\right]_n[\alpha\beta ypP;pP]_n\,q^n}{\left[y;\frac{P}{q}\right]_n\left[\beta;\frac{p}{q}\right]_n[\alpha\beta y;pP]_n}\frac{[ap,bp;p]_n\left[cq,\frac{aq}{bc};q\right]_n}{\left[\frac{ap}{c},bcp;p\right]_n\left[q,\frac{aq}{b};q\right]_n}. \tag{4.6}$$

(viii) Taking p=P=q in (4.6) we find

$$\frac{[\alpha q,\beta q,yq,\alpha\beta yq,aq,bq,cq,aq/bc;q]_\infty}{[q,q,\alpha\beta q,\alpha yq,\beta yq,aq/b,aq/c,bcq;q]_\infty}$$

$$+\ {}_{12}\Phi_{11}\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,a/bc,\alpha\beta y,q\sqrt{\alpha\beta y},-q\sqrt{\alpha\beta y},\alpha,y,\beta;q;q^2\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,bcq,q,\sqrt{\alpha\beta y},-\sqrt{\alpha\beta y},\beta yq,\alpha\beta q,\alpha yq\end{matrix}\right]$$

$$=\ {}_{10}\Phi_{9}\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,a/bc,\alpha\beta yq,\alpha q,\beta q,yq;q;q\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,bcq,q,\beta yq,\alpha yq,\alpha\beta q\end{matrix}\right]$$

$$+ {}_{10}\Phi_9 \left[\begin{array}{l} \alpha\beta y, q\sqrt{\alpha\beta y}, -q\sqrt{\alpha\beta y}, \alpha, \beta, y, aq, bq, cq, aq/bc; q; q \\ \sqrt{\alpha\beta y}, -\sqrt{\alpha\beta y}, \beta yq, \alpha yq, \alpha\beta q, q, aq/b, aq/c, bcq \end{array} \right] \quad (4.7)$$

which is believed to be new.

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