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ON MULTI-SET AND MULTI-SET FUNCTION

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Abstract: In this paper, we introduce the concept of α-levels of multi-set and study many of its properties, weak α-level and strong α -level of multi-set. Also we study the cartesian product of two multi-sets and some of its properties, Also we study the multi-set function and some of its properties and define the image and the inverse image of multi- sets and study of some properties.

Keywords and Phrases: Multi-set, Multi-set relation, Multi-set function.

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1. Introduction

In [13] Yager introduced the idea of multi-sets as an extension of classical set theory. In classical set theory, an element accurasy only one time in a set. In many real life applications an element accuresy more than one time in a set. This is why Yager et al defined a multi-set which is a generalization of an ordinary set.

In the classical set theory, a set is a well-defined collection of distinct objects and the operations are based on this definition. If the problem allowed to repeated occurrences of any object in a set then we need another mathematical structure, that which is recently known as multi-set (mset for short). The same idea are studied by Yager in [15] and Jena in [9], but under the name of bags and lists. Thus, a multiset differs from a set in the sense that each element has a multiplicity-a natural number not necessarily one-that indicates how many times it is the member of the multiset was studied by ([1], [5], [6], [7], [8]). Application of multiset theory indecision making can be seen in [16,13]. One of the most natural and simplest examples is the multiset of prime factors of a positive integer "n". The number 640 has the factorization $640 = 2^3 4^2 5^1$ which gives the multiset $\{2, 2, 2, 4, 4, 5\}$. The relation on a set is a simple mathematical model to which many real-life data can be connected. In fact, topological structures are generalized methods for measuring similarity and dissimilarity between objects in the universe. The relations are used the construction of topological structures in many fields such as dynamics, rough set theory and approximation spaces. In this paper we study the cartesian product of msets and some of its properties, we study the mset relations and some of its,the mset function between msets and we define the image, the inverse image of msets. Finally, we define the concepts of weak α-level and strong α -level sets of multi-set and weak α-level multi-sets and strong α -level multi-sets and study of its properties.

2. Preliminaries and Basic definitions

In this section the basic definitions and notations of multisets and the different types of collections of multisets and the basic definitions of functions in multiset context are given ([1],[5], [6], [7],[8]).

Definition 2.1. [6] *A mset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers. Here $C_M(x)$ is the number of occurrences of the element x in the mset M. Let$X = \{x_1, x_2, ..., x_n\}$, the mset M drawn from the set X is represented by $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, ..., n$ in the mset M.*

Example 2.2. Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 3/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3. [6,7] *Let M and N be two msets drawn from a set X. Then the following are defined:*

1. $M = N$ *if* $C_M(x) = C_N(x)$, $\forall x \in X$.
2. $M \subseteq N$ *if* $C_M(x) \leq C_N(x)$, $\forall x \in X$.
3. $P = M \bigcup N$ *if* $C_P(x) = Max\{C_M(x), C_N(x)\}$, $\forall x \in X$.
4. $P = M \bigcap N$ *if* $C_P(x) = Min\{C_M(x), C_N(x)\}$, $\forall x \in X$.
5. $P = M \oplus N$ *if* $C_P(x) = C_M(x) + C_N(x)$, $\forall x \in X$.
6. $P = M \ominus N$ *if* $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$, $\forall x \in X$.

Where $\oplus$ and $\ominus$ represent m-set addition and m-set subtraction, respectively.

Definition 2.4. [7,8] *Let M be any m-set drawn from a set X. The support set of M denoted by M^* is a subset of X and defined as $M^* = \{x \in X : C_M(x) > 0\}$ i.e. M^* is an ordinary set. M^* is also called root set.*

Definition 2.5. [7,8] *An m-set M is said to be an empty m-set if $C_M(x) = 0,\ \forall x \in X$.*

Definition 2.6. [7,8] *A domain X, is defined as a set of elements from which m-sets are constructed. The m-set space $[X]^w$ is the set of all m-sets whose elements are in X such that no element in the m-set occurs more than w times. The set $[X]^\infty$ is the set of all m-sets over a domain X such that there is no limit on the number of the occurrences of an element in an m-set. If $X = \{x_1, x_2, ..., x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} :$ for $i = 1, 2, ..., k\, ; m_i \in \{0, 1, 2, ..., w\}\}$.*

Definition 2.7. [7,8] *Let X be a support set and $[X]^w$ be the m-set space defined over X. Then for any m-set $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that*
$C_{M^c}(x) = w - C_M(x),\ \forall x \in X$.

Notation 2.8. [7,8] *Let M be an m-set from X with x appearing n times in M. It is denoted by $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ where M is an m-set with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[M]_x$ denotes that the element x belongs to the m-set M and $\mid [M]_x \mid$ denotes the cardinality of an element x in M.*

Definition 2.9. [6,8] *A submset N of M is said to be:*

1. *Whole m-subset of M if $C_N(x) = C_M(x)$, for every $x \in N$.*
2. *Partial whole m-subset of M if $C_N(x) = C_M(x)$, for some $x \in N$.*
3. *Full submset of M if $M^* = N^*$ and $C_N(x) \leq C_M(x)$, for every $x \in N$.*

Definition 2.10. [6,7,8] *(Power Mset) Let $M \in [X]^w$ be an mset. The power mset of M denoted by $P(M)$ is the set of all the submsets of M. i.e. $N \in P(M)$ if and only if $N \subseteq M$. If $N = \emptyset$, then $N \in^1 P(M)$, and if $N \neq \emptyset$, then $N \in^k P(M)$ where $k = \prod_z \binom{|[M]_z|}{|[N]_z|}$, the product $\prod_z$ is taken over by distinct of z of the mset N and $\mid [M]_z \mid = m$ iff $z \in^m M$, $\mid [N]_z \mid = n$ iff $z \in^n N$, then $\binom{|[M]_z|}{|[N]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$. The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.*

Definition 2.11. [7,8] *The maximum mset is defined as Z where $C_Z\ (x) =$*

$Max\{C_M\ (x) : x \in^k M,\ M \in [X]^w$ and $k \leq m\}$.

Definition 2.12. [6,7] *Let $[X]^w$ be an mset space, $\{M_1, M_2, ...\}$ be a collection of msets drawn from $[X]^w$, then the following operations are possible under an arbitrary collection of msets.*

1. *The union $\cup_{i\in I} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = Max\{C_{M_i}(x) : x \in X\}\}$.*

2. *The intersection $\cap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = Min\{C_{M_i}(x) : x \in X\}\}$.*

3. *The m-set addition $\oplus_{i\in I} M_i = \{C_{\oplus M_i}(x)/x : C_{\oplus M_i}(x) = \sum_{i\in I} C_{M_i}(x),\ x \in X\}$.*

4. *The m-set complement $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x),\ x \in X\}$.*

Definition 2.13. [8] *Let M_1 and M_2 be two m-sets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1,\ y \in^n M_2\}$.*

Definition 2.14. [8] *A sub-mset R of $M \times M$ is said to be an m-set relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/x\, R\, n/y$. The Domain and Range of the m-set relation R on M is defined as follows:*
Dom $R = \{x \in^k M : \exists\, y \in^r M$ such that $k/x\, R\, r/y\}$ where $C_{DomR}(x) = sup\{C_1\ (x, y) : x \in^k M\}$.
Ran $R = \{y \in^r M : \exists\, x \in^k M$ such that $k/x\, R\, r/y\}$ where $C_{RanR}(y) = sup\{C_2\ (x, y) : y \in^r M\}$.

Definition 2.15. [8] *Let R be a m-set relation on M .The family $R^{-1} = \{(n/y, m/x) : m/xRn/y\}$is said to be the inverse m-set relation of R.*

Definition 2.16. [7,8] *Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multi-set topology of M if τ satisfies the following properties.*

1. *The m-set M and the empty m-set $\emptyset$ are in τ.*

2. *the union of elements of any sub collection of τ is in τ.*

3. *the intersection of elements of any finite sub collection of τ is in τ.*

The pair (M, τ) is called multi-topological space (for short M-topological space). The elements of τ are called open m-sets.

3. α -level sets

In this section we define α-level set of a multi-set and we study of its properties.

Definition 3.1. *Let M be a multi-set drawn from a set X, $\alpha \in (0, w]$. The set $M_\alpha = \{x \in X : C_M(x) \geq \alpha\}$is called a weak α-level of a mset M.*

Definition 3.2. *Let M be a multi-set drawn from a set X, $\alpha \in [0, w)$. The set $M_{\overline{\alpha}} = \{x \in X : C_M(x) > \alpha\}$is called strong α-level of a mset M.*

Example 3.3. Let $x = \{a, b, c, d, e, f\}$, $M = \{3/a, 5/b, 7/c, 3/d, 2/e\}$ and let $\alpha_1 = 3, \alpha_2 = 6$ then $M_{\alpha_1} = \{a, b, c, d\}$, $M_{\alpha_2} = \{c\}$ and $M_{\overline{\alpha_1}} = \{b, c\}$

Proposition 3.3. *Let M and N be two msets drawn from a set X, then*

1. $M \subseteq N \Rightarrow M_\alpha \subseteq N_\alpha \forall \alpha \in (0, w]$.
2. $0 < \alpha_1 \leq \alpha_2 \leq w \Rightarrow M_{\alpha_2} \subseteq M_{\alpha_1}$.
3. $M = N \Leftrightarrow M_\alpha = N_\alpha \forall \alpha \in (0, w]$.
4. $M^\star = \cup\{M_\alpha : \alpha \in (0, w]\}$.

Proof.

1. Let $x \in M_\alpha \Rightarrow C_M(x) \geq \alpha \Rightarrow C_N(x) \geq \alpha \Rightarrow x \in N_\alpha$. Hence $M_\alpha \subseteq N_\alpha \forall \alpha \in (0, w]$.
2. Let $x \in M_{\alpha_2} \Rightarrow C_M(x) \geq \alpha_2 \Rightarrow C_M(x) \geq \alpha_1 \Rightarrow x \in M_{\alpha_1}$ .Hence $M_{\alpha_2} \subseteq M_{\alpha_1}$.
3. It obvious from above (2)
4. Let $x \in M^\star \Rightarrow C_M(x) > 0$. Then, $\exists \alpha \in (0, w]$ such that $C_M(x) \geq \alpha \Rightarrow x \in M_\alpha \Rightarrow x \in \cup\{M_\alpha : \alpha \in (0, w]\}$. Hence $M^\star \subseteq \cup\{M_\alpha : \alpha \in (0, w]\}$. Conversely, let $x \in \cup\{M_\alpha : \alpha \in (0, w]\} \Rightarrow \exists \alpha \in (0, w]$ such that $x \in M_\alpha \Rightarrow C_M(x) \geq \alpha \Rightarrow C_M(x) > 0 \Rightarrow x \in M^\star$. Hence, $\cup\{M_\alpha : \alpha \in (0, w]\} \subseteq M^\star$. Consequently we have $M^\star = \cup\{M_\alpha : \alpha \in (0, w]\}$.

Proposition 3.4. *Let M and N be two msets drawn from a set X, then*

1. $M \subseteq N \Rightarrow M_{\overline{\alpha}} \subseteq N_{\overline{\alpha}} \forall \alpha \in [0, w)$.
2. $0 < \alpha_1 \leq \alpha_2 \leq w \Rightarrow M_{\overline{\alpha_2}} \subseteq M_{\overline{\alpha_1}}$.
3. $M = N \Leftrightarrow M_{\overline{\alpha}} = N_{\overline{\alpha}} \forall \alpha \in [0, w)$.

Proof.

1. Let $x \in M_{\overline{\alpha}} \Rightarrow C_M(x) > \alpha \Rightarrow C_N(x) > \alpha \Rightarrow x \in N_{\overline{\alpha}}$ .Hence $M_{\overline{\alpha}} \subseteq N_{\overline{\alpha}} \forall \alpha \in [0, w)$.

2. Let $x \in M_{\overline{\alpha_2}} \Rightarrow C_M(x) > \alpha_2 \Rightarrow C_M(x) > \alpha_1 \Rightarrow x \in M_{\overline{\alpha_1}}$ .Hence $M_{\overline{\alpha_2}} \subseteq M_{\overline{\alpha_1}}$.

3. It obvious from above (2).

Proposition 3.5. *Let M and N be two msets drawn from a set X, then* $(M \cap N)_\alpha = M_\alpha \cap N_\alpha$.

Proof. Since $M \cap N \subseteq M$ and $M \cap N \subseteq N$ then $(M \cap N)_\alpha \subseteq M_\alpha \cap N_\alpha$. Conversely let $x \in M_\alpha \cap N_\alpha \Rightarrow x \in M_\alpha$ and $x \in N_\alpha \Rightarrow C_M(x) \geq \alpha$ and $C_N(x) \geq \alpha \Rightarrow \min\{C_M(x), C_N(x)\} \geq \alpha \Rightarrow C_{M \cap N}(x) \geq \alpha \Rightarrow x \in (M \cap N)_\alpha$. Hence we have$(M \cap N)_\alpha = M_\alpha \cap N_\alpha$.

Proposition 3.6. *Let M and N be two msets drawn from a set X, then* $(M \cup N)_\alpha = M_\alpha \cup N_\alpha$.

Proof. Since M and N are multi-subsets of $M \cup N$, then we have $M_\alpha \cup N_\alpha \subseteq (M \cup N)_\alpha$. Conversely, let $x \in (M \cup N)_\alpha \Rightarrow C_{M \cup N}(x) \geq \alpha \Rightarrow \max\{C_M(x), C_N(x)\} \geq \alpha \Rightarrow C_M(x) \geq \alpha$ or $C_N(x) \geq \alpha \Rightarrow x \in M_\alpha$or $x \in N_\alpha \Rightarrow x \in (M_\alpha \cup N_\alpha) \Rightarrow (M \cup N)_\alpha \subseteq (M_\alpha \cup N_\alpha)$. Hence we have $(M \cup N)_\alpha = M_\alpha \cup N_\alpha$.

Proposition 3.7. *Let M and N be two msets drawn from a set X, then* $(M \cap N)_{\overline{\alpha}} = M_{\overline{\alpha}} \cap N_{\overline{\alpha}}$.

Proof. Similar to the above proposition 3.5.

Proposition 3.8. *Let M and N be two msets drawn from a set X, then* $(M \cup N)_{\overline{\alpha}} = M_{\overline{\alpha}} \cup N_{\overline{\alpha}}$.

Proof. Similar to the above proposition 3.6.

Remark 3.9. *For any multi-set M drawn from a set X we have* $(M_\alpha)^c \neq (M^c)_\alpha$.

Example 3.10. Let$X = \{a, b, c, d, e\}$, $M = \{3/a, 5/b, 7/c, 4/d, 4/e\}$, $A = \{1/a, 2/b, 3/d\}$, $B = \{4/b, 3/c\}$,$\alpha = 2$ then we have $A_\alpha = \{b, d\}$, $(A_\alpha)^c = \{a, c, e\}$, $A^c = \{2/a, 3/b, 7/c, 1/d, 4/e\}$, $(A^c)_\alpha = \{a, b, c, e\}$ .Also let $\alpha = 3$, we have $A_\alpha = \{d\}$, $(A_\alpha)^c = \{a, c, e, b\}$, $A^c = \{2/a, 3/b, 7/c, 1/d, 4/e\}$, $((A^c))_\alpha = \{b, c, e\}$.

Example 3.11. Let$X = \{a, b, c, d, e\}$, $M = \{3/a, 5/b, 7/c, 4/d, 4/e\}$, $A = \{1/a, 2/b, 3/d\}$, $B = \{4/b, 3/c\}, \alpha_1 = 3$ then we have , $A_{\alpha_1} = \{d\}, B_{\alpha_1} = \{b, c\}$, $A \cup B = \{1/a, 4/b, 3/c, 3/d\}$, $(A \cup B)_{\alpha_1} = \{b, c, d\} = A_{\alpha_1} \cup B_{\alpha_1}$.

Definition 3.12. *Let X be a non-empty set ,a family* $\{A_i; i \in N\}$*of subsets of X* *is said a descending family if* $\alpha \leq \beta \Rightarrow A_\beta \subseteq A_\alpha \quad forall \quad \alpha, \beta \in N$.

Proposition 3.13. *Let* $M \in [X]^w$ *be any multi-set drawn from* X *.Then The family* $\{M_\alpha : \alpha \in (0, w]\}$ *of all weak* α *-levels of M is a descending family.*

Proof. Trivial from the proposition 3.3. Thus from a multi-set M we have a descending family of crisp subsets of X . The conversely are study in the following proposition.

Proposition 3.13. *Let a family* $\{A_{\alpha_i} : A_{\alpha_i} \subseteq X, i = 1, 2....n\}$ *which satisfy the condition* $A_{\alpha_i} \supseteq A_{\alpha_{i+1}} \ \forall \ \ i \in \{1, 2, ...n - 1\}$. *Then we have a multi-set from these sets.*

Proof. Define a function $C_{A^\star} : X \to N$ as follows

$$C_{A^\star}(x) = \begin{cases} \max\{\alpha_i\} - \alpha_i & if\ x \in A_{\alpha_i}, i \neq 1 \\ \max\{\alpha_i\} & if\ x \in A_{\alpha_1} \end{cases}$$

Then $A^\star$ is a multi-set induced by the family $\{A_{\alpha_i} : A_{\alpha_i} \subseteq X\}$.

Proposition 3.13. *The* α *-levels of* $A^\star$ *is the family* $\{A_{\alpha_i} : A_{\alpha_i} \subseteq X\}$.

Proof. Let $\max\{\alpha_i\} = \alpha^\star$ then , $(A^\star)_{\alpha^\star} = \{x : C_{A^\star}(x) \geq \alpha^\star\}$. Then $(A^\star)_{\alpha^\star} = A_{\alpha_1}$and $(A^\star)_{\alpha^\star - \alpha_2} = A_{\alpha_2}$also we have $A^\star_{\alpha^\star - \alpha_3} = A_{\alpha_3}$and so on. Hence we have the result.

4. α -level multi-sets

In this section we studyα -level mult-set from a mult-set and study some properties.

Definition 4.1. *Let M be a multi-set drawn from a set X, and* $A \subseteq M$ *we define a multi-subset* $\tilde{A}_\alpha$ *on M by*

$$c_{\tilde{A}_\alpha}(x) = \begin{cases} C_A(x) & if x \in A_\alpha \\ 0 & otherwise \end{cases}$$

It is called weak α*-level multi-set induced by A.*

Definition 4.2. *Let M be a multi-set on X, and* $A \subseteq M$ *we define a multi-subset* $\tilde{A}_{\overline{\alpha}}$ *on M by*

$$c_{\tilde{A}_{\overline{\alpha}}}(x) = \begin{cases} C_A(x) & if x \in A_{\overline{\alpha}} \\ 0 & otherwise \end{cases}$$

It is called strong α*-level multi-set induced by A.*

Proposition 4.3. *For any two msets* $A, B \in [X]^w$ *we have :*

1. $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha \Rightarrow A_\alpha \subseteq B_\alpha \forall \alpha \in (0, w]$.
2. $\tilde{A}_{\overline{\alpha}} \subseteq \tilde{B}_{\overline{\alpha}} \Rightarrow A_{\overline{\alpha}} \subseteq B_{\overline{\alpha}} \forall \alpha \in [0, w)$.

Proof.

1. If $\exists x \in A_\alpha, x \notin B_\alpha$,then $c_{\tilde{A}_\alpha}(x) = C_A(x) > 0$and $c_{\tilde{B}_\alpha}(x) = 0$ which is contradiction for $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$.
2. If $\exists x \in A_{\overline{\alpha}}, x \notin B_{\overline{\alpha}}$,then $c_{\tilde{A}_{\overline{\alpha}}}(x) = C_A(x) > 0$ and $c_{\tilde{B}_{\overline{\alpha}}}(x) = 0$ which is contradiction for $\tilde{A}_{\overline{\alpha}} \subseteq \tilde{B}_{\overline{\alpha}}$.

Proposition 4.4. *For any two msets $A, B \in [X]^w$ we have :*

1. $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha \Leftrightarrow A \subseteq B \forall \alpha \in (0, w]$.
2. $\tilde{A}_{\overline{\alpha}} \subseteq \tilde{B}_{\overline{\alpha}} \Leftrightarrow A \subseteq B \forall \alpha \in [0, w)$.

Proof. We only show that if $A \subseteq B$ then,$\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$

1. $\exists x \in X s.t C_{\tilde{A}_\alpha}(x) > C_{\tilde{B}_\alpha}(x)$,then $\exists m \in N$ s.t $c_{\tilde{A}_\alpha}(x) = C_A(x) \geq m > c_{\tilde{B}_\alpha}(x)$ which is contradiction for $A_\alpha \subseteq B_\alpha \forall \alpha$ the second direction is obvious from proposition 4.3.
2. suppose that $\tilde{A}_{\overline{\alpha}} \nsubseteq \tilde{B}_{\overline{\alpha}} \Rightarrow \exists x \in X$ s.t$c_{\tilde{A}_{\overline{\alpha}}}(x) = C_A(x) > m > c_{\tilde{B}_{\overline{\alpha}}}(x) \Rightarrow \exists x \in X$ s.t $x \in A_{\overline{\alpha}}, x \notin B_{\overline{\alpha}}$ which is contradiction because that $A_{\overline{\alpha}} \subseteq B_{\overline{\alpha}} \forall \alpha \in [0, w)$.

Proposition 4.5. *For any two msets A and B on X we have :*

1. $\tilde{A}_\alpha = \tilde{B}_\alpha \Leftrightarrow A = B \forall \alpha \in (0, w]$.
2. $\tilde{A}_{\overline{\alpha}} = \tilde{B}_{\overline{\alpha}} \Leftrightarrow A = B \forall \alpha \in [0, w)$.

Proof. It obvious from the above propositions.

Proposition 4.6. *For any two msets $A, B \in [X]^w$ we have :*

1. $\tilde{A}_\alpha \cap \tilde{B}_\alpha = (\widehat{A \cap B})_\alpha \forall \alpha \in (0, w]$.
2. $\tilde{A}_{\overline{\alpha}} \cap \tilde{B}_{\overline{\alpha}} = (\widehat{A \cap B})_{\overline{\alpha}} \forall \alpha \in [0, w)$.

Proof. Since

$$C_{\tilde{A}_\alpha \cap \tilde{B}_\alpha}(x) = \min\{C_{\tilde{A}_\alpha}(x), C_{\tilde{B}_\alpha}(x)\} = \begin{cases} \min\{C_A(x), C_B(x)\} & if\ x \in (A_\alpha \cap B_\alpha) \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} C_{A \cap B}(x) & if\ x \in (A \cap B)_\alpha \\ 0 & otherwise \end{cases}$$

$$= \ C_{(\widehat{A \cap B})_\alpha}(x)$$

Then we have the result.

Proposition 4.7. *For any two msets A and B on X we have:*

1. $\tilde{A}_\alpha \cup \tilde{B}_\alpha = (\widehat{A \cup B})_\alpha \forall \alpha \in (0, w]$.

2. $\tilde{A}_{\overline{\alpha}} \cup \tilde{B}_{\overline{\alpha}} = (\widehat{A \cup B})_{\overline{\alpha}} \forall \alpha \in [0, w)$.

Proof.

1. $C_{(\widehat{A \cup B})_\alpha}(x) = \begin{cases} C_{A\cup B}(x) & if\ x \in (A\cup B)_\alpha \\ 0 & otherwise \end{cases}$
   then $C_{(\widehat{A \cup B})_\alpha}(x) = \begin{cases} max\{C_A(x), C_B(x)\} & if\ x \in A_\alpha \cup B_\alpha \\ 0 & otherwise \end{cases}$
   then we have $C_{(\widehat{A \cup B})_\alpha}(x) = \begin{cases} max\{C_{\tilde{A}_\alpha}(x), C_{\tilde{B}_\alpha}(x)\} & if\ x \in A_\alpha \cup B_\alpha \\ 0 & otherwise \end{cases}$
   Then $C_{(\widehat{A \cup B})_\alpha}(x) = \begin{cases} C_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x) & if\ x \in A_\alpha \cup B_\alpha \\ 0 & otherwise \end{cases}$
   Hence we have $C_{(\widehat{A \cup B})_\alpha}(x) = C_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x)$

2. Similar to 1

5. Cartesian product and α -level multi-sets

In this section we study the Cartesian product of multi-sets and some of its properties also we study α -level of Cartesian product of multi-sets.

Definition 5.1. *Let A be a multi-set on X and B be a multi-set on Y .The product of two multi-sets A and B is define by $A \times B = \{(m/x, n/y)/mn : m/x \in A, n/y \in B\}$, where $C_{A\times B}(x, y) = C_A(x).C_B(y)$.*

Definition 5.2. *Let A be a multi-set on X and B be a multi-set on Y .The weak α -level of the multi-set $A \times B$ is define by*
$(A \times B)_\alpha = \{(x, y) : C_{A\times B}(x, y) \geq \alpha\}$
The strong α -level of the multi-set $A \times B$ is define by
$(A \times B)_{\overline{\alpha}} = \{(x, y) : C_{A\times B}(x, y) > \alpha\}$.

Proposition 5.3. *For any two msets A and B on X we have:*

1. $(A \times B)_\alpha \supseteq A_\alpha \times B_\alpha$

2. $(A \times B)_{\overline{\alpha}} \supseteq A_{\overline{\alpha}} \times B_{\overline{\alpha}}$

Proof.

1. Let $(x, y) \in A_\alpha \times B_\alpha$
   $\Rightarrow x \in A_\alpha, y \in B_\alpha$
   $\Rightarrow C_A(x) \geq \alpha, C_B(y) \geq \alpha$

$\Rightarrow C_A(x).C_B(y) \geq \alpha$
$\Rightarrow C_{A\times B}(x,y) \geq \alpha$
$\Rightarrow (x,y) \in (A\times B)_\alpha$
Then,$(A\times B)_\alpha \supseteq A_\alpha \times B_\alpha$

2. Similar to a weak α- level .

The conversely is not true
Example 4.8. Let$X = \{a,b,c,d,e\}$, $M = \{3/a, 4/b, 2/c, 5/d\}$, $A = \{3/a, 2/c\}, B = \{4/b, 5/d\}$ then $A\times B = \{(3/a,4/b)/12, (3/a,5/d)/15, (2/c,4/b)/8, (2/c,5/d)/15\}$ Let $\alpha = 4$ then we have$A_\alpha = \phi, B_\alpha = \{b,d\}, A_\alpha \times B_\alpha = \phi$ But $(A\times B)_\alpha = \{(a,b),(a,d),(c,d),(c,b)\}$.

Definition 5.4. *Let M be a multi-set on X ,and $A, B \subseteq M$we define a multi-subset $(\widetilde{A\times B})_\alpha$ on $M \times M$ by*
$$c_{(\widetilde{A\times B})_\alpha}(x,y) = \begin{cases} C_{A\times B}(x,y) & if\ (x,y)\in (A\times B)_\alpha \\ 0 & otherwise \end{cases}$$
It is called weak α-level multi-set product.

Definition 5.5. *Let M be a multi-set on X ,and $A, B \subseteq M$we define a multi-subset $(\widetilde{A\times B})_{\overline{\alpha}}$ on $M \times M$ by*

$$c_{(\widetilde{A\times B})_{\overline{\alpha}}}(x,y) = \begin{cases} C_{A\times B}(x,y) & if(x,y)\in (A\times B)_{\overline{\alpha}} \\ 0 & otherwise \end{cases}$$
It is called strong α-level multi-set product.

Proposition 5.3. *For any two msets A and B on X we have :*

1. $(\widetilde{A\times B})_\alpha \supseteq \widetilde{A}_\alpha \times \widetilde{B}_\alpha$.

2. $(\widetilde{A\times B})_{\overline{\alpha}} \supseteq \widetilde{A}_{\overline{\alpha}} \times \widetilde{B}_{\overline{\alpha}}$.

Proof.

1. Since
$$C_{(\widehat{A\times B})_\alpha}(x,y) = \begin{cases} C_{A\times B}(x,y) & if(x,y)\in (A\times B)_\alpha \\ 0 & otherwise \end{cases}$$
and $C_{\widetilde{A_\alpha}\times\widetilde{B_\alpha}}(x,y) = C_{\widetilde{A}_\alpha}(x)$.
$$C_{\widetilde{B}_\alpha}(y) = \begin{cases} C_{A(x)}.C_B(y) & if\ x\in A_\alpha and\ y\in B_\alpha \\ 0 & otherwise \end{cases}$$
$$= \begin{cases} C_{A(x)}.C_B(y) & if\ (x,y)\in A_\alpha\times B_\alpha \\ 0 & otherwise \end{cases}$$

$$\leq \begin{cases} C_{A(x)}.C_B(y) & if(x,y) \in (A \times B)_\alpha \\ 0 & otherwise \end{cases}$$
$$\leq C_{(\widehat{A \times B})_\alpha}(x,y)$$

Hence we have $(\widehat{A \times B})_\alpha \supseteq \tilde{A}_\alpha \times \tilde{B}_\alpha$

2. Similar to 1

6. Multi-set function

In this section we study the concept of multi-set function , the image and the inverse image of a multi-set.

Definition 6.1. [5,6] *An mset relation f is called an mset function if for every element m/x in domain f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x,y)$ and $C_2(x,y)$.*

Definition 6.2. [6,5] *A mset function f is called one-one (injective) if no two elements in Dom f have the same image under f with $C_1(x,y) \leq C_2(x,y), \quad forallx, y$.*

Definition 6.3. *Let M_1 be a mset drawn from a set X ,and M_2be a mset drawn from a set Y. Let $f : M_1 \to M_2$ be a mset function from M_1 to M_2 .Let A be a submset of M_1, the image of A under the function f denoted by $f(A)$ where $f(A) : Y \to \mathbf{N}$ define by*

$$C_{f(A)}(y) = \begin{cases} min\{M_2(y), sup\{C_A(x) : f(x) = y\}\} & if\ f^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

Example 6.4. Let$M = \{5/a, 4/b, 4/c, 3/d\}$ and N $= \{7/x, 5/y, 6/z, 4/w\}$ be two msets and let $f : M \to N$ given by $f = (5/a, 5/y)/25$, $(4/b, 6/z)/24$, $(4/c, 4/w)/16$, $(3/d, 6/z)/18$, $A_1 = \{2/a, 2/b\ , 3/c\}$, $A_2 = \{5/a\ , 2/b, 3/c\}$ then,the images of A_1 and A_2 are $f(A_1) = \{2/y\ , 2/z, 3/w\}$,$f(A_2) = \{5/y, 2/z, 3/w\}$.
In the following we study the properties of the mset function.

Proposition 6.5. *Let$f : M_1 \to M_2$ be a mset function and A_1, A_2 be submsets of M_1. Then,*

1. $A_1 \subseteq A_2 \Rightarrow f(A_1) \leq f(A_2)$
2. $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$
3. $f(A_1 \cap A_2) \leq f(A_1) \cap f(A_2)$
4. $f(A_1) \oplus f(A_2) = f(A_1 \oplus A_2)$
5. $f(A_1) \ominus f(A_2) \leq f(A_1 \ominus A_2)$

Proof.

1. $C_{f(A_1)}(y)) = min\{M_2(y), sup\{C_{A_1}(x) : f(x) = y\}\}$. Since, $C_{A_1}(x) \leq C_{A_2}(x)$ $\forall x \in X$, then we have $C_{f(A_1)}(y)) = min\{M_2(y), sup\{C_{A_1}(x) : f(x) = y\}\} \leq min\{M_2(y), sup\{C_{A_2}(x) : f(x) = y\}\} = C_{f(A_2)}(y))$. Thus, we have $f(A_1) \subseteq f(A_2)$

2. Since $A_1, A_2 \subseteq A_1 \cup A_2$ then, $f(A_1), f(A_2) \subseteq f(A_1 \cup A_2)$ $C_{f(A_1 \cup A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1 \cup A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{(A_1 \cup A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y),\ sup\{\max\{C_{(A_1)}(x), C_{(A_2)}(x)\}\} : f(x) = y\}\} = min\{C_{M_2}(y), \max(sup\ \{C_((A_1)(x) : f(x) = y\}, sup\ \{C_((A_2)(x) : f(x) = y\})\} = \max(min\{C_{M_2}(y), sup\ \{C_{(A_1)}(x) : f(x) = y\}, min\{C_{M_2}(y), sup\ \{ C_{(A_2)}(x) : f(x) = y\}) = \max\{C_{f(A_1)}(y), C_{f(A_2)}(y)\}$. Thus, $C_{f(A_1 \cup A_2)}(y) = \max\{C_{f(A_1)}(y), C_{f(A_2)}(y)\}$. Consequently, we have $f(A_1) \cup f(A_2) = f(A_1 \cup A_2)$

3. $C_{f(A_1 \cap A_2)}(y) = min\{C_{M_2}(y),$
$sup\{C_{(A_1 \cap A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{\min\{C_(A_1)(x), C_(A_2)(x)\} : f(x) = y\}\} \leq min\{C_{M_2}(y), \min\{\sup\{C_(A_1)(x) : f(x) = y\}, \sup\{C_(A_2)(x)\} : f(x) = y\}\} \leq min\{\min\{C_{M_2}(y), \sup\{C_(A_1)(x) : f(x) = y\}, \min\{\{C_{M_2}(y)$
$, \sup\{C_(A_2)(x)\} : f(x) = y\}\} \leq \min\{C_{f(A_1)}(y), C_{f(A_2)}(y)\} = C_{f(A_1) \cap f(A_2)}(x)$. Thus $f(A_1 \cap A_2) \leq f(A_1) \cap f(A_2)$

4. $C_{f(A_1 \oplus A_2)}(y) = min\{C_{M_2}(y), sup$
$\{C_{(A_1 \oplus A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{A_1}(x) + C_{A_2}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup\{C_{A_1}(x) : f(x) = y\}\} + \sup\{C_{A_2}(x) : f(x) = y\}\}$
$= min\{C_{M_2}(y), sup\{C_{A_1}(x) : f(x) = y\}\} + \min\{C_{M_2}(y), \sup\{C_{A_2}(x) : f(x) = y\}\} = C_{f(A_1)}(y) + C_{f(A_2)}(y)$. Consequently, $f(A_1) \oplus f(A_2) = f(A_1 \oplus A_2)$.

5. $C_{f(A_1 \ominus A_2)}(y) = min\{C_{M_2}(y), sup\{C_{(A_1 \ominus A_2)}(x) : f(x) = y\}\} = min\{C_{M_2}(y), sup$ $\{\max\{C_{A_1}(x) - C_{A_2}(x), 0\} : f(x) = y\}\} \geq \max\ \{C_{f(A_1)}(y) - C_{f(A_2)}(y), 0\}$. Consequently, $f(A_1) \ominus f(A_2) \leq f(A_1 \ominus A_2)$

Proposition 6.6. *For any mset A on X we have :*

1. $(f(A))_\alpha \supseteq f(A_\alpha)$

2. $(f(A))_{\overline{\alpha}} = f(A_{\overline{\alpha}})$

Proof.

1. Let $y \in f(A_\alpha) \Rightarrow \exists x \in A_\alpha s.t f(x) = y \Rightarrow \exists x \in X s.t. C_A(x) \geq \alpha, f(x) = y$
$\Rightarrow \sup\{C_A(x) : f(x) = y\} \geq \alpha \Rightarrow C_{f(A)}(y) \geq \alpha \Rightarrow y \in (f(A))_\alpha$. Hence we have $f(A_\alpha) \subseteq (f(A))_\alpha$

2. Similar to the above.

Proposition 6.7. *For any two msets A and B drawn from X we have :*

1. $\widetilde{(f(A))}_{\alpha} \supseteq f(\widetilde{A_{\alpha}})$

2. $\widetilde{(f(A))}_{\overline{\alpha}} = f(\widetilde{A}_{\overline{\alpha}})$

Proof.

1. $C_{f(\widetilde{A_{\alpha}})}(y) = \begin{cases} \sup\{C_{\widetilde{A_{\alpha}}}(x) : f(x) = y\} & if\ x \in A_{\alpha} \\ 0 & otherwise \end{cases}$
$\leq \begin{cases} C_{fA}(y) & if\ x \in f(A)_{\alpha} \\ 0 & otherwise \end{cases}$
$= C_{\widetilde{f(A))_{\alpha}}}(y).$
Thus,We have $C_{f(\widetilde{A_{\alpha}})}(y) \leq C_{\widetilde{f(A))_{\alpha}}}(y)\ \ \forall y.$

2. Similar to the weak case

Definition 6.8. *Let$f : X \to Y$ be a function from X into Y and let B be submset ofM_2,then ,the map $f^{-1} : M_1 \to M_2$define by $C_{f^{-1}B}(x) = min\{C_{M_1}(x), C_B(f(x))\}$is called the inverse image of a mset B of M_2 ,whereM_1 is a m-set on X andM_2 be a m-subset on Y.*

Example 6.9. Let $M = \{5/a, 4/b, 4/c, 3/d\}$ and N = $\{7/x, 5/y, 6/z, 4/w\}$ be two m-sets. Let $g : M \to N$ given by $g = \{(5/a, 7/y)/35,\ (4/b, 7/x)/28,\ (4/c, 6/z)/24,\ (3/d, 4/w)/12\}$, $B_1 = \{5/y,\ 6/z, 4/w\}$, $B_2 = \{2/y,\ 3/z, 2/w\}$ then, $f^{-1}B_1 = \{5/a,\ 4/c, 3/d\}$ and $f^{-1}B_2 = \{2/a, 3/c, 2/d\}$.
In the following we study the properties of the inverse of mset function.

Proposition 6.10. *Let $f : M_1 \to M_2$ be a mset function andB_1, B_2 be submsets of M_2. Then,*

1. $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

2. $f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2)$

3. $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$

4. $f^{-1}(B_1) \oplus f^{-1}(B_2) = f^{-1}(B_1 \oplus B_2)$

5. $f^{-1}(\phi) = \phi$,$f^{-1}((M_2) = M_1$

Proof.

1. $C_{f^{-1}(B_1)}(x) = \min\{C_{M_1}(x), C_{B_1}(f(x))\} \leq \min\{C_{M_1}(x), C_{B_2}(f(x))\} = C_{f^{-1}(B_1)}$ (x)

2. $C_{f^{-1}(B_1 \cup B_2)}(x) = \min\{C_{M_1}(x), C_{B_1 \cup B_2}(f(x))\} = \min\{C_{M_1}(x), \max\{C_{B_1}(f(x)),$ $C_{B_2}(f(x))\}\} = \max\{C_{f^{-1}(B_1)}(x), C_{f^{-1}(B_2)}(x)\} = C_{f^{-1}(B_1) \cup f^{-1}(B_2)}(x)$

3. $C_{f^{-1}(B_1 \cap B_2)}(x) = \min\{C_{M_1}(x), C_{B_1 \cap B_2}(f(x))\} = \min\{C_{M_1}(x), \min\{C_{B_1}(f(x)),$ $C_{B_2}(f(x))\}\} = \min\{C_{f^{-1}(B_1)}(x), C_{f^{-1}(B_2)}(x)\} = C_{f^{-1}(B_1) \cap f^{-1}(B_2)}(x)$

. The others are similar.

Proposition 6.11. *Let $f : M_1 \to M_2$ be a mset function and B be submset of M_2. Then*

1. $(f^{-1}(B))_\alpha = f^{-1}(B_\alpha)$

2. $(f^{-1}(B))_{\overline{\alpha}} = f^{-1}(B_{\overline{\alpha}})$

Proof.

1. Let $x \in (f^{-1}(B))_\alpha \Leftrightarrow C_{(f^{-1}(B))} \geq \alpha \Leftrightarrow C_B(f(x)) \geq \alpha \Leftrightarrow f(x) \in B_\alpha \Leftrightarrow x \in$ $f^{-1}(B_\alpha)$

2. Similar to the weak cut

Proposition 6.12. *For any two msets A and B on X we have :*

1. $(\widetilde{f^{-1}(B)})_\alpha = f^{-1}(\widetilde{B_\alpha})$

2. $(\widetilde{f^{-1}(B)})_{\overline{\alpha}} = f^{-1}(\widetilde{B}_{\overline{\alpha}})$

Proof.

1. $C_{f^{-1}(\widetilde{B_\alpha})}(x) = \begin{cases} \sup\{C_{\widetilde{A_\alpha}}(x) : f(x) = y\} & if\ x \in A_\alpha \\ 0 & otherwise \end{cases}$
$\leq \begin{cases} C_{fA}(y) & if\ x \in f(A)_\alpha \\ 0 & otherwise \end{cases}$
$= C_{\widetilde{f(A))_\alpha}}(y)$.
Thus, we have $C_{f(\widetilde{A_\alpha})}(y) \leq C_{\widetilde{f(A))_\alpha}}(y)\ \ \forall y$

2. Similar to the weak case

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