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Fractional q-Derivative of Generalized Miller-Ross Function

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Abstract: This paper is devoted to fractional q-derivative of special functions. To begin with the theorem on term by term q-fractional differentiation has been derived. The result is an extension of an earlier result due to Yadav and Purohit [8] As a special case, of fractional q-differentiation of Generalized Miller-Ross function has been obtained.
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Definition

1.1. q-Analogue of Differential Operator

Al-Salam [3], has given the q-analogue of differential operator as

$$D_q f(x) = \frac{f(xq) - f(x)}{x(q-1)} \tag{1.1}$$

This is an inverse of the q-integral operator defined as

$$\int_x^\infty f(t)d(t;q) = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k}) \tag{1.2}$$

where $0 < |q| < 1$.

1.2. Fractional q-Derivative of Order α

The fractional q-derivative of order α is defined as

$$D^{\alpha}_{x,q} f(x) = \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x-yq)_{-\alpha-1} f(y) d(y;q) \tag{1.2.1}$$

where $Re(\alpha) < 0$
As a particular case of (1.2.1), we have

$$D^{\alpha}_{x,q} x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1} \tag{1.2.2}$$

2. Main Results

In this section we drive the results on term by term q-fractional differentiation of a power series. As particular case we will the fractional q-differentiation of the Generalized M-Series and exponential series.

Theorem 1: If the Generalized Miller-Ross function ${}^{\alpha}N^{\alpha,\beta}_{p,q}(z)$ converges absolutely for $|q| < \rho$ then

$$D^{\mu}_{z,q}\left\{ z^{\lambda-1} \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k z^{k+\beta}}{\Gamma(\alpha k+\beta+1)} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k}{\Gamma(\alpha k+\beta+1)} D^{\mu}_{z,q} z^{k+\lambda+\beta-1} \tag{2.1}$$

where $Re(\lambda) > 0,\ Re(\mu) < 0,\ 0 < |q| < 1$

Proof: Starting from the left side and using equation (1.2.1), we have

$$D^{\mu}_{z,q}\left\{ z^{\lambda-1} \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k z^{k+\beta}}{\Gamma(\alpha k+\beta+1)} \right\}$$

$$= \frac{1}{\Gamma_q(-\mu)} \int_0^z (z-yq)_{-\mu-1} y^{\lambda-1} \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k y^{k+\beta}}{\Gamma(\alpha k+\beta+1)} d(y;q)$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \int_0^1 (1-tq)_{-\mu-1} t^{\lambda-1} \sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k t^{k+\beta} z^{k+\beta}}{\Gamma(\alpha k+\beta+1)} d(t;q) \tag{2.2}$$

Now the following observation are made

(i) $\sum_{k=0}^{\infty} \frac{(a_1)_k ...(a_p)_k}{(b_1)_k ...(b_q)_k} \frac{a^k t^{k+\beta} z^{k+\beta}}{\Gamma(\alpha k+\beta+1)}$ converges absolutely and therefore uniformly on domain of x over the region of integration.

(ii) $\int_0^1 |(1-tq)_{-\mu-1} t^{\lambda-1}| d(t;q)$ is convergent provided $Re(\lambda) > 0,\ Re(\mu) < 0,\ 0 < |q| < 1$

Therefore the order of integration and summation can be interchanged in (2.2) to obtain.

$$= \frac{z^{\lambda-\mu-1}}{\Gamma_q(-\mu)} \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{a^k z^{k+\beta}}{\Gamma(\alpha k+\beta+1)} \int_0^1 (1-tq)_{-\mu-1} t^{\lambda+k+\beta-1} d(t;q)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{a^k}{\Gamma(\alpha k+\beta+1)} D^{\mu}_{z,q} z^{k+\lambda+\beta-1}$$

Hence the statement (2.1) is proved.

3. Some Special Cases

(i) If we take $\alpha = 0, \beta = 0$ in equation (2.1) it becomes the fractional q-derivative of power series.

$$D^{\mu}_{z,q} \left\{ z^{\lambda-1} \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} a^k z^k \right\} = \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} a^k D^{\mu}_{z,q} z^{k+\lambda-1} \tag{3.1}$$

This equation (3.1) is known result given by Yadav and Purohit [8].

(ii) When $\alpha = 1, \beta = 0, a = 1$ and no upper or lower parameter in(2.1), we have

$$D^{\mu}_{z,q} \left\{ z^{\lambda-1} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} \right\} = \sum_{k=0}^{\infty} \frac{1}{k!} D^{\mu}_{z,q} \{z^{k+\lambda-1}\} \tag{3.2}$$

equivalently,

$$D^{\mu}_{z,q} \left\{ z^{\lambda-1} e^z \right\} = \sum_{k=0}^{\infty} \frac{1}{k!} D^{\mu}_{z,q} \{z^{k+\lambda-1}\} \tag{3.3}$$

Thus the equation reduces to fractional q-derivative of exponential function.

(iii) If no upper or lower parameter, we have

$$D^{\mu}_{z,q} \left\{ z^{\lambda-1} \sum_{k=0}^{\infty} \frac{a^k z^{k+\beta}}{\Gamma(\alpha k+\beta+1)} \right\} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta+1)} D^{\mu}_{z,q} z^{k+\lambda+\beta-1} \tag{3.3}$$

Hence the series convert in fractional q-derivative of Miller-Ross function. Thus it is the complete analysis of the statement (2.1).

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