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FURTHER PROPERTIES OF RIGHT DOWN OR LEFT DOWN IMPLICATION OPERATOR ON IFSs AND IFMs

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Abstract: In this paper, I study boundaries, regularities, first place anti-tonicity, second place isotonicity etc., using $\rightharpoondown$ operator. Further, I check distributive properties of $\rightharpoondown$ operator over $\vee$ and $\wedge$.

Keywords and Phrases: Intuitionistic Fuzzy Matrices (IFMs), Intuitionistic Fuzzy Set (IFS), Intuitionistic Fuzzy right down implication operator (IFRDIO).

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1. Introduction

After the introduction of fuzzy set theory by Zadeh [14] in 1965, fuzzy concept evolved in almost all fields. Hiroshi Hasimoto [4] used implication operator in fuzzy set and extended it to fuzzy Matrix. After the generalization of fuzzy theory Atanassov [2] as Intuitionistic fuzzy Set theory Im et. al., [5] extended it to Intuitionistic fuzzy Matrix. Meenakshi and Gandhimathi [7], Sriram and Murugadas [11, 12] developed this Intuitionistic fuzzy Matrix in finding the g-inverse, Intuitionistic fuzzy linear transformation etc. Sriram and Murugadas [13] extend the implicatin operator $\rightarrow$ to IFM and discussed several properties like sub-inverse, semi-inverse and obtained necessary and sufficient condition for the existence of g-inverse using the implication operator $\rightarrow$. The authors in [6,7,8,9,10] introduced right down or left down implication operators ($\leftharpoondown$ or $\rightharpoondown$) for IFS as well as IFM. I study some properties of it.

2. Preliminaries

We recollect some relevant basic definitions and results will be used later.

Definition 2.1. [2] *An Intuitionistic Fuzzy Set(IFS) A in E (universal set) is defined as an object of the following form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E\}$, *where the functions:* $\mu_A(x) : E \to [0,1]$ *and* $\nu_A(x) : E \to [0,1]$ *define the membership and non-membership functions of the element* $x \in E$ *respectively and for every* $x \in E : 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

For simplicity we consider the pair $\langle x, x'\rangle$ as membership and non-membership function of an IFS with $x + x' \leq 1$.

Definition 2.2. *For* $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, *define*
$\langle x, x'\rangle \vee \langle y, y'\rangle = \langle \max\{x, y\}, \min\{x', y'\}\rangle$.
$\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle \min\{x, y\}, \max\{x', y'\}\rangle$.
$\langle x, x'\rangle$ *and* $\langle y, y'\rangle$ *are comparable, that is* $\langle x, x'\rangle \geq \langle y, y'\rangle$, *if* $x \geq y$ *and* $x' \leq y'$.
$\langle x, x'\rangle^c = \langle x', x\rangle$.

Definition 2.3. *For any two comparable elements* $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, *define*

$$\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle or \langle y, y'\rangle \rightharpoondown \langle x, x'\rangle = \begin{cases} \langle 0, 1\rangle & if\ \langle x, x'\rangle \leq \langle y, y'\rangle \\ \langle x, x'\rangle & if\ \langle x, x'\rangle > \langle y, y'\rangle \end{cases}$$

Definition 2.4. *Let* $X = \{x_1, x_2, ... x_m\}$ *be a set of alternatives and* $Y = \{y_1, y_2, ... y_n\}$ *be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by* $A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j)\rangle)$ *for* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$, *where* $\mu_A : X \times Y \to [0,1]$ *and* $\nu_A : X \times Y \to [0,1]$ *satisfy the condition* $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. *For simplicity we denote an intuitionistic fuzzy matrix (IFM) as matrix of pairs* $A = (\langle a_{ij}, a'_{ij}\rangle)$ *of non negative real numbers satisfying* $a_{ij} + a'_{ij} \leq 1$ *for all* i, j.
For any two elements $A = (\langle a_{ij}, a'_{ij}\rangle)$, $B = (\langle b_{ij}, b'_{ij}\rangle) \in \mathscr{F}_{mn}$ *and* $C \in \mathscr{F}_{np}$, *define*

1. $A \oplus B = (\langle \max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\}\rangle)$, *(component wise addition).*

2. $A \odot B = (\langle \min\{a_{ij}, b_{ij}\}, \max\{a'_{ij}, b'_{ij}\}\rangle)$ *(component wise multiplication) for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$.

3. $J = (\langle 1, 0\rangle)$ *the Universal matrix (matrix in which all entries are* $\langle 1, 0\rangle$*).*

4. $I = (\langle \delta_{ij}, \delta'_{ij}\rangle)$ *(Identity Matrix) where* $\langle \delta_{ij}, \delta'_{ij}\rangle = \begin{cases} \langle 1, 0\rangle & if\ i = j \\ \langle 0, 1\rangle & if\ i \neq j \end{cases}$.

   The Zero matrix O is the matrix in which all the entries are $\langle 0, 1\rangle$.

5. $A \geq B$ *if* $a_{ij} \geq b_{ij}$ *and* $a'_{ij} \leq b'_{ij}$ *for all* i, j *and* $A > B$ *if* $a_{ij} > b_{ij}$ *or* $a'_{ij} < b'_{ij}$ *for all* i, j *in which case* A *and* B *are comparable.*

6. $A^c = (\langle a'_{ij}, a_{ij} \rangle)$ *(complement of* A*).*

7. $AC = (\langle \max_{k=1}^{n} \min\{a_{ik}, c_{kj}\}, \min_{k=1}^{n} \max\{a'_{ik}, c'_{kj}\} \rangle)$
$= (\langle \sum_{k=1}^{n} a_{ik}c_{kj}, \prod_{k=1}^{n}(a'_{ik} + c'_{kj}) \rangle)$.

Here $a_{ik}c_{kj} = \min\{a_{ik}, c_{kj}\}$ *and* $a'_{ik} + c'_{kj} = \max\{a'_{ik}, c'_{kj}\}$. *Some times we can use* $\bigwedge_{k=1}^{n}$ *for* $\prod_{k=1}^{n}$ *and* $\bigvee_{k=1}^{n}$ *for* $\sum_{k=1}^{n}$.

Definition 2.5. *For IFMs* $A = (\langle a_{ij}, a'_{ij} \rangle) \in \mathscr{F}_{mn}$, $B = (\langle b_{ij}, b'_{ij} \rangle) \in \mathscr{F}_{mn}$ *and* $C = (\langle c_{ij}, c'_{ij} \rangle) \in \mathscr{F}_{np}$, *define*
$A \vee B = (\langle a_{ij}, a'_{ij} \rangle \vee \langle b_{ij}, b'_{ij} \rangle)$, *(which is equivalent to* $A \oplus B$*).*
$A \wedge B = (\langle a_{ij}, a'_{ij} \rangle \wedge \langle b_{ij}, b'_{ij} \rangle)$, *(which is equivalent to* $A \odot B$*).*
$A \diamond C = (\bigwedge_k (\langle a_{ik}, a'_{ik} \rangle \vee \langle c_{kj}, c'_{kj} \rangle))$, *(min-max product).*

3. Results Using $\rightharpoondown$ Operator

The boundary and regularities of right down or left down implication operators are trivial from the definition of ($\leftharpoondown$ or $\rightharpoondown$). Throughout this section all the elements in an IFS and IFM are comparable by means of $\leq, <$ and $\geq$.

Theorem 3.1. *IFRDIO* $\rightharpoondown$ *satisfies boundaries*
1. $\langle 0, 1 \rangle \rightharpoondown \langle 0, 1 \rangle = \langle 0, 1 \rangle$. 2. $\langle 0, 1 \rangle \rightharpoondown \langle 1, 0 \rangle = \langle 1, 0 \rangle$.
3. $\langle 1, 0 \rangle \rightharpoondown \langle 0, 1 \rangle = \langle 0, 1 \rangle$. 4. $\langle 1, 0 \rangle \rightharpoondown \langle 1, 0 \rangle = \langle 0, 1 \rangle$.
The following theorem shows that neutrality property and dominance of falsity of antecedent fails.

Theorem 3.2. *Suppose* $\langle a, a' \rangle \neq \langle 1, 0 \rangle \in A$ *is an arbitrary IFS then the IFRDIO* $\rightharpoondown$ *has the following regularities*
1.$\langle a, a' \rangle \rightharpoondown \langle 1, 0 \rangle = \langle 1, 0 \rangle$. *2.*$\langle 1, 0 \rangle \rightharpoondown \langle a, a' \rangle = \langle 0, 1 \rangle$.
3.$\langle 0, 1 \rangle \rightharpoondown \langle a, a' \rangle = \langle a, a' \rangle$.

Lemma 3.3. *If* $\langle a, a' \rangle, \langle b, b' \rangle, \langle c, c' \rangle$*in IFS, then*
1. If $\langle a, a' \rangle \leq b, b' \rangle$ *then* $(\langle b, b' \rangle \rightharpoondown \langle c, c' \rangle) \leq \langle a, a' \rangle \rightharpoondown \langle c, c' \rangle$*(First Place Antitonicity).*
2. If $\langle b, b' \rangle \leq \langle c, c' \rangle$, *then* $(\langle a, a' \rangle \rightharpoondown \langle b, b' \rangle) \leq \langle a, a' \rangle \rightharpoondown \langle c, c' \rangle$ *(Second Place Isotonicity).*
Proof. 1. Case 1.$\langle a, a' \rangle \leq \langle b, b' \rangle \leq c, c' \rangle$,

$$\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle = \begin{cases} \langle 0, 1 \rangle & \text{if } \langle a, a' \rangle \geq \langle c, c' \rangle \\ \langle c, c' \rangle & \text{if } \langle a, a' \rangle < \langle c, c' \rangle \end{cases}$$

$$\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle b, b'\rangle \geq \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle b, b'\rangle < \langle c, c'\rangle. \end{cases}$$

In this case $\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \langle b, b'\rangle \rightharpoondown \langle c, c'\rangle$
Case 2. $\langle a, a'\rangle \leq \langle c, c'\rangle \leq \langle b, b'\rangle$
$\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$. Therefore $\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle \leq \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle$.
Case 3. $\langle c, c'\rangle \leq \langle a, a'\rangle \leq b, b'\rangle$.
Here $\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$ and $\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$.
Thus if $\langle a, a'\rangle \leq b, b'\rangle$ then $(\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle) \leq \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle$
2. Case 1. If $\langle b, b'\rangle \leq \langle c, c'\rangle \leq \langle a, a'\rangle$
In this case $\langle a, a'\rangle \rightharpoondown \langle b, b'\rangle = \langle 0, 1\rangle$ and $\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$
Case 2. $\langle b, b'\rangle \leq \langle a, a'\rangle \leq \langle c, c'\rangle$
Here $\langle a, a'\rangle \rightharpoondown \langle b, b'\rangle = \langle 0, 1\rangle$ and $\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$
Case 3. If $\langle a, a'\rangle \leq \langle b, b'\rangle \leq \langle c, c'\rangle$
In this case

$$\langle a, a'\rangle \rightharpoondown \langle b, b'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a, a'\rangle \geq \langle b, b'\rangle \\ \langle b, b'\rangle & \text{if } \langle a, a'\rangle < \langle b, b'\rangle \end{cases}$$

$$\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$$

Thus if $\langle b, b'\rangle \leq \langle c, c'\rangle$, then $(\langle a, a'\rangle \rightharpoondown \langle b, b'\rangle) \leq \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle$

Lemma 3.4. *If $\langle a, a'\rangle, \ \langle b, b'\rangle, \ \langle c, c'\rangle \in IFS$, then the IFRDIO $\rightharpoondown$ satisfies* $[\langle a, a'\rangle \vee \langle b, b'\rangle] \rightharpoondown \langle c, c'\rangle = (\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle) \wedge (\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle)$.
Proof. Case 1. Assume $\langle a, a'\rangle \leq \langle b, b'\rangle$
sub case (i). $\langle a, a'\rangle \leq \langle b, b'\rangle \leq \langle c, c'\rangle$

$$(\langle a, a'\rangle \vee \langle b, b'\rangle) \rightharpoondown \langle c, c'\rangle = \langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle b, b'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle b, b'\rangle < \langle c, c'\rangle \end{cases}$$

$$\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$$

$$\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle b, b'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle b, b'\rangle < \langle c, c'\rangle \end{cases}$$

sub case (ii). If $\langle a, a'\rangle \leq \langle c, c'\rangle < \langle b, b'\rangle$,
$(\langle a, a'\rangle \vee \langle b, b'\rangle) \rightharpoondown \langle c, c'\rangle = \langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$

$$\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle = \begin{cases} \langle 0, 1 \rangle & \text{if } \langle a, a' \rangle = \langle c, c' \rangle \\ \langle c, c' \rangle & \text{if } \langle a, a' \rangle < \langle c, c' \rangle \end{cases}$$

Clearly $\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle \wedge (\langle b, b' \rangle \rightharpoondown \langle c, c' \rangle) = \langle 0, 1 \rangle$.
sub case (iii). If $\langle c, c' \rangle < \langle a, a' \rangle \leq \langle b, b' \rangle$.
$(\langle a, a' \rangle \vee \langle b, b' \rangle) \rightharpoondown \langle c, c' \rangle = \langle b, b' \rangle \rightharpoondown \langle c, c' \rangle = \langle 0, 1 \rangle$ and $\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle = \langle 0, 1 \rangle$.
Thus the Lemma holds in all the cases. Similarly we can prove the Lemma when $\langle a, a' \rangle > \langle b, b' \rangle$.

Remark 3.5. *The above Lemma can be generalized as*
$\left(\bigvee_{i=1}^{n} \langle a_i, a_i' \rangle \right) \rightharpoondown \langle c, c' \rangle = \bigwedge_{i=1}^{n} (\langle a_i, a_i' \rangle \rightharpoondown \langle c, c' \rangle)$.
The following Lemma shows that the IFRDIO $\rightharpoondown$ is left distributive over $\wedge$.

Lemma 3.6. *If $\langle a, a' \rangle, \ \langle b, b' \rangle, \ \langle c, c' \rangle \in IFS$, then the IFRDIO $\rightharpoondown$ satisfies the distributive property $\langle a, a' \rangle \rightharpoondown [\langle b, b' \rangle \wedge \langle c, c' \rangle] = (\langle a, a' \rangle \rightharpoondown \langle b, b' \rangle) \wedge (\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle)$.*
Proof. Case 1. If $\langle b, b' \rangle \leq \langle c, c' \rangle$, then $(\langle b, b' \rangle \wedge \langle c, c, \rangle = \langle b, b' \rangle$.
sub case (i) If $\langle b, b' \rangle \leq \langle c, c' \rangle \leq \langle a, a' \rangle$.
Clearly $\langle a, a' \rangle \rightharpoondown (\langle b, b' \rangle \wedge \langle c, c' \rangle) = \langle 0, 1 \rangle$ and $(\langle a, a' \rangle \rightharpoondown \langle b, b' \rangle) \wedge (\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle) = \langle 0, 1 \rangle$.
sub case (ii) If $\langle b, b' \rangle \leq \langle a, a' \rangle < \langle c, c' \rangle$, then
$\langle a, a' \rangle \rightharpoondown (\langle b, b' \rangle \wedge \langle c, c' \rangle) = \langle a, a' \rangle \rightharpoondown \langle b, b' \rangle = \langle 0, 1 \rangle$ and it clearly indicate the right hand side value of the given equation is also $\langle 0, 1 \rangle$, as $(\langle b, b' \rangle \wedge \langle c, c' \rangle)$ is one of the term in the right hand side.
sub case (iii) If $\langle a, a' \rangle < \langle b, b' \rangle \leq \langle c, c' \rangle$, then $\langle a, a' \rangle \rightharpoondown (\langle b, b' \rangle \wedge \langle c, c' \rangle) = \langle b, b' \rangle$
$(\langle a, a' \rangle \rightharpoondown \langle b, b' \rangle) \wedge (\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle) = \langle b, b' \rangle \wedge \langle c, c' \rangle = \langle b, b' \rangle$.
Hence the Lemma holds in all the cases and one can easily check the result for $\langle b, b' \rangle > \langle c, c' \rangle$.

Remark 3.7. *The above Lemma can be generalized as*
$\langle a, a' \rangle \rightharpoondown \left(\bigwedge_{i=1}^{n} \langle b_i, b_i' \rangle \right) = \bigwedge_{i=1}^{n} (\langle a, a' \rangle \rightharpoondown \langle b_i, b_i' \rangle)$.
In the following remark, we prove that the IFRDIO $\rightharpoondown$ is not right distributive over $\wedge$.

Remark 3.8. $(\langle a, a' \rangle \wedge \langle b, b' \rangle) \rightharpoondown \langle c, c' \rangle \neq (\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle) \wedge (\langle b, b' \rangle \rightharpoondown \langle c, c' \rangle)$.
Let $\langle a, a' \rangle < \langle c, c' \rangle < \langle b, b' \rangle$.
$(\langle a, a' \rangle \wedge \langle b, b' \rangle) \rightharpoondown \langle c, c' \rangle = \langle a, a' \rangle \rightharpoondown \langle c, c' \rangle = \langle c, c' \rangle$, but $(\langle a, a' \rangle \rightharpoondown \langle c, c' \rangle) \wedge (\langle b, b' \rangle \rightharpoondown \langle c, c' \rangle) = \langle c, c' \rangle \wedge \langle 0, 1 \rangle = \langle 0, 1 \rangle$. Hence the remark.

Lemma 3.9. *If $\langle a, a' \rangle, \ \langle b, b' \rangle, \ \langle c, c' \rangle \in IFS$, then the IFRDIO $\rightharpoondown$ satisfies*

$[\langle a, a'\rangle \wedge \langle b, b'\rangle] \rightharpoondown \langle c, c'\rangle = (\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle) \vee (\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle)$.

Proof. Case 1. Assume $\langle a, a'\rangle \leq \langle b, b'\rangle$

sub case (i). If $\langle a, a'\rangle \leq \langle b, b'\rangle \leq \langle c, c'\rangle$

Now, $(\langle a, a'\rangle \wedge \langle b, b'\rangle) \rightharpoondown \langle c, c'\rangle = \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0, 1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$

and $\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0, 1\rangle & \text{if } \langle b, b'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle b, b'\rangle < \langle c, c'\rangle \end{cases}$

Clearly, $(\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle) \vee (\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle) = \begin{cases} \langle 0, 1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$

Therefore the Lemma holds.

sub case (ii). If $\langle a, a'\rangle \leq \langle c, c'\rangle < \langle c, c'\rangle$.

Now, $(\langle a, a'\rangle \wedge \langle b, b'\rangle) \rightharpoondown \langle c, c'\rangle = \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0, 1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$

$\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$. In this case also,

$$(\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle) \vee (\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle) = \begin{cases} \langle 0, 1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$$

sub case (iii). If $\langle c, c'\rangle < \langle a, a'\rangle \leq \langle b, b'\rangle$, then
$(\langle a, a'\rangle \wedge \langle b, b'\rangle) \rightharpoondown \langle c, c'\rangle = \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$ and $\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$ and $(\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle) \vee (\langle b, b'\rangle \rightharpoondown \langle c, c'\rangle) = \langle 0, 1\rangle$.

Therefore the Lemma holds in this case also.

Similarly we can prove when $\langle a, a'\rangle > \langle b, b'\rangle$.

Lemma 3.10. *The above Lemma can be generalized as*
$\left(\bigwedge_{i=1}^{n} \langle a_i, a_i'\rangle\right) \rightharpoondown \langle c, c'\rangle = \bigvee_{i=1}^{n} (\langle a_i, a_i'\rangle \rightharpoondown \langle c, c'\rangle)$

The following Lemma shows that the IFRDIO $\rightharpoondown$ is left distributive over $\vee$.

Lemma 3.11. *If* $\langle a, a'\rangle,\ \langle b, b'\rangle,\ \langle c, c'\rangle \in IFS$, *then the IFIO* $\rightharpoondown$ *satisfies*
$\langle a, a'\rangle \rightharpoondown [\langle b, b'\rangle \vee \langle c, c'\rangle] = (\langle a, a'\rangle \rightharpoondown \langle b, b'\rangle) \vee (\langle a, a'\rangle \rightharpoondown \langle c, c'\rangle)$.

Proof. Case 1. Assume $\langle b, b'\rangle \leq \langle c, c'\rangle$,
then $\langle b, b\rangle \vee \langle c, c'\rangle = \langle c, c'\rangle$.

sub case (i) If $\langle a, a'\rangle \leq \langle b, b'\rangle \leq \langle c, c'\rangle$, then

$$\langle a, a'\rangle \rightharpoondown [\langle b, b'\rangle \vee \langle c, c'\rangle] = \langle a, a'\rangle \rightharpoondown \langle c, c'\rangle = \begin{cases} \langle 0, 1\rangle & \text{if } \langle a, a'\rangle = \langle c, c'\rangle \\ \langle c, c'\rangle & \text{if } \langle a, a'\rangle < \langle c, c'\rangle \end{cases}$$

Now,

$$\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a,a'\rangle = \langle b,b'\rangle \\ \langle b,b'\rangle & \text{if } \langle a,a'\rangle < \langle b,b'\rangle \end{cases}$$

Clearly by this result the Lemma holds in this case.
sub case (ii). If $\langle b,b'\rangle < \langle a,a'\rangle \leq \langle c,c'\rangle$, then

$$\langle a,a'\rangle \rightharpoondown [\langle b,b'\rangle \vee \langle c,c'\rangle] = \langle a,a'\rangle \rightharpoondown \langle c,c'\rangle = \begin{cases} \langle 0,1\rangle & \text{if } \langle a,a'\rangle = \langle c,c'\rangle \\ \langle c,c'\rangle & \text{if } \langle a,a'\rangle < \langle c,c'\rangle \end{cases}$$

$\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle = \langle 0,1\rangle$. Now,

$$(\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle) \vee (\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle) = \begin{cases} \langle 0,1\rangle & \text{if } \langle a,a'\rangle = \langle c,c'\rangle \\ \langle c,c'\rangle & \text{if } \langle a,a'\rangle < \langle c,c'\rangle \end{cases}$$

Thus the Lemma holds in this case too.
sub case (iii). If $\langle b,b'\rangle \leq \langle c,c'\rangle < \langle a,a'\rangle$, then
$\langle a,a'\rangle \rightharpoondown [\langle b,b'\rangle \vee \langle c,c'\rangle] = \langle a,a'\rangle \rightharpoondown \langle c,c'\rangle = \langle 0,1\rangle$, and $\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle = \langle 0,1\rangle$.
Therefore the Lemma is true in whole.

Remark 3.12. *From the above Lemma we have* $\langle a,a'\rangle \rightharpoondown [\langle b,b'\rangle \vee \langle c,c'\rangle] = (\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle) \vee (\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle) \geq (\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle) \wedge (\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle$.

Remark 3.13. *The above Lemma can be generalized as*
$\langle a,a'\rangle \rightharpoondown \left(\bigvee_{i=1}^{n} (\langle b_i, b_i'\rangle \right) = \bigvee_{i=1}^{n} (\langle a,a'\rangle \rightharpoondown \langle b_i, b_i'\rangle)$.
The following Lemma shows that the IFRDIO $\rightharpoondown$ is not right distributive over $\wedge$.

Lemma 3.14. *If* $\langle a,a'\rangle$, $\langle b,b'\rangle$, $\langle c,c'\rangle \in IFS$, *then the IFRDIO* $\rightharpoondown$ *satisfies* $(\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle) \wedge (\langle b,b'\rangle \rightharpoondown \langle c,c'\rangle) \neq [(\langle a,a'\rangle \wedge \langle b,b'\rangle) \rightharpoondown \langle c,c'\rangle]$.
Proof. Consider $\langle a,a'\rangle < \langle c,c'\rangle < \langle b,b'\rangle$.
$(\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle) \wedge (\langle b,b'\rangle \rightharpoondown \langle c,c'\rangle) = \langle b,b'\rangle \rightharpoondown \langle c,c'\rangle = \langle 0,1\rangle$, but
$(\langle a,a'\rangle \rightharpoondown \langle c,c'\rangle) \vee (\langle b,b'\rangle \rightharpoondown \langle c,c'\rangle) = \langle c,c'\rangle \vee \langle 0,1\rangle = \langle c,c'\rangle$.
Therefore the IFRDIO $\rightharpoondown$ is not right distributive over $\wedge$.

Lemma 3.15. *If* $\langle a,a'\rangle$, $\langle b,b'\rangle \in IFS$, *then the IFRDIO* $\rightharpoondown$ *satisfies* $\langle b,b'\rangle \geq (\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle)$.
Proof. The Proof is evident from the definition of the IFRDIO $\rightharpoondown$.

Lemma 3.16. *If* $\langle a,a'\rangle$, $\langle b,b'\ \rangle , \langle c,c'\rangle \in IFS$, *then the IFRDIO* $\rightharpoondown$ *satisfies* $(\langle a,a'\rangle \wedge \langle b,b'\rangle) \geq \langle a,a'\rangle \wedge (\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle)$.
Proof. If $\langle a,a'\rangle < \langle b,b'\rangle$, $\langle a,a'\rangle \geq \langle a,a'\rangle \wedge \langle b,b'\rangle$, $\langle a,a'\rangle = \langle a,a'\rangle$.
If $\langle a,a'\rangle > \langle b,b'\rangle$, $\langle b,b'\rangle \geq \langle a,a'\rangle \wedge \langle 0,1\rangle$, $\langle b,b'\rangle > \langle 0,1\rangle$.
If $\langle a,a'\rangle = \langle b,b'\rangle$, $\langle a,a'\rangle \geq \langle a,a'\rangle \wedge \langle 0,1\rangle$, $\langle a,a'\rangle > \langle 0,1\rangle$.

Lemma 3.17. *If* $\langle a,a'\rangle$, $\langle b,b'\rangle$, $\langle c,c'\rangle \in IFS$, *then the IFRDIO* $\rightharpoondown$ *satisfies,*

If $\langle a,a'\rangle \le \langle b,b'\rangle$ *and* $\langle c,c'\rangle \le \langle d,d'\rangle$, *then* $\langle d,d'\rangle \rightharpoondown \langle a,a'\rangle \le \langle c,c'\rangle \rightharpoondown \langle b,b'\rangle$.
Proof. For $\langle c,c'\rangle \le \langle d,d'\rangle$, $\langle d,d'\rangle \rightharpoondown \langle a,a'\rangle \le \langle c,c'\rangle \rightharpoondown \langle a,a'\rangle$ for any $\langle a,a'\rangle \in IFS$, using first place anti-tonicity for $\rightharpoondown$, by Lemma 3.3(i). Again $\langle a,a'\rangle \le \langle b,b'\rangle$, then $\langle c,c'\rangle \rightharpoondown \langle a,a'\rangle \le \langle c,c'\rangle \rightharpoondown \langle b,b'\rangle$, by Lemma 3.3(ii).
Therefore $\langle d,d'\rangle \rightharpoondown \langle a,a'\rangle \le \langle c,c'\rangle \rightharpoondown \langle b,b'\rangle$.

Lemma 3.18. *As* $\langle a,a'\rangle \rightharpoondown \langle b,b'\rangle = \langle b,b'\rangle \leftharpoondown \langle a,a'\rangle$, *we can easily see that* $\leftharpoondown$ *is right distributive over* $\vee$ *and* $\wedge$, *but not left distributive over* $\vee$ *and* $\wedge$.

Theorem 3.19. *For IFMs* A, B *and* C *of compatible order the following results hold*
(i) $[A \vee B] \rightharpoondown C = (A \rightharpoondown C) \wedge (B \rightharpoondown C)$
(ii) $A \rightharpoondown [B \wedge C] = (A \rightharpoondown B) \wedge (A \rightharpoondown C)$
(iii) $[A \wedge B] \rightharpoondown C = (A \rightharpoondown C) \vee (B \rightharpoondown C)$
(iv) $A \rightharpoondown [B \vee C] = (A \rightharpoondown B) \vee (A \rightharpoondown C)$
(v) $(A \rightharpoondown C) \wedge (B \rightharpoondown C) \le (A \wedge B) \rightharpoondown C$
(vi) $(A \rightharpoondown B) \wedge (A \rightharpoondown C) \le (A \rightharpoondown (B \vee C))$
(vii) If $A \le B$ *and* $C \le D$, *then* $D \rightharpoondown A \le C \rightharpoondown B$
Proof.
(i) $(A \rightharpoondown C) \wedge (B \rightharpoondown C) = \bigvee_{k}^{n}(\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle) \wedge \bigvee_{k=1}^{n}(\langle b_{ik},b'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle)$

$= \bigvee_{k=1}^{n}(\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle) \wedge (\langle b_{ik},b'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle)$

$= \bigvee_{k=1}^{n}(\langle a_{ik},a'_{ik}\rangle \vee \langle b_{ik},b'_{ik}\rangle) \rightharpoondown \langle c_{kj},c'_{kj}\rangle)$, by Lemma 3.4.

$= [A \vee B] \rightharpoondown C$.

(ii) $A \rightharpoondown (B \wedge C) = \bigvee_{k=1}^{n}\left[\langle a_{ik},a'_{ik}\rangle \rightharpoondown [\langle b_{kj},b'_{kj}\rangle \wedge \langle c_{kj},c'_{kj}\rangle]\right]$

$= \bigvee_{k=1}^{n}(\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle b_{kj},b'_{kj}\rangle) \wedge (\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle)$, by Lemma 3.6

$= \bigvee_{k=1}^{n}\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle b_{kj},b'_{kj}\rangle \wedge \bigvee_{k=1}^{n}\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle$

$= (A \rightharpoondown B) \wedge (A \rightharpoondown C)$.

(iii) $(A \rightharpoondown C) \vee (B \rightharpoondown C) = \bigvee_{k=1}^{n}(\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle) \vee \bigvee_{k=1}^{n}(\langle b_{kj},b'_{kj}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle)$

$= \bigvee_{k=1}^{n}\left[(\langle a_{ik},a'_{ik}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle) \vee (\langle b_{kj},b'_{kj}\rangle \rightharpoondown \langle c_{kj},c'_{kj}\rangle)\right]$

$= \bigvee_{k=1}^{n}(\langle a_{ik},a'_{ik}\rangle \wedge \langle b_{kj},b'_{kj}\rangle) \rightharpoondown \langle c_{kj},c'_{kj}\rangle$, by Lemma 3.9

$= (A \wedge B) \rightharpoondown C$.

(iv) $A \rightharpoondown (B \vee C) = \bigvee_{k=1}^{n} \langle a_{ik}, a'_{ik}\rangle \rightharpoondown (\langle b_{kj}, b'_{kj}\rangle \vee \langle c_{kj}, c'_{kj}\rangle)$
$= \bigvee_{k=1}^{n} \langle a_{ik}, a'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle \vee \bigvee_{k=1}^{n} \langle a_{ik}, a'_{ik}\rangle \rightharpoondown \langle c_{kj}, c'_{kj}\rangle$, by Lemma 3.11
$= (A \rightharpoondown B) \vee (A \rightharpoondown C)$.
(v) Comes directly from (iii), since $(A \rightharpoondown C) \vee (B \rightharpoondown C) \geq (A \rightharpoondown C) \wedge (B \rightharpoondown C)$.
(vi) Comes directly from (iv), since $(A \rightharpoondown B) \vee (A \rightharpoondown C) \geq (A \rightharpoondown B) \wedge (A \rightharpoondown C)$.
(vii) $A \leq B$ then $\langle a_{ij}, a'_{ij}\rangle \leq \langle b_{ij}, b'_{ij}\rangle$ for all i, j.
In particular $\langle a_{ik}, a'_{ik}\rangle \leq \langle b_{kj}, b'_{kj}\rangle$, by Lemma 3.3(ii),
for any $\langle d_{ik}, d'_{ik}\rangle$, $\langle d_{ik}, d'_{ik}\rangle \rightharpoondown \langle a_{kj}, a'_{kj}\rangle \leq \langle d_{ik}, d'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle$.
Again by $C \leq D$, $\langle c_{ik}, c'_{ik}\rangle \leq \langle d_{ik}, d'_{ik}\rangle$ by Lemma 3.3(i), for any
$\langle b_{kj}, b'_{kj}\rangle$, $\langle d_{ik}, d'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle \leq \langle c_{ik}, c'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle$.
Thus $\langle d_{ik}, d'_{ik}\rangle \rightharpoondown \langle a_{kj}, a'_{kj}\rangle \leq \langle c_{ik}, c'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle$.
That is $\bigvee_{k=1}^{n} (\langle d_{ik}, d'_{ik}\rangle \rightharpoondown \langle a_{kj}, a'_{kj}\rangle) \leq \bigvee_{k=1}^{n} (\langle c_{ik}, c'_{ik}\rangle \rightharpoondown \langle b_{kj}, b'_{kj}\rangle)$.
Therefore, $D \rightharpoondown A \leq C \rightharpoondown B$.

Remark 3.20. *$B \leq (A \rightharpoondown B)$ (consequent boundary) is not true in the case of IFM is illustrated through the following example.*

$$A \rightharpoondown B = \begin{pmatrix} \langle .5, .3\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .7, 0\rangle \end{pmatrix} \rightharpoondown \begin{pmatrix} \langle .2, .4\rangle & \langle .3, .3\rangle \\ \langle .5, .2\rangle & \langle .4, .2\rangle \end{pmatrix} = \begin{pmatrix} \langle .5, .2\rangle & \langle .4, .2\rangle \\ \langle .2, .4\rangle & \langle .3, .3\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle .2, .4\rangle & \langle .3, .3\rangle \\ \langle .5, .2\rangle & \langle .4, .2\rangle \end{pmatrix} \nleq \begin{pmatrix} \langle .5, .2\rangle & \langle .4, .2\rangle \\ \langle .2, .4\rangle & \langle .3, .3\rangle \end{pmatrix} \qquad B \nleq (A \rightharpoondown B)$$

Remark 3.21. *The following example shows that the inequality $(A \wedge B) \geq (A \wedge (A \rightharpoondown B))$ fails.*

$$A \wedge B = \begin{pmatrix} \langle .2, .4\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .4, .2\rangle \end{pmatrix} \qquad A \rightharpoondown B = \begin{pmatrix} \langle .5, .2\rangle & \langle .4, .2\rangle \\ \langle .2, .4\rangle & \langle .3, .3\rangle \end{pmatrix}$$

$$A \wedge (A \rightharpoondown B) = \begin{pmatrix} \langle .5, .3\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .7, 0\rangle \end{pmatrix} \wedge \begin{pmatrix} \langle .5, .2\rangle & \langle .4, .2\rangle \\ \langle .2, .4\rangle & \langle .3, .3\rangle \end{pmatrix} = \begin{pmatrix} \langle .5, .3\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .3, .3\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle .2, .4\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .4, .2\rangle \end{pmatrix} \ngeq \begin{pmatrix} \langle .5, .3\rangle & \langle .2, .4\rangle \\ \langle 0, 1\rangle & \langle .3, .3\rangle \end{pmatrix}$$

Therefore $(A \wedge B) \ngeq (A \wedge (A \rightharpoondown B))$.
Now we give some results about the IFRDIO $'\leftharpoondown'$.

Remark 3.22. *The $'\leftharpoondown'$ fails to satisfy first place anti-tonicity and second place isotonicity. Let $\langle a, a'\rangle \leq \langle b, b'\rangle$ and $\langle c, c'\rangle$ be any element in IFS such that $\langle a, a'\rangle < \langle c, c'\rangle < \langle b, b'\rangle$, then $\langle b, b'\rangle \leftharpoondown \langle c, c'\rangle = \langle b, b'\rangle$, and $\langle a, a'\rangle \leftharpoondown \langle c, c'\rangle = \langle 0, 1\rangle$.*
Therefore, $\langle b, b'\rangle \leftharpoondown \langle c, c'\rangle \nleq \langle a, a'\rangle \leftharpoondown \langle c, c'\rangle$.
Again, $\langle c, c'\rangle \leftharpoondown \langle a, a'\rangle = \langle c, c'\rangle$, and $\langle c, c'\rangle \leftharpoondown \langle b, b'\rangle = \langle 0, 1\rangle$.

Therefore, $\langle c, c' \rangle \leftharpoondown \langle a, a' \rangle \nleq \langle c, c' \rangle \leftharpoondown \langle b, b' \rangle$.
One can easily check that the following results for $' \leftharpoondown '$.

Theorem 3.23. *For IFMs* A, B *and* C *of compatible order the following results hold*
(i) $[A \vee B] \leftharpoondown C = (A \leftharpoondown C) \vee (B \leftharpoondown C)$.
(ii) $A \leftharpoondown [B \wedge C] = (A \leftharpoondown B) \vee (A \leftharpoondown C)$.
(iii) $[A \wedge B] \leftharpoondown C = (A \leftharpoondown C) \wedge (B \leftharpoondown C)$.
(iv) $A \leftharpoondown [B \vee C] = (A \leftharpoondown B) \wedge (A \leftharpoondown C)$.

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