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A NOTE ON SYMMETRIC BILATERAL BAILEY TRANSFORM

Satya Prakash Singh, S.N. Singh and Vijay Yadav

Department of Mathematics,
Tilak Dhari PG College,
Jaunpur, Uttar Pradesh, INDIA

Email : snsp39@gmail.com

*Department of Mathematics,
S.P.D.T. College, Andheri (E), Mumbai, INDIA

Email : vijaychottu@yahoo.com

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Abstract: In this paper, using symmetric Bailey transform and some known summation formulas of bilateral basic hypergeometric series, certain transformations of bilateral q-series into unilateral q-series have been established.

Keywords and Phrases: Bailey transform, symmetric bilateral Bailey transform, asymmetric bilateral Bailey transform, bilateral hypergeometric series, q-series, summation formula, transformation formula.

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1. Notations and Definitions

Let q be a complex number such that $0 < |q| < 1$. We define the q-shifted factorial for all integers k by

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \tag{1.1}$$

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}. \tag{1.2}$$

For brevity, we employ the condensed notation,

$$(a_1, a_2, a_3, ..., a_r; q)_k = (a_1; q)_k (a_2; q)_k ... (a_r; q)_k, \tag{1.3}$$

where k is an integer or infinity.
Following [3; (1.2.22) p. 4] we define generalized basic hypergeometric series as,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, a_3, ..., a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, ..., a_r; q)_n z^n}{(q, b_1, b_2, b_3, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r} \tag{1.4}$$

where $1 + s \geq r$ and $|z| < 1$.
Again, following [3; (5.1.1) p. 125] the general bilateral hypergeometric series is defined by,

$${}_r\Psi_s \left[\begin{matrix} a_1, a_2, a_3, ..., a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, a_3, ..., a_r; q)_n z^n}{(b_1, b_2, b_3, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{s-r}, \tag{1.5}$$

where $s \geq r$ and $\left|\frac{b_1 b_2 ... b_s}{a_1 a_2 ... a_r}\right| < |z| < 1$.

2. Introduction

In 1947, W.N. Bailey established a simple but widely useful transform known as Bailey transform. It states,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{2.2}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{2.3}$$

subject to conditions on the four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

Andrews [2] generalized the Bailey transform as the following two bilateral versions.

(a) Symmetric Bilateral Bailey Transform
If
$$\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.4}$$
and
$$\gamma_n = \sum_{r=|n|}^{\infty} \delta_r u_{r-n} v_{r+n} \tag{2.5}$$
then
$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.6}$$
subject to conditions on four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

(b) Asymmetric Bilateral Bailey Transform
Let $m = \max.(n, -n-1)$.
If
$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} v_{n+r} \tag{2.7}$$
and
$$\gamma_n = \sum_{r=m}^{\infty} \delta_r u_{r-n} v_{r+n+1} \tag{2.8}$$
then
$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.9}$$
subject to the conditions on four sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n which make all the infinite series absolutely convergent.

Here, we shall make use of symmetric bilateral Bailey transform in our analysis.

Choosing $u_r = v_r = \dfrac{1}{(q;q)_r}$, n a positive integer and $\delta_r = (\alpha, \beta; q)_r \left(\dfrac{q}{\alpha\beta}\right)^r$ in (2.4) and (2.5) we get,
$$\beta_n = \frac{1}{(q;q)_n^2} \sum_{r=-n}^{n} \frac{(-1)^r \alpha_r (q^{-n};q)_r q^{nr}}{(q^{1+n};q)_r q^{r(r-1)/2}}, \tag{2.10}$$
$$\gamma_n = \frac{(\alpha, \beta; q)_n \left(\frac{q}{\alpha\beta}\right)^n}{(q;q)_{2n}} {}_2\Phi_1 \left[\begin{matrix} \alpha q^n, \beta q^n; q; q/\alpha\beta \\ q^{1+2n} \end{matrix} \right],$$

which gives,

$$\gamma_n = \frac{(q/\alpha, q/\beta; q)_\infty}{(q, q/\alpha\beta; q)_\infty} \frac{(\alpha, \beta; q)_n (q/\alpha\beta)^n}{(q/\alpha, q/\beta; q)_n}. \tag{2.11}$$

Putting these values in symmetric Bailey transform it takes the following new form, If

$$\beta_n = \frac{1}{(q;q)_n^2} \sum_{r=-n}^{n} \frac{(q^{-n};q)_r (-q)^r \alpha_r}{q^{r(r-1)/2} (q^{1+n};q)_r} \tag{2.12}$$

then

$$\frac{(q/\alpha, q/\beta; q)_\infty}{(q, q/\alpha\beta; q)_\infty} \sum_{n=-\infty}^{\infty} \alpha_n \frac{(\alpha, \beta; q)_n (q/\alpha\beta)^n}{(q/\alpha, q/\beta; q)_n} = \sum_{n=0}^{\infty} \beta_n (\alpha, \beta; q)_n \left(\frac{q}{\alpha\beta}\right)^n. \tag{2.13}$$

We shall make use of (2.12), (2.13) and following summation formulas in further analysis.

(i)

$${}_1\Psi_1 \left[\begin{matrix} a; q; z \\ b \end{matrix} \right] = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}. \tag{2.14}$$

[3; App. II (II. 29) p. 239]

Putting $a = q^{-n}, b = q^{1+n}$ and zq^n for z in (2.14) we get,

$$\sum_{r=-n}^{n} \frac{(q^{-n};q)_r (zq^n)^r}{(q^{1+n};q)_r} = \frac{(q;q)_n^2 (z, q/z; q)_n}{(q;q)_{2n}}. \tag{2.15}$$

(ii)

$${}_3\Psi_3 \left[\begin{matrix} b, c, d; q; q/bcd \\ q/b, q/c, q/d \end{matrix} \right] = \frac{(q, q/bc, q/bd, q/cd; q)_\infty}{(q/b, q/c, q/d, q/bcd; q)_\infty}. \tag{2.16}$$

[3; App. II (II. 31) p. 239]

Taking $b = q^{-n}$ in (2.16) we have

$$\sum_{r=-n}^{n} \frac{(q^{-n}, c, d; q)_r (q^{1+n}/cd)^r}{(q^{1+n}, q/c, q/d; q)_r} = \frac{(q, q/cd; q)_n}{(q/c, q/d; q)_n}. \tag{2.17}$$

(iii)

$${}_2\Psi_2 \left[\begin{matrix} b, c; q; -aq/bc \\ aq/b, aq/c \end{matrix} \right] = \frac{\left(\frac{aq}{bc}; q\right)_\infty \left(\frac{aq^2}{b^2}, \frac{aq^2}{c^2}, q^2, aq, \frac{q}{a}; q^2\right)_\infty}{\left(\frac{aq}{b}, \frac{aq}{c}, \frac{q}{b}, \frac{q}{c}, -\frac{aq}{bc}; q\right)_\infty} \tag{2.18}$$

[3; App. II (II. 30) p. 239]

Taking $a = 1, b = q^{-n}$ in (2.18) we have,

$$\sum_{r=-n}^{n} \frac{(q^{-n}, c; q)_r \left(-\frac{q^{1+n}}{c}\right)^r}{\left(q^{1+n}, \frac{q}{c}; q\right)_r} = \frac{(q;q)_n(-q/c;q)_n}{(-q;q)_n(q/c;q)_n}. \tag{2.19}$$

(iv)

$${}_4\Psi_4\left[\begin{array}{l} -q\sqrt{a}, b, c, d; q; \dfrac{a^{3/2}q}{bcd} \\ -\sqrt{a}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{aq}{d} \end{array}\right] = \frac{\left(aq, \dfrac{aq}{bc}, \dfrac{aq}{bd}, \dfrac{aq}{cd}, \dfrac{q\sqrt{a}}{b}, \dfrac{q\sqrt{a}}{c}, \dfrac{q\sqrt{a}}{d}, q, \dfrac{q}{a}; q\right)_\infty}{\left(\dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{aq}{d}, \dfrac{q}{b}, \dfrac{q}{c}, \dfrac{q}{d}, q\sqrt{a}, \dfrac{q}{\sqrt{a}}, \dfrac{a^{3/2}q}{bcd}; q\right)_\infty}. \tag{2.20}$$

[3; App. II (II. 32) p. 239]

Taking $a = 1, b = q^{-n}$ in (2.20) we get,

$$\sum_{r=-n}^{n} \frac{(q^{-n}, -q, c, d; q)_r \left(\dfrac{q^{1+n}}{cd}\right)^r}{\left(q^{1+n}, -1, \dfrac{q}{c}, \dfrac{q}{d}; q\right)_r} = \frac{(q, q/cd; q)_n}{(q/c, q/d; q)_n}. \tag{2.21}$$

[3; App. II (II. 28) p. 239]

We shall also need Jacobi's triple product identity, viz.,

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -q/z; q^2)_\infty. \tag{2.22}$$

3. Main Results

In this section following transformations have been established.

(a)

$${}_2\Psi_3\left[\begin{array}{l} \alpha, \beta; q; zq/\alpha\beta \\ q/\alpha, q/\beta, 0 \end{array}\right] = \frac{(q, q/\alpha\beta; q)_\infty}{(q/\alpha, q/\beta; q)_\infty}\, {}_4\Phi_3\left[\begin{array}{l} \alpha, \beta, z, q/z; q; q/\alpha\beta \\ -q, \sqrt{q}, -\sqrt{q} \end{array}\right], \tag{3.1}$$

where $|q/\alpha\beta| < 1$.

(b)

$${}_4\Psi_5\left[\begin{array}{l} \alpha, \beta, c, d; q; q^2/\alpha\beta cd \\ q/\alpha, q/\beta, q/c, q/d, 0 \end{array}\right] = \frac{(q, q/\alpha\beta; q)_\infty}{(q/\alpha, q/\beta; q)_\infty}\, {}_3\Phi_2\left[\begin{array}{l} \alpha, \beta, q/cd; q; q/\alpha\beta \\ q/c, q/d \end{array}\right], \tag{3.2}$$

where $|q/\alpha\beta| < 1$.

(c)

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,c;q;q^2/\alpha\beta c\\ q/\alpha,q/\beta,q/c\end{array}\right]=\frac{(q,q/\alpha\beta;q)_\infty}{(q/\alpha,q/\beta;q)_\infty}\,{}_3\Phi_2\left[\begin{array}{l}\alpha,\beta,-q/c;q;q/\alpha\beta\\ -q,q/c\end{array}\right], \tag{3.3}$$

where $|q/\alpha\beta| < 1, |q^3/\alpha^2\beta^2c^2| < |q^2/\alpha\beta c| < 1$.

(d)

$$ {}_5\Psi_6\left[\begin{array}{l}\alpha,\beta,c,d,-q;q;q^2/\alpha\beta cd\\ q/\alpha,q/\beta,q/c,q/d,-1,0\end{array}\right]=\frac{(q,q/\alpha\beta;q)_\infty}{(q/\alpha,q/\beta;q)_\infty}\,{}_3\Phi_2\left[\begin{array}{l}\alpha,\beta,q/cd;q;q/\alpha\beta\\ q/c,q/d\end{array}\right], \tag{3.4}$$

where $|q/\alpha\beta| < 1$.

Proof of (3.1)-(3.4): Choosing $\alpha_r = (-z)^r q^{r(r-1)/2}$, $\dfrac{(c,d;q)_r(-q/cd)^r q^{r(r-1)/2}}{(q/c,q/d;q)_r}$, $\dfrac{(c;q)_r(q/c)^r q^{r(r-1)/2}}{(q/c;q)_r}$ and $\dfrac{(-q,c,d;q)_r(-q/cd)^r q^{r(r-1)/2}}{(-1,q/c,q/d;q)_r}$ in (2.12) and using the summation formulas (2.15), (2.17), (2.19) and (2.21) respectively one can calculate β_n in each case. Putting these values of α_n and β_n in (2.13) we get transformations (3.1), (3.2), (3.3) and (3.4) respectively.

4. Special Cases

In this section we shall discuss some special cases of the results (3.1)-(3.4).

(i) Taking $\alpha, \beta \to \infty$ in (3.1) we get,

$$\sum_{n=-\infty}^{\infty} q^{\frac{3}{2}n^2}\left(-zq^{-1/2}\right)^n = (q;q)_\infty \sum_{n=0}^{\infty}\frac{(z,q/z;q)_n q^{n^2}}{(q;q)_{2n}}. \tag{4.1}$$

Applying Jacobi's triple product identity (2.22) we have

$$\sum_{n=0}^{\infty}\frac{(z,q/z;q)_n q^{n^2}}{(q;q)_{2n}} = \frac{(zq,q^2/z,q^3;q^3)_\infty}{(q;q)_\infty}. \tag{4.2}$$

For $z = -1$, (4.2) yields,

$$\sum_{n=0}^{\infty}\frac{(-1,-q;q)_n q^{n^2}}{(q;q)_{2n}} = \frac{(-q,-q^2;q^3)_\infty}{(q,q^2;q^3)_\infty}. \tag{4.3}$$

For $z = 1$, (4.1) yields the Euler's identity,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{4.4}$$

Taking $\alpha, \beta, c, d \to \infty$ in (3.2) we get

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{5}{2}n^2} q^{-\frac{1}{2}n}, \tag{4.5}$$

Applying Jacobi's triple product identity (2.22) on the right hand side of (4.5) we get Rogers-Ramanujan identity,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4; q^5)_\infty}. \tag{4.6}$$

Taking $\alpha, \beta \to \infty$ in (3.2) we have,

$$\sum_{n=0}^{\infty} \frac{(q/cd;q)_n q^{n^2}}{(q, q/c, q/d; q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(c,d;q)_n \left(-\frac{q^2}{cd}\right)^n q^{\frac{3}{2}n(n-1)}}{(q/c, q/d; q)_n}. \tag{4.7}$$

For $c = d = 1$, (4.7) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}, \tag{4.8}$$

which is a known identity.
For $c = d = -1$, (4.7) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \frac{4}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(1+q^n)^2}, \tag{4.9}$$

which gives bilateral representation of mock-theta function $f(q)$.
Taking $\alpha, \beta, c \to \infty$ in (3.3) we find,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2;q^4)_\infty}{(q, q^3; q^4)_\infty} = (-q;q^2)_\infty, \tag{4.10}$$

which is a known identity.
For $\alpha, \beta \to \infty$ (3.3) yields,

$$\sum_{n=0}^{\infty} \frac{(-q/c;q)_n q^{n^2}}{(q^2;q^2)_n (q/c;q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(c;q)_n q^{n(3n+1)/2}}{(q/c;q)_n c^n}. \tag{4.11}$$

Replacing q by q^2 and then taking $c = q$ in (4.11) we get,

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n^2}}{(q^4;q^4)_n (q;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{3n^2}. \tag{4.12}$$

Applying Jacobi's identity (2.22) on the right hand side of (4.12) we get,

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{2n^2}}{(q^4;q^4)_n (q;q^2)_n} = \frac{(-q^3;q^6)_\infty^2}{(q^2,q^4;q^6)_\infty}. \tag{4.13}$$

Replacing q by $-q$ in (4.13) we have,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n^2}}{(q^4;q^4)_n (-q;q^2)_n} = \frac{(q^3;q^6)_\infty^2}{(q^2,q^4;q^6)_\infty}, \tag{4.14}$$

For $c = -1$, (4.11) yields,

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(1+q^n)}, \tag{4.15}$$

which is another bilateral representation of mock theta function $f(q)$.

References

[1] Andrews, G.E. and Berndt, B.C., Ramanujan's Lost Notebook, Part I, Springer 2005.

[2] Andrews, G.E., The Bailey transform and false theta functions, The Ramanujan Journal, Vol. 14, No. 1 (2007), 173-184.

[3] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, Cambridge (1991).