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ON THE EXISTENCE OF GENERALISED FIX POINTS OF FUNCTIONS OF CLASS II

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Abstract: Introducing the idea of generalised iterations of three functions we prove fixed point theorem for functions of class II to generalise some earlier results.

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1. Introduction

A single valued function $f(z)$ is said to belong to class I if $f(z)$ is entire transcendental and class II if it is regular in the complex plane punctured at a, $b\,(a \neq b)$ and has an essential singularity at b and a singularity at a and if $f(z)$ omits the values a and b except possible at a.
To normalise the functions in class II we take $a = 0$ and $b = \infty$.

The iterations of the complex function $f(z)$ are defined by

$$f_0(z) = z \text{ and } f_{n+1}(z) = f(f_n(z))\,;\ n = 0, 1, 2, ... \quad .$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z$, $k = 1, 2, 3, ..., n-1$.

Baker [1] proved the following theorem for functions of class I.

Theorem 1.1. [1] *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n except for at most one value of n.*

Bhattacharyya [5] extended this theorem to functions in class II.

Theorem 1.2. [5] *If $f(z)$ belongs to class II, then $f(z)$ has an infinity of fix points of exact order n, for every positive integer n.*

In [8] Lahiri and Banerjee generalised the theorem in another direction. They introduced the concept of relative fix points defined as follows.

Let $f(z)$ and $g(z)$ be functions of complex variable z.
Let

$$\begin{aligned} f_1(z) &= f(z) \\ f_2(z) &= f(g(z)) = f(g_1(z)) \\ f_3(z) &= f(g(f(z))) = f(g(f_1(z))) \\ &\vdots \\ f_n(z) &= f(g(f(g...(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even})...))) \\ &= f(g_{n-1}(z)) = f(g(f_{n-2}(z))) \end{aligned}$$

and so

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f_2(z)) = g(f(g_1(z))) \\ &\vdots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly $f_n(z)$ and $g_n(z)$ are functions in class II, if $f(z)$ and $g(z)$ are so.

A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$, if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$, $k = 1, 2, 3, ..., n-1$. Such points α are also called relative fix points.

Theorem 1.3. [8] *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ is bounded.*

In [3] Banerjee and Mandal proved the result of Lahiri and Banerjee [8] by introducing the idea of relative fix point of exact factor order n.
A point α is called a relative fix point of $f(z)$ of exact factor order n if $f_n(\alpha)=\alpha$ but $f_k(\alpha)\neq\alpha$ and $g_k(\alpha)\neq\alpha$ for all divisors $k\ (<n)$ of n.

Theorem 1.4. [3] *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has an infinity of relative fix points of exact factor order n for every positive integer n, provided $\frac{T(r,f_{n-1})}{T(r,f_n)}$ is bounded.*
Banerjee and Mondal [2] introduced the idea of generalised iteration as follows.

Let $f(z)$ and $g(z)$ be two entire functions and $\alpha\in(0,1]$ be any number. Then the generalised iteration of $f(z)$ with respect to $g(z)$ is defind as follows.

$$\begin{aligned}
f_1(z) &= (1-\alpha)z+\alpha f(z)\\
f_2(z) &= (1-\alpha)g_1(z)+\alpha f(g_1(z))\\
f_3(z) &= (1-\alpha)g_2(z)+\alpha f(g_2(z))\\
&\vdots\\
f_n(z) &= (1-\alpha)g_{n-1}(z)+\alpha f(g_{n-1}(z))
\end{aligned}$$

and

$$\begin{aligned}
g_1(z) &= (1-\alpha)z+\alpha g(z)\\
g_2(z) &= (1-\alpha)f_1(z)+\alpha g(f_1(z))\\
g_3(z) &= (1-\alpha)f_2(z)+\alpha g(f_2(z))\\
&\vdots\\
g_n(z) &= (1-\alpha)f_{n-1}(z)+\alpha g(f_{n-1}(z)).
\end{aligned}$$

Clearly if $f(z)$ and $g(z)$ are functions in class II, then so also are $f_n(z)$ and $g_n(z)$.
Note 1.1. The generalised iteration reduces to relative itaration if $\alpha=1$.
A point β is called a generalised fix point of $f(z)$ of order n if $f_n(\beta)=\beta$ and a generalised fix point of exact order n if $f_n(\beta)=\beta$ but $f_k(\beta)\neq\beta$, $k=1,2,3,...,n-1$. β is called a generalised fix point of $f(z)$ of exact factor order n if $f_n(\beta)=\beta$ but $f_k(\beta)\neq\beta$ and $g_k(\beta)\neq\beta$ for all divisors $k\ (<n)$ of n.

Theorem 1.5. [4] *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has an infinity of generalised fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ is bounded.*

Theorem 1.6. [4] *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has an infinity of generalised fix points of exact factor order n for every positive integer n, provided*

$\frac{T(r,g_n)}{T(r,f_n)}$ *is bounded.*
Now we introduce the generalised iteration of three functions.

Let $f(z)$, $g(z)$ and $h(z)$ be three entire functions and $\alpha \in (0,1]$ be any number. Then the generalised iteration of $f(z)$ with respect to $g(z)$ is defined as follows.

$$\begin{aligned} f_1(z) &= (1-\alpha)z + \alpha f(z) \\ f_2(z) &= (1-\alpha)g_1(z) + \alpha f(g_1(z)) \\ f_3(z) &= (1-\alpha)g_2(z) + \alpha f(g_2(z)) \\ f_4(z) &= (1-\alpha)g_3(z) + \alpha f(g_3(z)) \\ &\vdots \\ f_n(z) &= (1-\alpha)g_{n-1}(z) + \alpha f(g_{n-1}(z)). \end{aligned}$$

Similarly

$$\begin{aligned} g_1(z) &= (1-\alpha)z + \alpha g(z) \\ g_2(z) &= (1-\alpha)h_1(z) + \alpha g(h_1(z)) \\ g_3(z) &= (1-\alpha)h_2(z) + \alpha g(h_2(z)) \\ g_4(z) &= (1-\alpha)h_3(z) + \alpha g(h_3(z)) \\ &\vdots \\ g_n(z) &= (1-\alpha)f_{n-1}(z) + \alpha g(f_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} h_1(z) &= (1-\alpha)z + \alpha h(z) \\ h_2(z) &= (1-\alpha)f_1(z) + \alpha h(f_1(z)) \\ h_3(z) &= (1-\alpha)f_2(z) + \alpha h(f_2(z)) \\ h_4(z) &= (1-\alpha)f_3(z) + \alpha h(f_3(z)) \\ &\vdots \\ h_n(z) &= (1-\alpha)f_{n-1}(z) + \alpha h(f_{n-1}(z)). \end{aligned}$$

Note 1.2. The generalised iteration reduces to relative iteration if $\alpha = 1$. Clearly if $f(z)$, $g(z)$ and $h(z)$ are functions in class II, then so also are $f_n(z)$, $g_n(z)$ and $h_n(z)$.
Now we introduce the following definition.

Definition 1.7. *A point β is called generalised fix point of $f(z)$ of order n if*

$f_n(\beta) = \beta$ and a generalised fix point of $f(z)$ of exact order n if $f_n(\beta) = \beta$ but $f_k(\beta) \neq \beta$, $k = 1, 2, ..., n-1$. β is called a generalised fix point of $f(z)$ with respect to $g(z)$ and $h(z)$ of exact factor order n if $f_n(\beta) = \beta$ but $f_k(\beta) \neq \beta$, $g_k(\beta) \neq \beta$ and $h_k(\beta) \neq \beta$ for all divisors $k\,(k < n)$ of n.

Example 1.8. Let $f(z) = 2z+2$, $g(z) = 2z-2$, $h(z) = 2z+1$ and $\alpha \in (0, 1]$. Then $z = -\frac{2\alpha^4+7\alpha^3+6\alpha^2+3\alpha}{\alpha^4+4\alpha^3+6\alpha^2+4\alpha}$ is generalised fix point of exact order 4 and also generalised fix point of exact factor order 4 of $f(z)$.

Let $f(z)$ be meromorphic in $r_0 \leq |z| < \infty, r_0 > 0$.

From the first fundamental theorem we have

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(\log r), \tag{1}$$

where $r_0 \leq |z| < \infty$, $r_0 > 0$.

Suppose that $f(z)$ is non-constant. Let $a_{1,}a_{2,...,}a_q; q \geq 2$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\upsilon| \geq \delta$ for $1 \leq \mu \leq \upsilon \leq q$. Then

$$m(r, f) + \sum_{\upsilon=1}^{q} m(r, a_\upsilon, f) \leq 2T(r, f) - N_1(r) + S(r), \tag{2}$$

where

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f'),$$

and

$$S(r) = m\left(r, \frac{f'}{f}\right) + \sum_{\upsilon=1}^{q} m\left(r, \frac{f'}{f - a_\upsilon}\right) + O(\log r).$$

Adding $N(r, f) + \sum_{\upsilon=1}^{q} N(r, a_\upsilon, f)$ to both sides of (2) and using (1), we obtain

$$(q-1)\,T(r, f) \leq \overline{N}(r, f) + \sum_{\upsilon=1}^{q} \overline{N}(r, a_\upsilon, f) + S_1(r), \tag{3}$$

where $S_1(r) = O(\log T(r, f))$ and $\overline{N}$ corresponds to distinct roots.

Again if f_n has an essential singularity at ∞, we have $\frac{\log r}{T(r, f_n)} \to 0$ as $r \to \infty$.

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. *If f, g and h are functions in class II, then for any $r_0 > 0$ and M, a positive constant $\frac{T(r,f(g))}{T(r,g)} > M$, $\frac{T(r,g(h))}{T(r,h)} > M$ and $\frac{T(r,h(f))}{T(r,f)} > M$ for all large r, except a set of r intervals of total finite length.*

This follows from the lemma of Lahiri and Banerjee [8].

Lemma 2.2. *If n is any positive integer and $f(z)$, $g(z)$ and $h(z)$ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant*

$$\frac{T(r, f_{n+p})}{T(r, f_n)} > M_1, \ \frac{T(r, g_{n+p})}{T(r, f_n)} > M_1 \ \text{and} \ \frac{T(r, h_{n+p})}{T(r, f_n)} > M_1$$

according as $p = 3m$ or $3m-1$ or $3m-2$; $m \in \mathbb{N}$, for all large r except a set of r intervals of total finite length.

Proof. For $j = 1, 2, ..., n$ and for all large r, by using Lemma 2.1, we get

$$\begin{aligned} T(r, f_{j+1}) &\leq T(r, (1-\alpha) g_j) + T(r, \alpha f(g_j)) + O(1) \\ &\leq T(r, g_j) + T(r, f(g_j)) + O(1) \\ &= T(r, f(g_j)) \left[1 + \frac{T(r, g_j)}{T(r, f(g_j))} + \frac{O(1)}{T(r, f(g_j))}\right] \\ &= (1 + O(1)) T(r, f(g_j)) \quad . \end{aligned} \tag{4}$$

Again $f(g_j(z)) = \frac{1}{\alpha} f_{j+1}(z) - \frac{1-\alpha}{\alpha} g_j(z)$ and so for large r

$$T(r, f(g_j)) \leq T(r, f_{j+1}) + T(r, g_j) + O(1) \quad .$$

Therefore

$$\begin{aligned} T(r, f_{j+1}) &\geq T(r, f(g_j)) - T(r, g_j) + O(1) \\ &= T(r, f(g_j)) \left[1 - \frac{T(r, g_j)}{T(r, f(g_j))} + \frac{O(1)}{T(r, f(g_j))}\right] \\ &= (1 + O(1)) T(r, f(g_j)) \quad . \end{aligned} \tag{5}$$

From (4) and (5) for all large r, we have

$$T(r, f_{j+1}) = (1 + O(1)) T(r, f(g_j)) \quad . \tag{6}$$

Similarly for large r, we have

$$T(r, g_{j+1}) = (1 + O(1)) T(r, g(h_j)) \tag{7}$$

and

$$T(r, h_{j+1}) = (1 + O(1)) T(r, h(f_j)) \quad . \tag{8}$$

Case I. When $p = 3m, m \in \mathbb{N}$. For all large r except a set of r intervals of total finite length, we have from (6), (7) and (8) by using Lemma 2.1

$$\begin{aligned}
\frac{T(r, f_{n+p})}{T(r, f_n)} &= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, g_{n+p-1})} \frac{T(r, g_{n+p-1})}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, g_{n+p-1})} \frac{(1 + O(1))T(r, g(h_{n+p-2}))}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, g_{n+p-1})} \frac{T(r, g(h_{n+p-2}))}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, g_{n+p-1})} \frac{T(r, g(h_{n+p-2}))}{T(r, h_{n+p-2})} \frac{T(r, h_{n+p-2})}{T(r, f_n)} \\
&\vdots \\
&= (1 + O(1)) \frac{T(r, f(g_{n+p-1}))}{T(r, g_{n+p-1})} \frac{T(r, g(h_{n+p-2}))}{T(r, h_{n+p-2})} \frac{T(r, h(f_{n+p-3}))}{T(r, f_{n+p-3})} \\
&\quad \ldots \frac{T(r, h(f_n))}{T(r, f_n)} \\
&> (1 + O(1)) M^p \\
&= M_1 \text{ say, where } M_1 = (1 + O(1)) M^p, \text{ a positive constant}
\end{aligned}$$

i.e,

$$\frac{T(r, f_{n+p})}{T(r, f_n)} > M_1$$

for all large r except a set of r intervals of total finite length.

Case II. When $p = 3m - 1$, $m \in \mathbb{N}$. For all large r except a set of r intervals of total finite length, we have from (6), (7) and (8) by using Lemma 2.1

$$\begin{aligned}
\frac{T(r, g_{n+p})}{T(r, f_n)} &= (1 + O(1)) \frac{T(r, g(h_{n+p-1}))}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, g(h_{n+p-1}))}{T(r, h_{n+p-1})} \frac{T(r, h_{n+p-1})}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, g(h_{n+p-1}))}{T(r, h_{n+p-1})} \frac{(1 + O(1))T(r, h(f_{n+p-2}))}{T(r, f_n)} \\
&= (1 + O(1)) \frac{T(r, g(h_{n+p-1}))}{T(r, h_{n+p-1})} \frac{T(r, h(f_{n+p-2}))}{T(r, f_n)}
\end{aligned}$$

$$
\begin{aligned}
&= (1+O(1))\frac{T(r,g(h_{n+p-1}))}{T(r,h_{n+p-1})}\frac{T(r,h(f_{n+p-2}))}{T(r,f_{n+p-2})}\frac{T(r,f_{n+p-2})}{T(r,f_n)} \\
&\vdots \\
&= (1+O(1))\frac{T(r,g(h_{n+p-1}))}{T(r,h_{n+p-1})}\frac{T(r,h(f_{n+p-2}))}{T(r,f_{n+p-2})}\frac{T(r,f(g_{n+p-3}))}{T(r,g_{n+p-3})} \\
&\quad \ldots\frac{T(r,h(f_n))}{T(r,f_n)} \\
&> (1+O(1))M^p \\
&= M_1 \text{ say, where } M_1=(1+O(1))M^p,\ \text{a positive constant}
\end{aligned}
$$

i.e,

$$
\frac{T(r,g_{n+p})}{T(r,f_n)} > M_1
$$

for all large r except a set of r intervals of total finite length.

Case III. When $p = 3m-2, m \in \mathbb{N}$. For all large except a set of r interval of total finite length, we have from (6), (7) and (8) by using Lemma 2.1

$$
\begin{aligned}
\frac{T(r,h_{n+p})}{T(r,f_n)} &= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_n)} \\
&= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_{n+p-1})}\frac{T(r,f_{n+p-1})}{T(r,f_n)} \\
&= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_{n+p-1})}\frac{(1+O(1))T(r,f(g_{n+p-2}))}{T(r,f_n)} \\
&= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_{n+p-1})}\frac{T(r,f(g_{n+p-2}))}{T(r,f_n)} \\
&= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_{n+p-1})}\frac{T(r,f(g_{n+p-2}))}{T(r,g_{n+p-2})}\frac{T(r,g_{n+p-2})}{T(r,f_n)} \\
&\vdots \\
&= (1+O(1))\frac{T(r,h(f_{n+p-1}))}{T(r,f_{n+p-1})}\frac{T(r,f(g_{n+p-2}))}{T(r,g_{n+p-2})}\frac{T(r,g(h_{n+p-3}))}{T(r,h_{n+p-3})} \\
&\quad \ldots\frac{T(r,h(f_n))}{T(r,f_n)} \\
&> (1+O(1))M^p \\
&= M_1 \text{ say, where } M_1=(1+O(1))M^p,\ \text{a positive constant}
\end{aligned}
$$

i.e,

$$\frac{T(r,h_{n+p})}{T(r,f_n)} > M_1$$

for all large r except a set of r intervals of total finite length.

Lemma 2.3. *If n is any positive integer and $f(z)$, $g(z)$ and $h(z)$ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant*

$$\frac{T(r,g_{n+p})}{T(r,g_n)} > M_1,\ \frac{T(r,h_{n+p})}{T(r,g_n)} > M_1 \text{ and } \frac{T(r,f_{n+p})}{T(r,g_n)} > M_1$$

according as $p = 3m$ or $3m-1$ or $3m-2; m \in \mathbb{N}$, for all large r except a set of r intervals of total finite length.

Lemma 2.4. *If n is any positive integer and $f(z)$, $g(z)$ and $h(z)$ are functions in class II, then for any $r_0 > 0$ and M_1, a positive constant*

$$\frac{T(r,h_{n+p})}{T(r,h_n)} > M_1,\ \frac{T(r,f_{n+p})}{T(r,h_n)} > M_1 \text{ and } \frac{T(r,g_{n+p})}{T(r,h_n)} > M_1$$

according as $p = 3m$ or $3m-1$ or $3m-2; m \in \mathbb{N}$, for all large r except a set of r intervals of total finite length.

3. Theorem

Theorem 3.1. *If $f(z)$, $g(z)$ and $h(z)$ belong to class II, then $f(z)$ has infinity of generalised fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded.*

Proof. We consider the function

$$\phi(z) = \frac{f_n(z)}{z}\ ,\ r_0 < |z| < \infty\ .$$

Then

$$T(r,\phi) = T(r,f_n) + O(\log r)\,. \tag{9}$$

We assume that $f(z)$ has only a finite number of generalised fix points of exact order n. From (3) by taking $q = 2$, $a_1 = 0$, $a_2 = 1$, we obtain

$$T(r,\phi) \le \overline{N}(r,\infty,\phi) + \overline{N}(r,0,\phi) + \overline{N}(r,1,\phi) + S_1(r,\phi)\,, \tag{10}$$

where $S_1(r,\phi) = O(\log T(r,\phi))$ outside a set of r intervals of finite length [7]. Now we have

$$\overline{N}(r,0,\phi) = \int_{r_0}^{r} \frac{\overline{n}(t,0,\phi)}{t}dt$$

where $\overline{n}(t,0,\phi)$ is the number of roots of $\phi(z)=0$ in $r_0<|z|\leq t$, each multiple root taken once at a time. The distinct roots of $\phi(z)=0$ in $r_0<|z|\leq t$ are the roots of $f_n(z)=0$ in $r_0<|z|\leq t$. Now $f_n(z)$ has a singularity at $z=0$, an essential singularity at $z=\infty$, and $f_n(z)\neq 0,\infty$. So $\overline{n}(t,0,\phi)=0$. So $\overline{N}(r,0,\phi)=0$. By similar argument $\overline{N}(r,\infty,\phi)=0$. So

$$T(r,\phi)\leq \overline{N}(r,1,\phi)+S_1(r,\phi)$$

We now calculate $\overline{N}(r,1,\phi)$. If $\phi(z)=1$, then $f_n(z)=z$.

So

$$\begin{aligned}\overline{N}(r,1,\phi) &= \overline{N}(r,0,f_n-z)\\ &\leq \sum_{j=1}^{n-1}\overline{N}(r,0,f_j-z)+O(\log r),\end{aligned}$$

the term $O(\log r)$ arises due to the assumption that $f(z)$ has only a finite number of generalised fix points of exact order n.

Now from (10), we have

$$\begin{aligned}T(r,\phi) &\leq \sum_{j=1}^{n-1}\overline{N}(r,0,f_j-z)+O(\log r)+O(\log T(r,\phi))\\ &\leq \sum_{j=1}^{n-1}T(r,f_j)+O(\log T(r,\phi))+O(\log r)\\ &= T(r,f_n)[\frac{T(r,f_{j_3})}{T(r,f_n)}+\frac{T(r,f_{j_6})}{T(r,f_n)}+...+\frac{T(r,f_{j_{3s}})}{T(r,f_n)}\\ &\quad +\{\frac{T(r,f_{j_1})}{T(r,g_n)}+\frac{T(r,f_{j_4})}{T(r,g_n)}+...+\frac{T\left(r,f_{j_{3p-2}}\right)}{T(r,g_n)}\}\frac{T(r,g_n)}{T(r,f_n)}\\ &\quad +\{\frac{T(r,f_{j_2})}{T(r,h_n)}+\frac{T(r,f_{j_5})}{T(r,h_n)}+...+\frac{T\left(r,f_{j_{3q-1}}\right)}{T(r,h_n)}\}\frac{T(r,h_n)}{T(r,f_n)}\\ &\quad +\frac{O\left(\log\{T(r,f_n)\left(1+\frac{O(\log r)}{T(r,f_n)}\right)\}\right)}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)}],\end{aligned}$$

where $j_1,j_4,...,j_{3p-2};\ j_2,j_5,...,j_{3q-1};$

$j_3,j_6,...,j_{3s}$ are divisors of n and are strictly less than n and are of the forms $3p-2, 3q-1$ and $3s \ (p,q,s \in \mathbb{N})$

$$< \ T(r,f_n)\left[\frac{n-1}{6n}+\frac{n-1}{6n}+\frac{n-1}{6n}\right], \text{ for all large } r \text{ by Lemma 2.2}$$

and since $\frac{T(r,g_n)}{T(r,f_n)}, \frac{T(r,h_n)}{T(r,f_n)}$ are bounded

$$< \ \frac{1}{2}T(r,f_n) \ .$$

Therefore $T(r,\phi) < \frac{1}{2}T(r,f_n)$, for all large r. This contradicts (9).
Hence $f(z)$ has infinitely many generalised fix points of exact order n.
This proves the theorem.

Theorem 3.2. *If $f(z)$, $g(z)$ and $h(z)$ belong to class II, then $f(z)$ has an infinity of generalised fix points of exact factor order n, for any positive integer n, provided $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded.*

Proof. As in Theorem 3.1, we assume that $f(z)$ has only a finite number of generalised fix points of exact factor order n.
Considering the function

$$w(z) = \frac{f_n(z)}{z} \ , \ r_0 < |z| < \infty \ .$$

We have

$$T(r,w) = T(r,f_n) + O(\log r) \ . \tag{11}$$

Now $\overline{N}(r,0,w) = 0$ and $\overline{N}(r,\infty,w) = 0$.
We consider following three cases to calculate $\overline{N}(r,1,w)$.
Case I. When $n = 3m, m \in \mathbb{N}$.

$$\begin{aligned}
\overline{N}(r, \ 1, \ w) &= \overline{N}(r,0,f_n - z) \\
&\le \sum_{j/n, j=1}^{n-2} [\overline{N}(r,0,f_j - z) + \overline{N}(r,0,g_j - z) + \overline{N}(r,0,h_j - z) + O(\log r)] \\
&\le \sum_{j/n, j=1}^{n-2} [T(r,f_j) + T(r,g_j) + T(r,h_j)] + O(\log r) \\
&= T(r,f_n)\left[\frac{T(r,f_{j_3})}{T(r,f_n)} + \frac{T(r,f_{j_6})}{T(r,f_n)} + \ldots + \frac{T(r,fj_{3s})}{T(r,f_n)} + \frac{T(r,g_{j_1})}{T(r,f_n)}\right. \\
&\quad + \frac{T(r,g_{j_4})}{T(r,f_n)} + \ldots + \frac{T\left(r,g_{j_{3p-2}}\right)}{T(r,f_n)} + \frac{T(r,h_{j_2})}{T(r,f_n)} + \frac{T(r,h_{j_5})}{T(r,f_n)} + \ldots
\end{aligned}$$

$$+\frac{T\left(r,h_{j_{3q-1}}\right)}{T\left(r,f_n\right)}]+T\left(r,g_n\right)[\frac{T\left(r,f_{j_2}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,f_{j_5}\right)}{T\left(r,g_n\right)}+...+\frac{T\left(r,f_{j_{3q-1}}\right)}{T\left(r,g_n\right)}$$
$$+\frac{T\left(r,g_{j_3}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,g_{j_6}\right)}{T\left(r,g_n\right)}+...+\frac{T\left(r,g_{3s}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,h_{j_1}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,h_{j_4}\right)}{T\left(r,g_n\right)}$$
$$+...+\frac{T\left(r,h_{j_{3p-2}}\right)}{T\left(r,g_n\right)}]+T\left(r,h_n\right)[\frac{T\left(r,f_{j_1}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,f_{j_4}\right)}{T\left(r,h_n\right)}+...+\frac{T\left(r,f_{j_{3p-2}}\right)}{T\left(r,h_n\right)}$$
$$+\frac{T\left(r,g_{j_2}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,g_{j_5}\right)}{T\left(r,h_n\right)}+...+\frac{T\left(r,g_{j_{3q-1}}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,h_{j_3}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,h_{j_6}\right)}{T\left(r,h_n\right)}$$
$$+...+\frac{T\left(r,h_{j_{3s}}\right)}{T\left(r,h_n\right)}]+O\left(\log r\right),\ \text{where}\ \ j_1,j_4,...,j_{3p-2};\ j_2,j_5,...,j_{3q-1};$$

$j_3,j_6,...,j_{3s}$ are divisors of n and are strictly less than n and are of the forms $3p-2, 3q-1$ and $3s$ $(p,q,s\in\mathbb{N})$

$$<\ \frac{n-1}{6n}T\left(r,f_n\right)+\frac{n-1}{6n}T\left(r,g_n\right)+\frac{n-1}{6n}T\left(r,h_n\right)+O\left(\log r\right),\ \text{for}$$

all large r, by Lemma 2.2, Lemma 2.3 and Lemma 2.4.

Case II. When $n=3m+1, m\in\mathbb{N}$.

$$\begin{aligned}\overline{N}(r,\ 1,\ w) &= \overline{N}\left(r,0,f_n-z\right)\\
&\leq \sum_{j/n,j=1}^{n-1}\left[\overline{N}\left(r,0,f_j-z\right)+\overline{N}\left(r,0,g_j-z\right)+\overline{N}\left(r,0,h_j-z\right)\right]+O\left(\log r\right)\\
&\leq \sum_{j/n,j=1}^{n-2}\left[T\left(r,f_j\right)+T\left(r,g_j\right)+T\left(r,h_j\right)\right]+O\left(\log r\right)\\
&= T\left(r,f_n\right)[\frac{T\left(r,f_{j_1}\right)}{T\left(r,f_n\right)}+\frac{T\left(r,f_{j_4}\right)}{T\left(r,f_n\right)}+...+\frac{T\left(r,f_{j_{3p-2}}\right)}{T\left(r,f_n\right)}+\frac{T\left(r,g_{j_2}\right)}{T\left(r,f_n\right)}\\
&\quad+\frac{T\left(r,g_{j_5}\right)}{T\left(r,f_n\right)}+...+\frac{T\left(r,g_{j_{3q-1}}\right)}{T\left(r,f_n\right)}]+T\left(r,g_n\right)[\frac{T\left(r,g_{j_1}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,g_{j_4}\right)}{T\left(r,g_n\right)}\\
&\quad+...+\frac{T\left(r,g_{j_{3p-2}}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,h_{j_2}\right)}{T\left(r,g_n\right)}+\frac{T\left(r,h_{j_5}\right)}{T\left(r,g_n\right)}+...+\frac{T\left(r,h_{j_{3q-1}}\right)}{T\left(r,g_n\right)}]\\
&\quad+T\left(r,h_n\right)[\frac{T\left(r,h_{j_1}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,h_{j_4}\right)}{T\left(r,h_n\right)}+...+\frac{T\left(r,h_{j_{3p-2}}\right)}{T\left(r,h_n\right)}+\frac{T\left(r,f_{j_2}\right)}{T\left(r,h_n\right)}\\
&\quad+\frac{T\left(r,f_{j_5}\right)}{T\left(r,h_n\right)}+...+\frac{T\left(r,f_{j_{3q-1}}\right)}{T\left(r,h_n\right)}]+O\left(\log r\right),\ \text{where}\ j_1,j_4,...,j_{3p-2}\end{aligned}$$

and $j_2, j_5, ..., j_{3q-1}$ are divisors of n and are strictly less than n and are of the forms $3p-2$ and $3q-1$ $(p, q, \in \mathbb{N})$

$$< \frac{n-1}{6n}T(r, f_n) + \frac{n-1}{6n}T(r, g_n) + \frac{n-1}{6n}T(r, h_n) + O(\log r).$$

Case III. When $n = 3m+2, m \in \mathbb{N}$, we have

$$\begin{aligned}
\overline{N}(r, 1, w) &= \overline{N}(r, 0, f_n - z) \\
&\leq \sum_{j/n, j=1}^{n-2} [\overline{N}(r, 0, f_j - z) + \overline{N}(r, 0, g_j - z) + \overline{N}(r, 0, h_j - z)] + O(\log r) \\
&\leq \sum_{j/n, j=1}^{n-2} [T(r, f_j) + T(r, g_j) + T(r, h_j)] + O(\log r) \\
&= T(r, f_n)[\frac{T(r, f_{j_2})}{T(r, f_n)} + \frac{T(r, f_{j_5})}{T(r, f_n)} + ... + \frac{T\left(r, f_{j_{3q-1}}\right)}{T(r, f_n)} + \frac{T(r, h_{j_1})}{T(r, f_n)} \\
&\quad + \frac{T(r, h_{j_4})}{T(r, f_n)} + ... + \frac{T\left(r, h_{j_{3p-2}}\right)}{T(r, f_n)}] + T(r, g_n)[\frac{T(r, f_{j_1})}{T(r, g_n)} + \frac{T(r, f_{j_4})}{T(r, g_n)} \\
&\quad + ... + \frac{T\left(r, f_{j_{3p-2}}\right)}{T(r, g_n)} + \frac{T(r, g_{j_2})}{T(r, g_n)} + \frac{T(r, g_{j_5})}{T(r, g_n)} + ... + \frac{T\left(r, g_{j_{3q-1}}\right)}{T(r, g_n)}] \\
&\quad + T(r, h_n)[\frac{T(r, g_{j_1})}{T(r, h_n)} + \frac{T(r, g_{j_4})}{T(r, h_n)} + ... + \frac{T\left(r, g_{j_{3p-2}}\right)}{T(r, h_n)} + \frac{T(r, h_{j_2})}{T(r, h_n)} \\
&\quad + \frac{T(r, h_{j_5})}{T(r, h_n)} + ... + \frac{T\left(r, h_{j_{3q-1}}\right)}{T(r, h_n)}] + O(\log r), \text{ where } j_1, j_4, ..., j_{3p-2}
\end{aligned}$$

and $j_2, j_5, ..., j_{3q-1}$ are divisors of n and are strictly less than n and are of the form $3p-2$ and $3q-1$, $(p, q \in \mathbb{N})$

$$< \frac{n-1}{6n}T(r, f_n) + \frac{n-1}{6n}T(r, g_n) + \frac{n-1}{6n}T(r, h_n) + O(\log r).$$

Therefore in any case

$$\overline{N}(r, 1, w) < \frac{n-1}{6n}T(r, f_n) + \frac{n-1}{6n}T(r, g_n) + \frac{n-1}{6n}T(r, h_n) + O(\log r).$$

Since $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded, we have

$$\begin{aligned}
T(r,w) &\leq \overline{N}(r,1,w)+S_1(r) \\
&< \frac{n-1}{6n}T(r,f_n)+\frac{n-1}{6n}T(r,g_n)+\frac{n-1}{6n}T(r,h_n)+O(\log r) \\
&\quad +O(\log T(r,w)) \\
&= T(r,f_n)\,[\frac{n-1}{6n}+\frac{n-1}{6n}\frac{T(r,g_n)}{T(r,f_n)}+\frac{n-1}{6n}\frac{T(r,h_n)}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)} \\
&\quad +\frac{O(\log T(r,w))}{T(r,f_n)}] \\
&\leq T(r,f_n)\,[\frac{n-1}{6n}+\frac{n-1}{6n}+\frac{n-1}{6n}+\frac{O(\log r)}{T(r,f_n)} \\
&\quad +\frac{O(\log(T(r,f_n)+O(\log r)))}{T(r,f_n)}] \\
&= T(r,f_n)\,[\frac{(n-1)}{2n}+\frac{O(\log r)}{T(r,f_n)}+\frac{O\left(\log\left(T(r,f_n)\left(1+\frac{O(\log r)}{T(r,f_n)}\right)\right)\right)}{T(r,f_n)}] \\
&< \frac{1}{2}T(r,f_n)\,,\ \text{for all large } r.
\end{aligned}$$

Therefore, $T(r,w)<\frac{1}{2}T(r,f_n)$ for all large r. This contradicts (11).
Hence $f(z)$ has infinitely many generalised fix points of exact factor order n.
This proves the theorem.

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