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CERTAIN ROGERS-RAMANUJAN TYPE MULTI SUM IDENTITIES AND RATIO OF INFINITE PRODUCTS

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Abstract: Some Rogers-Ramanujan type multi sum identities can be expressed in terms of infinite products. In this paper, an attempt has been made to establish the certain results involving the multi summation expressions and ratio of infinite products by using well known m dissections of the power series.

Keywords and Phrases: Ratio's of infinite products, Bailey pair's, Bailey lemma, Rogers-Ramanujan type multi sum identities.

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1. Introduction

The $m-$ dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the representation of P as $p = p_0 + P_1 + ... + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$ Andrews [2] and Hirschhorn [7] have given the 2 dissection and 5 dissection of the continued fraction $C(q)$ and

$C(q)^{-1}$ which is the ratio of basic infinite products. Lewis et al [10] obtained a conjecture of Hirschhorn on 4 dissection of Ramanujan's Continued Fraction. Denis et al [9] gave equivalent continued fraction representations for ratio's of infinite products.

$$\begin{aligned} S(q) &= \frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} \\ &= 1+\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^5}{1+}\frac{q^8}{1+}\frac{q^5+q^{10}}{1+}\frac{q^{12}}{1+...} \\ &= 1+\frac{q+q^2}{1-q+}\frac{q+q^4}{1-q+}\frac{q+q^6}{1-q+...} \\ &= 1+q+\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^5}{1+q^7+...} \end{aligned}$$

The Bailey chain is a well-known and frequently used technique in the theory of partitions. It arose from W.N. Bailey's realization [11] that the Rogers-Ramanujan identities could be derived from the simple observation that if $\{\alpha_0,\alpha_1,...\}$ and $\{\delta_0,\delta_1,...\}$ are sequences that satisfy.

$$\beta_k=\sum_{r=0}^{k}\alpha_r u_{k-r}v_{k+r}$$

and

$$\gamma_k=\sum_{r=k}^{\infty}\delta_r u_{r-k}v_{r+k},$$

then

$$\sum_{k=0}^{\infty}\alpha_k\gamma_k=\sum_{k=0}^{\infty}\beta_k\delta_k,$$

provided all infinite sums converge uniformly. L.J. Slater used his idea to produce her list of 130 identities of the Rogers-Ramanujan type [5,6].
A pair of sequences $(\alpha_n(a,q),\beta_n(a,q))$ is called a Bailey pair with parameters (a,q) if

$$\beta_n(a,q)=\sum_{r=0}^{n}\frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}},\qquad \text{for } n\geq 0.$$

The unit Bailey pair [3,4]

$$\beta_n(a,q)=\begin{cases}1, & \text{if } n=0\\ 0, & \text{if } n>0\end{cases}\qquad \alpha_n(a,q)=\frac{(a;q)_n(1-aq^{2n})}{(q;q)_n(1-a)}(-1)^n q^{(n^2-n)/2}.$$

In 2001 D. Bressoud established some interested theorem involving change of base in Bailey pairs, which are

(a) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then, $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a,q)

$$\alpha'_r(a,q) = a^r q^{r^2} \alpha_r(a,q),$$

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q;q)_{n-k}} \beta_k(a,q). \tag{1.1}$$

(b) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then, $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a^2,q^2)

$$\alpha'_r(a,q) = a^r \alpha_r(a^2,q^2),$$

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}} q^{n-k} \beta_k(a^2,q^2). \tag{1.2}$$

(c) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then, $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a^3,q^3)

$$\alpha'_r(a,q) = a^r q^{r^2} \alpha_r(a,q),$$

$$\beta'_n(a,q) = \frac{1}{(a^3q^3;q^3)_{2n}} \sum_{k=0}^{n} \frac{(aq;q)_{3n-k} a^k q^{k^2}}{(q^3;q^3)_{n-k}} \beta_k(a,q). \tag{1.3}$$

(d) Suppose that $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then, $(\alpha'_n(a,q), \beta'_n(a,q))$ is a Bailey pair with parameters (a,q)

$$\alpha'_r(a,q) = a^{-r} q^{-r^2} \alpha_r(a^3,q^3),$$

$$\beta'_n(a,q) = \frac{1}{(aq;q)_{2n}} \sum_{k=0}^{n} \frac{(aq^{2n+1};q^{-1})_{3k}(a^3q^3;q^3)_{2(n-k)}(-1)^k q^{3\binom{k}{2}-n^2} a^{-n}}{(q^3;q^3)_k} \times \beta_{n-k}(a^3,q^3). \tag{1.4}$$

Here an attempt has been made to obtain certain results involving the multi summation expressions and ratio of infinite products by using well known m-dissections of the power series.

2. Notations

Suppose that $|q| < 1$, where q is non-zero complex number. We will use the notations,

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \tag{2.1}$$

$$[z;q]_\infty = (z;q)_\infty(z^{-1}q;q)_\infty, \quad (\text{for} \quad z \neq 0) \quad \text{and often we write} \tag{2.2}$$

$$[z_1, z_2, ..., z_n; q]_\infty = [z_1;q]_\infty[z_2;q]_\infty...[z_n;q]_\infty, \tag{2.3}$$

The following fact can be easily verified;

$$[z^{-1};q]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty, \tag{2.4}$$

$$[z, zq; q^2]_\infty = [z;q]_\infty, \tag{2.5}$$

$$[z, -z; q]_\infty = [z^2; q^2]_\infty \tag{2.6}$$

$$[z^{-1}q;q]_\infty = [z;q]_\infty \tag{2.7}$$

$$[-1;q]_\infty[q;q^2]_\infty = 2. \tag{2.8}$$

We have the following general relations;
Suppose $a_1, a_2, ..., a_n; b_1, b_2, ..., b_n \in C\{0\}$ satisfy
(i) $a_i \neq q^n a_j$ for $i \neq j$ and $n \in z$,
(ii) $a_1, a_2, ..., a_n = b_1, b_2, ..., b_n$.
Then

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n}[a_i^{-1}b_j;q]_\infty}{\prod_{j=1, j\neq i}^{n}[a_i^{-1}a_j;q]_\infty} = 0 \tag{2.9}$$

This theorem appears without proof as given by Slater [4] and with a proof as given by Lewis [8]. Also, we have the following well known Rogers- Ramanujan type multi sum identity.

$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}} = \frac{(q^{16}, q^7, q^9; q^{16})_\infty}{(q^3, q^9, q^{12}; q^{12})_\infty}. \tag{2.10}$$

[D. Bressoud, 4; p.452]

3. Main Results

In this section we prove our main results:

$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}$$

$$= \frac{(q^6;q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty}{(q^3;q^3)_\infty}\Bigg[\frac{(q,q^{17},q^{15},q^{31};q^{32})_\infty(q^{26},q^{38};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{12},q^{14},q^{32},q^{32},q^{50},q^{52};q^{64})_\infty}$$
$$+\frac{q(q^3,q^{13},q^{19},q^{29};q^{32})_\infty(q^{30},q^{34};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{20},q^{32},q^{32},q^{44},q^{50};q^{64})_\infty}$$
$$+\frac{q^2(q^4,q^5,q^7,q^{25},q^{27},q^{28};q^{32})_\infty(q^{16},q^{24},q^{40},q^{48};q^{64})_\infty}{(q^8,q^{12},q^{20},q^{24};q^{32})_\infty(q^8,q^{12},q^{20},q^{32},q^{32},q^{44},q^{52},q^{56};q^{64})_\infty}$$
$$-\frac{q^2(q^4,q^4,q^8,q^{24},q^{28},q^{28};q^{32})_\infty(q^{18},q^{22},q^{42},q^{46},q^{60};q^{64})_\infty}{(q^9,q^{11},q^{21},q^{23};q^{32})_\infty(q^8,q^{12},q^{20},q^{32},q^{32},q^{44},q^{52},q^{56},q^{68};q^{64})_\infty}\Bigg]. \tag{3.1}$$

$$\frac{(q^3;q^3)_\infty}{(q^6;q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty} = \Bigg[\frac{(q,q^{17},q^{15},q^{31};q^{32})_\infty(q^{20},q^{26},q^{38},q^{44};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{14},q^{18},q^{32},q^{32},q^{46},q^{50},q^{50};q^{64})_\infty}$$
$$+\frac{q(q^3,q^{13},q^{19},q^{29};q^{32})_\infty(q^{12},q^{30},q^{34},q^{52};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{14},q^{18},q^{32},q^{32},q^{46},q^{50},q^{50};q^{64})_\infty}$$
$$-\frac{q^2(q^4,q^5,q^7,q^{25},q^{27},q^{28};q^{32})_\infty(q^{16},q^{24},q^{40},q^{48};q^{64})_\infty}{(q^8,q^{12},q^{20},q^{24};q^{32})_\infty(q^8,q^{14},q^{18},q^{32},q^{32},q^{46},q^{50},q^{56};q^{64})_\infty}$$
$$+\frac{q^2(q^4,q^4,q^8,q^{24},q^{28},q^{28};q^{32})_\infty(q^{22},q^{42},q^{60};q^{64})_\infty}{(q^9,q^{11},q^{21},q^{23};q^{32})_\infty(q^8,q^{14},q^{32},q^{32},q^{50},q^{56},q^{68};q^{64})_\infty}\Bigg]$$
$$\times\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}. \tag{3.2}$$

4. Proof of 3.1.

Considering $B(q)=\dfrac{(q^7,q^9;q^{16})_\infty}{(q^6,q^{10};q^{16})_\infty}$ and $B(q)=\dfrac{(q^7,q^9;q^{16})_\infty}{(q^6,q^{10};q^{16})_\infty}$ can be written as $B(q)=\dfrac{(q^7,q^{23};q^{32})_\infty}{(q^6,q^{22};q^{32})_\infty}$ by using (2.2).

Now, setting $(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4)=(1,-1,q^6,q^{10};q^7,q^{23},-q^{-7},q^{-7})$ and taking q^{32} for q in (2.9),

$$\frac{[q^7,q^{23},-q^{-7},q^{-7};q^{32}]_\infty}{[-1,q^6,q^{10};q^{32}]_\infty}+\frac{[-q^7,-q^{23},-q^{-7},q^{-7};q^{32}]_\infty}{[-1,-q^6,-q^{10};q^{32}]_\infty}$$
$$+\frac{[q,q^{17},-q^{-13},q^{-13};q^{32}]_\infty}{[q^{-6},-q^{-6},q^4;q^{32}]_\infty}+\frac{[q^{-3},q^{13},-q^{-17},q^{-17};q^{32}]_\infty}{[q^{-10},-q^{-10},q^4;q^{32}]_\infty}=0. \tag{4.1}$$

By using (2.6) and (2.8) in (4.1)

$$\frac{[q^7,q^{23};q^{32}]_\infty}{[q^6,q^{10};q^{32}]_\infty}+\frac{[-q^7,-q^{23};q^{32}]_\infty}{[-q^6,-q^{10};q^{32}]_\infty}$$

$$= \frac{2}{[q^{14}, q^{32}; q^{64}]_\infty(-q^{-14})} \left[-\frac{[q, q^{17}; q^{32}]_\infty [q^{-26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{-12}; q^{64}]_\infty} - \frac{[q^{-3}, q^{13}; q^{32}]_\infty [q^{-34}; q^{64}]_\infty}{[q^{-4}; q^{32}]_\infty [q^{-20}; q^{64}]_\infty} \right].$$

By applying (2.4),

$$\frac{[q^7, q^{23}; q^{32}]_\infty}{[q^6, q^{10}; q^{32}]_\infty} + \frac{[-q^7, -q^{23}; q^{32}]_\infty}{[-q^6, -q^{10}; q^{32}]_\infty} = \frac{2}{[q^{14}, q^{32}; q^{64}]_\infty(-q^{-14})}$$

$$\times \left[-\frac{[q, q^{17}; q^{32}]_\infty [q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{12}; q^{64}]_\infty}(q^{-14}) - \frac{[q^3, q^{13}; q^{32}]_\infty [q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{20}; q^{64}]_\infty}(q^{-13}) \right].$$

$$B(q) + B'(q) = \frac{2[q, q^{17}; q^{32}]_\infty [q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{12}, q^{14}, q^{32}; q^{64}]_\infty} + \frac{2q[q^3, q^{13}; q^{32}]_\infty [q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{20}, q^{32}; q^{64}]_\infty} \quad (4.2)$$

$$\alpha_1(q) = \frac{1}{2}[B(q)+B'(q)] = \frac{[q, q^{17}; q^{32}]_\infty [q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{12}, q^{14}, q^{32}; q^{64}]_\infty} + \frac{q[q^3, q^{13}; q^{32}]_\infty [q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{20}, q^{32}; q^{64}]_\infty} \quad (4.3)$$

Again setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = (1, -1, q^{30}, -q^{34}; q^{25}, q^{23}, -q^6, -q^{-10})$ and taking q^{32} for q in (2.9),

$$\frac{[q^{25}, q^{23}, -q^6, q^{-10}; q^{32}]_\infty}{[-1, q^{30}, -q^{34}; q^{32}]_\infty} + \frac{[-q^{25}, -q^{23}, q^6, q^{10}; q^{32}]_\infty}{[-1, -q^{30}, q^{34}; q^{32}]_\infty}$$

$$+ \frac{[q^{-5}, q^{-7}, -q^{-24}, -q^{-20}; q^{32}]_\infty}{[q^{-30}, -q^{-30}, -q^4; q^{32}]_\infty} + \frac{[-q^{-9}, -q^{-11}, q^{-28}, q^{-24}; q^{32}]_\infty}{[q^{-34}, -q^{-34}, -q^{-4}; q^{32}]_\infty} = 0.$$

By using (2.4) and (2.7) in the above

$$\frac{q^2}{[-1, q^2, -q^2; q^{32}]_\infty}[[q^7, q^{23}, -q^6, -q^{10}; q^{32}]_\infty - [-q^7, -q^{23}, q^6, q^{10}; q^{32}]_\infty]$$

$$-\frac{q^4[q^5, q^7, -q^{24}, -q^{20}; q^{32}]_\infty}{[q^{30}, -q^{30}, -q^4; q^{32}]_\infty} - \frac{[-q^9, -q^{11}, q^{28}, q^{24}; q^{32}]_\infty}{[q^{34}, -q^{34}, -q^4; q^{32}]_\infty} = 0.$$

By using (2.6), we get

$$[[q^7, q^{23}, -q^6, -q^{10}; q^{32}]_\infty - [-q^7, -q^{23}, q^6, q^{10}; q^{32}]_\infty] = \frac{2[q^4; q^{64}]_\infty}{q^2[q^{32}; q^{64}]_\infty}$$

$$\times \left[\frac{q^4[q^4, q^5, q^7; q^{32}]_\infty [q^{40}, q^{44}; q^{64}]_\infty}{[q^{24}, q^{20}; q^{32}]_\infty [q^8, q^{60}; q^{64}]_\infty} + \frac{[q^4, q^{24}, q^{28}; q^{32}]_\infty [q^{18}, q^{22}; q^{64}]_\infty}{[q^9, q^{11}; q^{32}]_\infty [q^8, q^{68}; q^{64}]_\infty} \right].$$

Dividing by $[-q^6, q^6, q^{10}, -q^{10}; q^{32}]_\infty$ and using (2.6),

$$\frac{[q^7, q^{23}; q^{32}]}{[q^6, q^{10}; q^{32}]} - \frac{[-q^7, -q^{23}; q^{32}]_\infty}{[-q^6, -q^{10}; q^{32}]_\infty}$$

$$= \frac{2q^2[q^4, q^5, q^7; q^{32}]_\infty[q^4, q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{60}; q^{64}]_\infty} - \frac{2[q^4, q^{24}, q^{28}; q^{32}]_\infty[q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{68}; q^{64}]_\infty}$$

$B(q) - B'(q)$

$$= \frac{2q^2[q^4, q^5, q^7; q^{32}]_\infty[q^4, q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{60}; q^{64}]_\infty} - \frac{2[q^4, q^{24}, q^{28}; q^{32}]_\infty[q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{68}; q^{64}]_\infty} \tag{4.4}$$

$\beta_1(q) = \frac{1}{2}[B(q) - B'(-q)]$

$$= \frac{q^2[q^4, q^5, q^7; q^{32}]_\infty[q^4, q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{60}; q^{64}]_\infty} - \frac{[q^4, q^{24}, q^{28}; q^{32}]_\infty[q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{68}; q^{64}]_\infty} \tag{4.5}$$

By adding (4.3) and (4.5), in the above
$B(q) = \alpha_1(q) + \beta_1(q)$

$$B(q) = \frac{[q, q^{17}; q^{32}]_\infty[q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty[q^{12}, q^{14}, q^{32}; q^{64}]_\infty} + \frac{q[q^3, q^{13}; q^{32}]_\infty[q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty[q^{14}, q^{20}, q^{32}; q^{64}]_\infty}$$

$$= \frac{q^2[q^4, q^5, q^7; q^{32}]_\infty[q^4, q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{60}; q^{64}]_\infty} - \frac{[q^4, q^{24}, q^{28}; q^{32}]_\infty[q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty[q^8, q^{12}, q^{20}, q^{32}, q^{68}; q^{64}]_\infty} \tag{4.6}$$

By applying (2.2) in (4.6),

$$\begin{aligned} B(q) = {} & \frac{(q, q^{17}, q^{15}, q^{31}; q^{32})_\infty(q^{26}, q^{38}; q^{64})_\infty}{(q^4, q^{28}; q^{32})_\infty(q^{12}, q^{14}, q^{32}, q^{32}, q^{50}, q^{52}; q^{64})_\infty} \\ & + \frac{q(q^3, q^{13}, q^{19}, q^{29}; q^{32})_\infty(q^{30}, q^{34}; q^{64})_\infty}{(q^4, q^{28}; q^{32})_\infty(q^{14}, q^{20}, q^{32}, q^{32}, q^{44}, q^{50}; q^{64})_\infty} \\ & + \frac{q^2(q^4, q^5, q^7, q^{25}, q^{27}, q^{28}; q^{32})_\infty(q^{16}, q^{24}, q^{40}, q^{48}; q^{64})_\infty}{(q^8, q^{12}, q^{20}, q^{24}; q^{32})_\infty(q^8, q^{12}, q^{20}, q^{32}, q^{32}, q^{44}, q^{52}, q^{56}; q^{64})_\infty} \\ & - \frac{q^2(q^4, q^4, q^8, q^{24}, q^{28}, q^{28}; q^{32})_\infty(q^{18}, q^{22}, q^{42}, q^{46}, q^{60}; q^{64})_\infty}{(q^9, q^{11}, q^{21}, q^{23}; q^{32})_\infty(q^8, q^{12}, q^{20}, q^{32}, q^{32}, q^{44}, q^{52}, q^{56}, q^{68}; q^{64})_\infty}. \end{aligned} \tag{4.7}$$

By taking known identity (2.10),

$$\sum_{s_1, s_2 \geq 0} \frac{q^{3s_1^2+2s_2^2}(-q^3; q^6)_{s_1}(q^2; q^2)_{3s_1-s_2}}{(q^6; q^6)_{2s_1}(q^6; q^6)_{s_1-s_2}(q^2; q^2)_{s_2}} = \frac{(q^{16}, q^7, q^9; q^{16})_\infty}{(q^3, q^9, q^{12}; q^{12})_\infty}.$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}=\frac{(q^6,q^{12})_\infty(q^6,q^7,q^9,q^{10},q^{16};q^{16})_\infty}{(q^3;q^3)_\infty(q^6,q^{10};q^{16})_\infty}$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}=\frac{(q^6,q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty}{(q^3;q^3)_\infty}B(q) \tag{4.8}$$

where $B(q)=\dfrac{(q^7,q^9;q^{16})_\infty}{(q^6,q^{10};q^{16})_\infty}$.
Now, by putting the value of $B(q)$ from (4.7) in (4.8),

$$\begin{aligned}&\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}\\ &\quad=\frac{(q^6;q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty}{(q^3;q^3)_\infty}\left[\frac{(q,q^{17},q^{15},q^{31};q^{32})_\infty(q^{26},q^{28};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{12},q^{14},q^{32},q^{32},q^{50},q^{52};q^{64})_\infty}\right.\\ &\quad+\frac{q(q^3,q^{13},q^{19},q^{29};q^{32})_\infty(q^{30},q^{34};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{20},q^{32},q^{32},q^{44},q^{50};q^{64})_\infty}\\ &\quad+\frac{q^2(q^4,q^5,q^7,q^{25},q^{27},q^{28};q^{32})_\infty(q^{16},q^{24},q^{40},q^{48};q^{64})_\infty}{(q^8,q^{12},q^{20},q^{24};q^{32})_\infty(q^8,q^{12},q^{20},q^{32},q^{32},q^{44},q^{52},q^{56};q^{64})_\infty}\\ &\quad\left.-\frac{q^2(q^4,q^4,q^8,q^{24},q^{28},q^{28};q^{32})_\infty(q^{18},q^{22},q^{42},q^{46},q^{60};q^{64})_\infty}{(q^9,q^{11},q^{21},q^{23};q^{32})_\infty(q^8,q^{12},q^{20},q^{32},q^{32},q^{44},q^{52},q^{56},q^{68};q^{64})_\infty}\right].\end{aligned} \tag{4.9}$$

This proves (3.1).

Proof of (3.2).

Considering (4.2),

$$B(q)+B'(q)=\frac{2[q,q^{17};q^{32}]_\infty[q^{26};q^{64}]_\infty}{[q^4;q^{32}]_\infty[q^{12},q^{14},q^{32};q^{64}]_\infty}+\frac{2q[q^3,q^{13};q^{32}]_\infty[q^{34};q^{64}]_\infty}{[q^4;q^{32}]_\infty[q^{14},q^{20},q^{32};q^{64}]_\infty}$$

Multiplying (4.2) by $\dfrac{[q^{12},q^{20};q^{64}]_\infty}{[q^{14},q^{46};q^{64}]_\infty}$,

$$\begin{aligned}B(q)^{-1}+B'(q)^{-1}&=\frac{2[q,q^{17};q^{32}]_\infty[q^{20},q^{26};q^{64}]_\infty}{[q^4;q^{32}]_\infty[q^{14},q^{14},q^{32},q^{46};q^{64}]_\infty}\\ &\quad+\frac{2q[q^3,q^{13};q^{32}]_\infty[q^{12},q^{34};q^{64}]_\infty}{[q^4;q^{32}]_\infty[q^{14},q^{14},q^{32},q^{46};q^{64}]_\infty}\end{aligned} \tag{4.10}$$

$$\alpha_2(q) = \frac{1}{2}[B(q)^{-1} + B'(q)^{-1}] = \frac{[q, q^{17}; q^{32}]_\infty [q^{20}, q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} + \frac{q[q^3, q^{13}; q^{32}]_\infty [q^{12}, q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{14}, q^{32}, q^{46}; q^{64}]_\infty}. \tag{4.11}$$

Again, multiplying (4.4) by $\frac{[q^{12}, q^{20}; q^{64}]_\infty}{[q^{14}, q^{46}; q^{64}]_\infty}$, we get

$$B(q)^{-1} - B'(q)^{-1} = -\frac{2q^2[q^4, q^5, q^7; q^{32}]_\infty [q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} - \frac{2[q^4, q^{24}, q^{28}; q^{32}]_\infty [q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}, q^{68}; q^{64}]_\infty}. \tag{4.12}$$

$$\begin{aligned}\beta_2(q) &= \frac{1}{2}[B(q)^{-1} - B'(q)^{-1}] \\ &= -\frac{q^2[q^4, q^5, q^7; q^{32}]_\infty [q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} \\ &\quad - \frac{[q^4, q^{24}, q^{28}; q^{32}]_\infty [q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}, q^{68}; q^{64}]_\infty}.\end{aligned} \tag{4.13}$$

By adding (4.11) and (4.13), we get

$$\begin{aligned}B(q)^{-1} &= \alpha_2(q) + \beta_2(q) \\ B(q)^{-1} &= \frac{[q, q^{17}; q^{32}]_\infty [q^{20}, q^{26}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} + \frac{q[q^3, q^{13}; q^{32}]_\infty [q^{12}, q^{34}; q^{64}]_\infty}{[q^4; q^{32}]_\infty [q^{14}, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} \\ &\quad - \frac{q^2[q^4, q^5, q^7; q^{32}]_\infty [q^{40}, q^{48}; q^{64}]_\infty}{[q^{20}, q^{24}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}; q^{64}]_\infty} \\ &\quad - \frac{[q^4, q^{24}, q^{28}; q^{32}]_\infty [q^4, q^{18}, q^{22}; q^{64}]_\infty}{q^2[q^9, q^{11}; q^{32}]_\infty [q^8, q^{14}, q^{32}, q^{46}, q^{68}; q^{64}]_\infty}.\end{aligned} \tag{4.14}$$

By applying (2.2) in (4.14), we get

$$\begin{aligned}B(q)^{-1} &= \frac{(q, q^{17}, q^{15}, q^{31}; q^{32})_\infty (q^{20}, q^{26}, q^{38}, q^{44}; q^{64})_\infty}{(q^4, q^{28}; q^{32})_\infty (q^{14}, q^{14}, q^{18}, q^{32}, q^{32}, q^{46}, q^{50}, q^{50}; q^{64})_\infty} \\ &\quad + \frac{q(q^3, q^{13}, q^{19}, q^{29}; q^{32})_\infty (q^{12}, q^{30}, q^{34}, q^{52}; q^{64})_\infty}{(q^4, q^{28}; q^{32})_\infty (q^{14}, q^{14}, q^{32}, q^{32}, q^{46}, q^{50}, q^{50}; q^{64})_\infty} \\ &\quad - \frac{q^2(q^4, q^5, q^7, q^{25}, q^{27}, q^{28}; q^{32})_\infty (q^{16}, q^{24}, q^{40}, q^{48}; q^{64})_\infty}{(q^8, q^{12}, q^{20}, q^{24}; q^{32})_\infty (q^8, q^{14}, q^{18}, q^{32}, q^{32}, q^{46}, q^{50}, q^{56}; q^{64})_\infty} \\ &\quad + \frac{q^2(q^4, q^4, q^8, q^{24}, q^{28}, q^{28}; q^{32})_\infty (q^{22}, q^{42}, q^{60}; q^{64})_\infty}{(q^9, q^{11}, q^{21}, q^{23}; q^{32})_\infty [q^8, q^{14}, q^{32}, q^{32}, q^{50}, q^{56}, q^{68}; q^{64})_\infty}.\end{aligned} \tag{4.15}$$

Now, from (4.8), can be written as

$$B(q)^{-1}\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}=\frac{(q^3;q^3)_\infty}{(q^6;q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty}. \tag{4.16}$$

By using (4.15) in (4.16)

$$\begin{aligned}
&\frac{(q^3;q^3)_\infty}{(q^6;q^{12})_\infty(q^6,q^{10},q^{16};q^{16})_\infty}\\
&\quad=\Big[\frac{(q,q^{17},q^{15},q^{31};q^{32})_\infty(q^{20},q^{26},q^{38},q^{44};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{14},q^{18},q^{32},q^{32},q^{46},q^{50},q^{50};q^{64})_\infty}\\
&\quad+\frac{q(q^3,q^{13},q^{19},q^{29};q^{32})_\infty(q^{12},q^{30},q^{34},q^{52};q^{64})_\infty}{(q^4,q^{28};q^{32})_\infty(q^{14},q^{14},q^{32},q^{32},q^{46},q^{50},q^{50};q^{64})_\infty}\\
&\quad-\frac{q^2(q^4,q^5,q^7,q^{25},q^{27},q^{28};q^{32})_\infty(q^{16},q^{24},q^{40},q^{48};q^{64})_\infty}{(q^8,q^{12},q^{20},q^{24};q^{32})_\infty(q^8,q^{14},q^{18},q^{32},q^{32},q^{46},q^{50},q^{56};q^{64})_\infty}\\
&\quad+\frac{q^2(q^4,q^4,q^8,q^{24},q^{28},q^{28};q^{32})_\infty(q^{22},q^{42},q^{60};q^{64})_\infty}{(q^9,q^{11},q^{21},q^{23};q^{32})_\infty(q^8,q^{14},q^{32},q^{32},q^{50},q^{56},q^{68};q^{64})_\infty}\Big]\\
&\quad\times\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+2s_2^2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(q^2;q^2)_{s_2}}
\end{aligned}\tag{4.17}$$

This proves (3.2).

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