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ON HOMOGENEOUS EINSTEIN KROPINA SPACE

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Abstract: In this paper, we study homogeneous Einstein Kropina metric. First, we characterize the sufficient and necessary condition for a homogeneous Kropina metric to be Einstein and with vanishing S-curvature.Further, we study the conformal deformation of the metric $F(\alpha,\beta) = K(\alpha,\beta) + \varepsilon(x)\beta$, where $K(\alpha,\beta) = \dfrac{\alpha^2}{\beta}$ and $\varepsilon(x)$ depends on the position only. Finally, we prove that the conformal deformation of Kropina metric is single colored.

Keywords and Phrases: Homogeneous Finsler space, Einstein space, Ricci curvature, S-curvature, Berwald space, Single colored.

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1. Introduction

It is important to study the Einstein manifolds in Riemannian -Finsler geometry. A Finsler metric $F(x, y)$ on an n-dimensional manifold M is called an Einstein metric [28], if there exists a scalar function $\lambda(x)$ on M such that

$$Ric = \lambda(x)F^2.$$

Recentlty, some progress has been made on Einstein Finsler metrics of (α, β) type. In [3], the authors D. Bao and C. Robles have shown that every Randers metric of dimension $(n \geq 3)$ is necessarily Ricci constant. A 3-dimensional Randers metric is Einstein iff it is constant flag curvature, see more in ([2], [5], [7], [13], [21],

[23]). The invariant Einstein Finsler metrics on homogeneous manifolds are very interesting in Finsler geometry; see [10] for some results on homogeneous Einstein Randers metrics. In 2012, the author S. Deng studied the homogeneous Finsler spaces. The (α, β)-metric form an important class of Finsler metrics, which is of the form $F = \alpha\phi(\beta/\alpha)$ is a positive definite with $||\beta||_\alpha < b_0$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition [17]:

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''} > 0, \ \ |s| \leq b < b_0. \tag{1.1}$$

In this paper we consider the homogeneous (α, β)-metrics, i.e., $F = \dfrac{\alpha^2}{\beta}$ Kropina metric. The Kropina metric is just the special (α, β)-metric with $\phi(s) = \dfrac{1}{s}$. Therefore, $\phi(s)$ satisfies (1.1). It was considered by V.K. Kropina firstly [14]. Since, then many authors have been investigated the geometric properties of Kropina metric ([20], [26]).

The Ricci curvature plays an important role in Finsler geometry and is defined as the trace of the Riemannian curvature on each tangent space. In [27], L. Zhou first gave the formula of Riemannian curvature and Ricci curvature for (α, β)-metrics. Later, X. Cheng, Z. Shen and Y. Tian found some error of those formulas [7]. They also proved that if $\phi(s)$ is a polynomial in s, then the (α, β)-metric is Einstein if and only if it is Ricci flat and in [28], the authors, X. Zhang and Yi-Bing Shen studied on Einstein Kropina metric.

The present paper is organized as following: Based on the formula of the Ricci curvature for Kropina metric [28], we give a formula of Ricci curvature for homogeneous Kropina metric. Using this formula, we find a necessary condition related ϕ for F to be Einstein. Then, we show that if ϕ is normal. Moreover, on the compactness, we obtain a sufficient and necessary condition for a homogeneous Kropina metric to be Einstein with vanishing S-curvature. Further, we will study the conformal deformation of this metric. Under this, we prove that the conformal deformation of Kropina metric is single colored.

2. Preliminaries

In this section, we present some fundamental definitions and facts of Finsler geometry, see([1], [6], [7], [24]).

Definition 2.1. *Let V be an n-dimensional real vector space. A Minkowski norm on V is a real function F on V which is smooth on $V\backslash\{0\}$ and satisfies the following conditions:*

1. $F(u) \geq 0$, $\forall u \in V$,

2. $F(\lambda u) = \lambda F(u),\ \forall \lambda > 0,$

3. *Given any basis* $u_1, u_2, ..., u_n$ *of* V*, write* $F(y) = F(y^1, y^2, ..., y^n)$ *for* $y = y^1 u_1 + y^2 u_2 + ... + y^n u_n$*. Then the Hessian matrix*

$$g_{ij} = \left[\frac{1}{2}F^2\right]_{y^i y^j}$$

is positive definite at any point of $V \backslash \{0\}$*.*

For example, let $<, >$ be an inner product on V. Define $F(y) = \sqrt{< y, y >}$. Then F is a Minkowski norm. In this case it is called Euclidean.

A Finsler metric on a smooth manifold M is a function $F : TM \to [0, \infty)$ which is C^∞ on the slit tangent bundle $TM \backslash \{0\}$ and whose restriction to any tangent space $T_x M$, $x \in M$ is a Minkowski norm.

Every Finsler metric F induces a spray G on M defined by [22]

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i(x, y) = \frac{1}{4} g^{il}(x, y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\right\} y^j y^k.$$

G is globally defined vector field on TM. The notion of Riemannian curvature for Riemann metrics can be extended to Finsler metrics. For a non zero vector $y \in T_x M \backslash \{0\}$ the Riemannian curvature $R_y : T_x M \to T_x M$ is linear map defined by

$$R_y(u) = R^i_k(y) u^k \frac{\partial}{\partial x^i},\ \ u = u^i \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The trace of Riemann curvature R_y is scalar function Ric on TM defined by

$$Ric(y) = tr(R_y),$$

which is called the Ricci curvature of (M, F).

Recall, the notion of S-curvature of a Finsler space. In [10], [24]), the author Z. Shen introduced the notion of S-curvature of a Finsler spaces. It is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. S-curvature is a non-Riemannian quantity, i.e., any Riemannian manifold has vanishing S-curvature. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{e_i\}$ of V, let

$$\sigma_F = \frac{Vol(B^n)}{Vol\{(y^i) \in \mathbb{R}^n : F(y^i e_i) < 1\}}$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{e_i\}$. But it is easily seen that

$$\tau(y) = ln\frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}, \;\; y \in V\backslash\{0\}$$

is independent of the choice of the basis. $\tau = \tau(y)$ is called the distortion of (V, F). Now, let (M, F) be a Finsler space and $\tau(x, y)$ be the distortion of the Minkowski norm F_x on T_xM. For $y \in T_x\backslash\{0\}$, let $\sigma(t)$ be the geodesics with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Then, the quantity

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]|_{t=0},$$

is called the S-curvature of the Finsler space (M, F).

Now, we use the notations as in [7] to give a formula of the Ricci curvature of the given metric, let

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \;\; s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ';' denote the covariant derivative with respect to the Levi-Civita connection of α. Let

$$r^i_j = a^{im} r_{mj}, \;\; s^i_j = a^{im} s_{mj},$$
$$r_j = b^m r_{mj} = b_i r^i_j, \;\; s_j = b^m s_{mj} = b_i s^i_j,$$
$$r = r_{ij} b^i b^j = b^j r_j,$$

where $(a^{ij}) = (a_{ij})^{-1}$ and $b^i = a^{ij} b_j$. Further, let $r^i = a^{ij} r_j$, $s^i = a^{ij} s_j$. Denote $r_{i0} = r_{ij} y^j$, $s_{i0} = s_{ij} y^j$ and $r_{00} = r_{ij} y^i y^j$, $s_0 = s_i y^i$. Then, the author X. Zhang and Yi-Bing Shen proved the following;

Proposition 2.1. *([28]), For the Kropina metric $F = \frac{\alpha^2}{\beta}$, the Ricci curvature of F is given by*

$$Ric = \overline{Ric} + T, \tag{2.1}$$

where

$$\begin{aligned} T &= -\frac{\alpha^2}{b^4\beta}s_0 - \frac{r}{b^4}r_{00} + \frac{\alpha^2}{b^2\beta}b^k s_{0;k} + \frac{1}{b^2}b^k r_{00;k} + \frac{n-2}{b^2}s_{0;0} + \frac{n-1}{b^2\alpha^2}\beta r_{00;0} \\ &\quad + \frac{1}{b^2}(\frac{\alpha^2}{\beta}s_0 + r_{00})r^k_k - \frac{\alpha^2}{\beta}s^k_{0;k} - \frac{1}{b^2}r_{0;0} - \frac{2(2n-3)}{b^4}r_0 s_0 - \frac{n-2}{b^4}s_0^2 \\ &\quad - \frac{4(n-1)}{b^4\alpha^2}\beta r_{00}r_0 + \frac{2(n-1)}{b^4\alpha^2}\beta r_{00}s_0 + \frac{3(n-1)}{b^4\alpha^4}\beta^2 r_{00}^2 + \frac{2n}{b^2}s^k_0 r_{0k} + \frac{1}{b^4}r_0^2 \\ &\quad - \frac{\alpha^2}{b^2\beta}s^k_0 r_k + \frac{n-1}{b^2\beta}\alpha^2 s^k_0 s_k - \frac{\alpha^4}{2b^2\beta^2}s^k s_k - \frac{\alpha^2}{b^2\beta}s^k r_{0k} - \frac{\alpha^4}{4\beta^2}s^j_k s^k_j. \end{aligned}$$

Recall that the group of isometries of a Finsler space (M, F) is a Lie transformation group of M [9]. A Finsler metric M is called Homogeneous, if its isometry group acts transitivity on M. A homogeneous Finsler space can be expressed as $(G/H, F)$, where G is a connected Lie group, H is a compact subgroup of G and F is invariant under the action of G . Moreover, the action of G on G/H is almost effective and the Lie algebra g of G has a reductive decomposition

$$g = h + m,$$

where h is the Lie algebra of H and m is a subspace of g satisfying $ad(h)(m) \subset m$, $\forall h \in H$. We identify m with the tangent space $T_o(G/H)$ of G/H at the origin o through the mapping $X \to \frac{d}{dt}(exp(tX))|_{t=0}$. Under this identification, G invariant Finsler metric on G/H is in one-to-one correspondence with H invariant Minkowski norm on m, see [9] for more information on invariant metrics. However, under these, that the Kropina metric is Homogeneous. Also, according to [12], Riemannian metric α and 1-form β are invariant under action of G and induces a Minkowski norm. In the following, we adopt some ideas from [11] to deal with invariant Kropina metric.

Let $u_1, u_2, ..., u_n$ be an orthonormal basis of m with respect to $\langle , \rangle$. Then, there exists a local co-ordinate system on a neighborhood of $o = H$ which is defined by the mapping

$$(exp(x^1u_1), exp(x^2u_2), ..., exp(x^nu_n))H \to (x^1, x^2, ..., x^n).$$

Let Γ^k_{ij} be the christiffel symbols under the co-ordinate system, i.e., $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$. To compute the value of Γ^k_{ij} at the origin o, we need the following notations.

Let C^k_{ij} $(1 \leq i, j, k \leq n)$ be the structure constants of g, i.e., $C^k_{ij} = \langle [u_i, u_j]_m, u_k\rangle$, where $[u_i, u_j]_m$ denotes the projection of $[u_i, u_j]$ to m and $f(k, i)$ be defined by

$$f(k,i) = \begin{cases} 1, & k < i; \\ 0, & k \geq i. \end{cases}$$

Given $v \in g$, let $\hat{v}$ denote the fundamental Killing vector field generated by v, i.e., $\hat{v}_g H = \frac{d}{dt} exp(tv)H|_{t=0}$, $\forall g \in G$. Then, we use the following quantities proved by S. Deng and Z. Hu;

$$\left.\begin{array}{c} \Gamma^l_{ij}(o) = f(i,j)C^l_{ij} + < \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l >, \\ \frac{\partial \Gamma^l_{ij}}{\partial x^k}|_o = -\Gamma^s_{ij}(\Gamma^l_{ks} + < \nabla_{\hat{u}_k}\hat{u}_l, \hat{u}_s >) + f(k,j)C^a_{kj} < \nabla_{\hat{u}_i}\hat{u}_a, \hat{u}_l > \\ +f(k,i)C^s_{ki} < \nabla_{\hat{u}_s}\hat{u}_j, \hat{u}_l > + \hat{u}_k < \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l >, \quad i \geq j, \\ < \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l > |_o = -\frac{1}{2}(C^l_{ij} + C^j_{il} + C^i_{jl}), \\ \hat{u}_k < \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l > |_o = \frac{1}{2}(C^l_{ka}C^a_{ij} + C^j_{ka}C^a_{il} + C^i_{ka}C^a_{jl} + C^s_{ij}C^s_{kl} + C^s_{il}C^s_{kj} + C^s_{jl}C^s_{ki}). \end{array}\right\} \tag{2.2}$$

Here we have used simplified symbols, namely, when the upper index are repeated, it automatically takes the summation of all the products in the range of the index, i.e., $C^s_{ij}C^s_{kl} = \sum_{s=1}^n C^s_{ij}C^s_{kl}$. In the following, we will still use these simplified symbols.

Let $(M = G/H, F = \frac{\alpha^2}{\beta})$ be homogeneous Kropina space. According to [12], Riemannian metric α and 1-form β are both invariant. Therefore, we suppose that the invariant vector $\tilde{u}$ generated by $u = cu_n$ corresponds to the invariant 1-form β (i.e., $\alpha(\tilde{u}, X) = \beta(X)$ for any vector field X on M). To find the quantities involved in (2.1), we also need the following lemma, for further computations.

Lemma 2.1. *[11] For u_i, u_j, u_k, $u_l \in m$ and the value at the origin, we have*

$$b_i = c\delta_{ni}, \quad s_{ij} = \frac{c}{2}C^n_{ij}, \quad s_j = \frac{c^2}{2}C^n_{nj},$$

$$r_{ij} = -\frac{c}{2}(C^j_{ni} + C^i_{nj}), \quad s_{i;j} = cs_{ni;j} + \frac{c^2}{2}C^n_{li}\Gamma^l_{nj},$$

$$s_{ij;k} = \frac{c}{4}C^n_{lj}(C^l_{ki} + C^k_{il} + C^i_{kl}) + \frac{c}{4}C^n_{il}(C^l_{kj} + C^k_{jl} + C^j_{kl}) + \frac{c}{2}C^l_{ji}C^n_{kl},$$

$$\frac{1}{c}b_{i;j;k} = -\Gamma^s_{nj}\langle\nabla_{\hat{u}_k}\hat{u}_i, \hat{u}_s\rangle - \Gamma^i_{ns}\langle\nabla_{\hat{u}_k}\hat{u}_j, \hat{u}_s\rangle + C^s_{kn}\langle\nabla_{\hat{u}_s}\hat{u}_j, \hat{u}_i\rangle + \hat{u}_k\langle\nabla_{\hat{u}_n}\hat{u}_j, \hat{u}_i\rangle.$$

3. Ricci Curvature of Homogeneous Kropina metric

In this section, we compute the Ricci curvature of Homogeneous Kropina metric.
Using the quantities in (2.2) and lemma 2.1, we obtain the values of the quantities involved in (2.1) at the origin and direct computations, we get the following notations;

$$r_{00} = cC^0_{0n},\ \ s_0 = \frac{c^2}{2}C^n_{n0},\ \ r_0 = -\frac{c^2}{2}C^n_{n0},\ \ r = 0,\ \ r_{00;0} = -cC^0_{0s}(C^0_{ns} + C^s_{n0}),$$

$$r^m_m = -cC^m_{nm},\ \ r_{0m}r^m_0 = r_{0m}r_{m0} = \frac{c^2}{4}(C^m_{n0} + C^0_{nm}),$$

$$r_{00;m}b^m = cb_{00;n} = -\frac{c^2}{2}(C^s_{n0} + C^0_{ns})(-C^s_{n0} + C^0_{ns} + C^n_{0s}),$$

$$r_{0m;0}b^m = cr_{0n;0} = \frac{c^2}{2}[C^n_{ns}C^0_{s0} + \frac{1}{2}(C^0_{ns} + C^s_{n0})(C^s_{0n} + C^n_{s0} + C^0_{ns})],$$

$$r_{0m}s^m_0 = r_{0m}s_{m0} = -\frac{c^2}{2}C^n_{m0}(C^m_{n0} + C^0_{nm}),$$

$$s_{0;0} = \frac{c^2}{2}C^n_{ns}C^0_{0s},\ \ s_{0m}s^m_0 = s_{0m}s_{m0} = -\frac{c^2}{4}(C^n_{m0})^2,$$

$$s_m s^m_0 = \frac{c^3}{4}C^n_{nm}C^n_{m0},\ \ s_m r^m_0 = -\frac{c^3}{4}C^n_{nm}(C^0_{nm} + C^m_{n0}),\ \ r_m s^m_0 = -\frac{c^3}{4}C^n_{nm}C^n_{m0},$$

$$s_{0;m}b^m = \frac{c^3}{4}C^n_{ns}(C^s_{n0} + C^n_{0s} + C^0_{ns}),\ \ s_{m;0}b^m = \frac{c^3}{4}C^n_{ns}(C^s_{n0} + C^0_{ns} + C^n_{0s}),$$

$$s^m_{0;m} = s_{m0;m} = \frac{c}{2}C^n_{s0}C^m_{ms} + \frac{c}{4}C^n_{ms}(C^0_{ms} + C^m_{0s} + C^s_{0m}),$$

$$s_m s^m = \frac{c^4}{4}(C^n_{nm})^2,\ \ s^i_m s^m_i = s_{im}s_{mi} = -\frac{c^2}{4}(C^n_{mi})^2,$$

here the superscript 0 on the C's means to take the inner product with y, e.g., $C^0_{nm} = \langle [u_n, u_m]_m, y\rangle$. Summarize the above computations, we have the following;

Theorem 3.1. *Let G/H, α, β, $F = \frac{\alpha^2}{\beta}$, m, u, u_i are given. Then the Ricci scalar of homogeneous Kropina metric is given by*

$$\begin{aligned}Ric(y) = \overline{Ric} &+ \frac{\alpha^2c^3}{b^2\beta^4}[C^n_{0s}(C^s_{0n} + C^n_{0s} + C^0_{ns}) + C^n_{nm}((n-1)C^n_{m0} - 2)]\\
&- (n-1)\frac{\beta c}{b^2\alpha^2}C^0_{0s}(C^0_{0s} + C^s_{n0}) - \frac{\alpha^2 c}{2\beta}C^n_{s0}C^m_{ms} + \frac{\alpha^2 c}{4\beta}C^n_{ms}(C^0_{ms} + C^m_{0s} + C^s_{nm})\\
&+ \frac{c^2}{4b^2}[2\{(C^s_{n0})^2 - C^0_{ns}C^n_{0s} + (n-1)C^n_{ns}C^0_{0s}\} - 2nC^n_{m0}(C^m_{n0} + C^s_{nm})]\\
&+ \frac{(3n-4)c^4}{4b^4}(C^n_{n0})^2 + (n-1)\frac{3\beta c^3}{b^4\alpha^2}C^0_{0s}C^n_{ns} + (n-1)\frac{3\beta^2c^2}{b^4\alpha^4}(C^0_{0n})^2\\
&+ \frac{c^4}{4b^4}(C^n_{n0})^2 + \frac{\alpha^2c^3}{4b^2\beta^2}(C^n_{nm} + C^0_{m0}) - \frac{\alpha^4c^4}{8b^2\beta^2}(C^n_{nm})^2\\
&+ \frac{\alpha^2c^2}{4b^2\beta}C^n_{m0}(C^m_{n0} + C^0_{nm}) + \frac{\alpha^4c^2}{16\beta^2}(C^n_{mi})^2. \qquad (3.1)\end{aligned}$$

Let $m = m_0 + \mathbb{R}u_n$ be the orthogonal space decomposition with respect to the inner product $<,>$, then $u_1, u_2, ..., u_{n-1}$ form an orthonormal basis of m_0 with respect to $<,>$. Obviously, $Ad(h)(m_0) \subset m_0$, $\forall h \in H$, since the inner product $<,>$ on m is Ad H-invariant. In the following, we will use the symbol C^w_{uv}, C^m_{uv} to represent $\langle [u,v]_m, w\rangle$ and $\langle [u,v]_m, u_m\rangle$ respectively, where $u, v, w \in m$, $1 \le m \le n$.

Definition 3.2. *([29]) We say that the smooth function $\phi(s)$ is normal if it satisfies the following condition*

$$\sum_{i=1}^{n} k_i\phi_i(s) = const \Longleftrightarrow k_i = 0,\ \forall i = 1, 2, ...$$

Note that, if $\phi(s)$ satisfies (1.1), since by above definition, it is normal. It is easy to check that $\phi(s)$ is normal or not. However, without using this definition for a function ϕ, it is not easy to determine whether it is normal or not. As notice these concepts, we prove the following;

Proposition 3.2. *Let $F = \frac{\alpha^2}{\beta}$ be a homogeneous Kropina metric. If F is Einstein then either ϕ is not normal or there exists $\mu \in \mathbb{R}$ such that $\langle [u,y]_m, y\rangle = 0$ and $\langle [u,y]_m, y\rangle = \mu < y, y >$, $\forall y \in m_0$, where u is the AdH-invariant vector in m corresponding to β.*

Proof. Since in [28] F is Einstein. Let $y = \cos\theta w + \sin\theta u_n$, where $w \in m_0$ and $\langle w, w\rangle = 1$. Then, we have $\langle y, y\rangle = 1$ and $C^y_{yn} = \cos^2\theta C^w_{wn} + \cos\theta\sin\theta C^n_{wn}$, $C^n_{ny} = \cos\theta C^n_{nw}$. We compute the coefficients term of $Ric(y)$ in (3.1). By direct

computations, we get the following results;
The coefficient of $\frac{\alpha^2 c^3}{b^2\beta^4}$ is given by

$$\cos^2\theta(C^n_{ws}C^s_{wn}+(C^n_{ws})^2+C^w_{ns}C^n_{ws})+\cos\theta\sin\theta \\ (C^n_{ns}C^s_{wn}+3C^n_{ws}C^n_{ns}+C^n_{ns}C^w_{ns})+(n-1) \\ (\cos\theta C^n_{mw}C^n_{nm}+\sin\theta(C^n_{mn})^2)-2C^n_{nm}.$$

The coefficient of $\frac{-(n-1)\beta c}{b^2\alpha^2}$ is given by

$$\cos^3\theta(C^w_{ws}C^s_{nw}+\cos\theta(C^w_{ws})^2+2\sin\theta C^w_{ws}) \\ +\cos^2\theta\sin\theta(\sin\theta(C^s_{ws})^2+C^s_{ws}C^s_{nw}).$$

The coefficient of $\frac{-\alpha^2 c}{2\beta}$ is given by

$$\cos\theta C^n_{ms}C^m_{ms}+\sin\theta C^n_{sn}C^m_{ms}.$$

The coefficient of $\frac{-\alpha^2 c}{4\beta}$ is given by

$$C^n_{ms}[\cos\theta(C^w_{ms}+C^m_{ws}+C^s_{wm})+\sin\theta(C^n_{ms}+C^m_{ns}+C^s_{nm})].$$

The coefficient of $\frac{c^2}{2b^2}$ is given by

$$\cos^2\theta(C^s_{nw})^2-\sin^2\theta(C^n_{ns})^2-\cos^2\theta C^w_{ns}C^n_{ws} \\ -\sin\theta\cos\theta(C^w_{ns}C^n_{ns}+C^n_{ns}C^n_{ws}).$$

The coefficient of $\frac{(n-1)c^2}{2b^2}$ is given by

$$C^n_{ns}(\cos^2\theta C^w_{ws}+\cos\theta\sin\theta C^s_{ws}).$$

The coefficient of $\frac{-nc^2}{2b^2}$ is given by

$$\cos^2\theta C^n_{mw}(C^m_{nw}+C^w_{nw})+\cos\theta\sin\theta[C^n_{mw}C^n_{nm} \\ +C^n_{mn}(C^m_{nw}+C^w_{nw})]\sin^2\theta C^n_{mn}C^n_{nm}.$$

The coefficient of $\frac{(3n-4)c^4}{4b^4}$ is given by

$$\cos^2\theta(C^n_{nw})^2.$$

The coefficient of $\frac{(n-1)3\beta c^3}{\alpha^2 b^4}$ is given by

$$C^n_{ns}(\cos^2\theta C^w_{ws}+\cos\theta\sin\theta C^s_{ws}).$$

The coefficient of $\frac{3(n-1)\beta^2c^2}{\alpha^4b^4}$ is given by

$$\cos^4\theta(C^w_{wn})^2+\cos^2\theta\sin^2\theta(C^n_{wn})^2+2\cos^3\theta\sin\theta C^w_{wn}C^n_{wn}.$$

The coefficient of $\frac{c^4}{4b^4}$ is given by

$$\cos^2\theta(C^n_{nw})^2.$$

The coefficient of $\frac{\alpha^2c^3}{4\beta^2b^2}$ is given by

$$C^n_{nm}-\cos^2\theta C^w_{wm}-\cos\theta\sin\theta C^m_{wm}.$$

The coefficient of $\frac{\alpha^2c^2}{4\beta b^2}$ is given by

$$\begin{aligned}\cos^2\theta C^n_{mw}(C^m_{nw}+C^w_{nm})+\cos\theta\sin\theta[C^n_{mw}C^n_{nm}\\ +C^n_{mn}(C^m_{nw}+C^w_{nm})]+\sin^2\theta(C^n_{nm})^2.\end{aligned}$$

Replacing y by $\bar{y}=-\cos\theta w+\sin\theta u_n$ in (3.1), we get

$$\begin{aligned}Ric(y)+Ric(\bar{y})&=\overline{Ric}(y)+\overline{Ric}(\bar{y})+2\cos^4\theta((C^w_{ws})^2+(C^w_{wn})^2)\\&+\cos^2\theta\phi_1(c\sin\theta)+2\cos^2\theta C^n_{mw}\phi_2(c\sin\theta)\\&+\cos\theta\phi_3(c\sin\theta)+2C^n_{nm}+\varphi(\theta),\end{aligned}\tag{3.2}$$

where $\varphi(\theta)$ is a function of θ and $s=c\sin\theta$.
From the result of [5], the Ricci curvature of the homogeneos Riemannian manifold $(G/H,\alpha)$ is given by

$$\begin{aligned}{}^{\alpha}Ric(y)=&-\frac{1}{2}\sum_{l=1}^{n}\langle[y,u_l]_m,[y,u_l]_m\rangle+\frac{1}{4}\sum_{k,l=1}^{n}\langle[u_k,u_l]_m,y\rangle^2\\&-\frac{1}{2}K(y,y)-\langle[Z,y]_m,y\rangle,\end{aligned}\tag{3.3}$$

where Z is the unique vector in m defined by $\langle Z,X\rangle=tr(adX)$, $\forall X\in m$ and K is the Killing form of g. Therefore, it follows that

$$\begin{aligned}\overline{Ric}(y)+\overline{Ric}(\bar{y})=&\cos^2\theta[-(C^s_{wm})^2+\frac{1}{2}(C^w_{ms})^2]\\&-K(w,w)-2\langle[Z,w]_m,w\rangle+\psi(\theta),\end{aligned}\tag{3.4}$$

where $\psi(\theta)$ is function of θ.
Since F is Einstein, i.e., $Ric(X) = \lambda F^2(X)$, $\forall X \in m$, for some real number $\lambda \in \mathbb{R}$. Then, we have

$$Ric(y) = Ric(\bar{y}) = \lambda\phi^2(c\sin\theta). \tag{3.5}$$

Substitute (3.4) and (3.5) in (3.1), we obtain

$$\begin{aligned} 2\lambda\phi^2(c\sin\theta) - \varphi(\theta) - \psi(\theta) = \cos^2\theta[-(C^s_{wm})^2 + \frac{1}{2}(C^w_{ms})^2] \\ - K(w,w) - 2\langle[Z,w]_m, w\rangle + \cos^2\theta\phi_1(c\sin\theta) \\ + 2\cos^2\theta C^n_{mw}\phi_2(c\sin\theta) + \cos\theta\phi_3(c\sin\theta). \end{aligned} \tag{3.6}$$

Since equation (3.6) valid for any unit vector w in m_0, we choose another vector w' in (3.6). Then, we have

$$T(w) - T(w') + ((C^n_{mw})^2 - (C^n_{mw'})^2)\phi_2(s) = 0, \tag{3.7}$$

where $T(w) = -(C^s_{ws})^2 + 1/2(C^w_{ms})^2 - \cos\theta K(w,w) - 2\langle[Z,w]_m, w\rangle$.
Suppose ϕ is normal. Then, mention above different vectors w, w' in the unit ball m_0, we have

$$(C^n_{mw})^2 - (C^n_{mw'})^2 = 0.$$

Which implies that $C^n_{mw} = 0$ for any $w \in m$. Hence, there is a number $\lambda \in \mathbb{R}$ such that $(C^n_{mw})^2 = \lambda^2$. Since C^n_{mw} is a continuous function on m_0. Then, there exists a real number $\mu \in \mathbb{R}$ such that $C^n_{mw} = \mu$. According to properties of topology such μ does not exists, then the result has reverse and so μ must be exists and real number. This completes the proof.

4. Einstein Kropina metric with S-curvature

In this section, we find the formula of Ricci scalar of homogeneous Kropina metric with vanishing S-curvatute and we will give a sufficient and necessary condition on ϕ such that F is Einstein.
Note that, the authors Bao and Roble have proved in that a connected compact Einstein Randers space with negative Ricci scalar must be Riemannaian [9].

Here, we study the compact case: Suppose $M = G/H$ is compact and connected. Then G is compact. In particular, G is Unimodular, i.e.,$tr(Adx) = 0$, $\forall x \in g$, $g = Lie(G)$, on this we have the following

Proposition 4.3. *Let $F = \frac{\alpha^2}{\beta}$ is an invariant Kropina metric on the compact connected coset space G/H. Then, F has vanishing S-curvature if and only if $\langle[u,x]_m, x\rangle = 0$, $\forall x = 0$.*

Further, if F is Einstein, then either ϕ is not normal or F has vanishing S-curvature.

Proof. Since, $F = \alpha\phi(\beta/\alpha)$ has vanishing S-curvature if and only if $r_{ij} = 0$ and $s_j = 0$ in the local co-ordinate system.
Using the formula of S-curvature from theorem (2.1) in [10]. By direct computation, we obtain the S-curvature of homogeneous Kropina metric at the origin is given by

$$S(0, y) = \frac{(n-1)c}{F(y)b^2}\{\langle [u_n, y]_m, y\rangle - \alpha^2\langle [u_n, y], y\rangle\},\ y \in m. \tag{4.1}$$

Letting $x = 0$ and $y = u$ in (4.1), we get $c(o) = 0$. Hence, $S(0, y) = 0$, $\forall y \in m$.
Now, lemma 2.1, since $c(o) = 0$, which implies that r_{ij} and $s_j = 0$. Therefore, which asserts that F has vanishing S-curvature iff $\langle [u, x]_m, x\rangle = 0, \forall x \in m$.

Further, suppose F is Einstein and ϕ is normal, then proposition 3.5 shows that the invariant vector u satisfies $\langle [u, y]_m, y\rangle = 0$, $\langle [u, y]_m, y\rangle = \mu < y, y >$, $\forall y \in m$ for some $\mu \in \mathbb{R}$. Since G is unimodular $tr(Adu) = 0$. Thus, $C^n_{mw} = 0$. From section-3, we know that $C^n_{mw} = \mu = 0$, it implies $\mu = 0$. Therefore, $ad_m u$ is skew-symmetric with respect to $<, >$, that is $\langle [u, x]_m, x\rangle = 0$, $\forall x \in m$. Hence, F has vanishing S-curvature.
The main result of this section is the following:

Assume that F has vanishing S-curvature, $\langle [u, x]_m, x\rangle = 0$, $\forall x \in m$. Then, the formula of the Ricci scalar of homogeneous Kropina metric (3.1) can be simplied as the following

Theorem 4.2. *Let G/H, α, β, $F = \frac{\alpha^2}{\beta}$, m, u, u_i are given. Assume that F has vanishing S-curvature. Then the Ricci scalar of the homogeneous Kropina metric is given by*

$$\begin{aligned} {}^{\alpha}Ric(y) = &-\frac{1}{2}\sum_{l=1}^{n}\langle [y, u_l]_m, [y, u_l]_m\rangle + \frac{1}{4}\sum_{k,l=1}^{n}\langle [u_k, u_l]_m, y\rangle^2 - \frac{1}{2}K(y, y) - \langle [Z, y]_m, y\rangle \\ &+ \frac{\alpha^2 c^2}{4b^2\beta}\{\langle [u_m, u_n]_m, u_n\rangle(\langle [u_n, u_n]_m, u_m\rangle + \langle [u_n, u_m]_m, u_n\rangle)\}, \end{aligned} \tag{4.2}$$

where Z is the unique vector in m defined by $< Z, X >= tr(adX)$, $\forall X \in m$, K is the Killing vector field of g and $y \neq 0 \in m$, where m is subspace of group g

Proof. Assume that F has vanishing S-curvature, by lemma 3.2 we have, $C^n_{nj} = 0$, $C^j_{ni} + C^i_{nj} = 0$, $\forall i \leq i, j \leq n$. From these conditions, the coefficients of (3.1) all C_i's are zeros. Then, the result follows from theorem 3.1 and the formula of Ricci curvature of homogeneous Riemannian manifold $(G/H, \alpha)$ [5].

5. Fundamental Applications:Ricci quadratic homogeneous Kropina metric

In this section, we characterize the Ricci quadratic homogeneous Kropina metric. We glance that a Finsler space is called R-quadratic if the Riemannian curvature R_y is quadratic in y. It is obvious that a Riemannian manifold or a Berwald space must be R-quadratic. However, there exist many examples of non-Berwaldian spaces which are R-quadratic. A Finsler space is called Ricci quadratic if its Ricci scalar is quadratic in y.

In 2009, the authors Li and Z. Shen were studied Ricci-quadratic Randers space and they got a characterization of such spaces, using local co-ordinate system. In ([9], [12]), the authors S.Deng and Z. Hu proved that a homogeneous Randers space is Ricci-quadratic if and only if it is Berwald space. In the following above and by using the formula of Ricci scalar of homogeneous Kropina metric (3.1), we prove main result

Theorem 5.3. *A homogeneous Kropina metric is Ricci-quadratic if and only if it is Berwaldian.*

We prove this theorem, first we prove the following lemma:

Lemma 5.2. *Fix $x \in M$ and α, β, F as function of $y \in T_xM$. If f^*, g^* and h^* are polynomial of y such that $\frac{f^*}{F^2} + \frac{g^*}{F} + Fh^*$ is a polynomial, then $(\alpha^4/\beta^2)/f^*$.*

Proof. The polynomial Fh^* can be written as,

$$Fh^* = \frac{(\alpha^2 - \beta^2)h^*}{\beta} + \beta h^*.$$

Here, we see that if $\frac{f^*}{F^2} + \frac{g^*}{F} + Fh^*$ is a polynomial of y, then there is a polynomial g_1^* of y such that

$$h_1^* = \frac{f^*}{F^2} + \frac{g_1^*}{F},$$

is a polynomial of y, which implies that,

$$(\frac{\alpha^2}{\beta})^2 h_1^* - f^* = g_1^*(\frac{\alpha^2}{\beta}).$$

Since g_1^* is polynomial of y and α is the square root of the positive definite quadratic form, both sides of above equation must be vanishing, we have

$$g_1^*(\frac{\alpha^2}{\beta}) = 0,$$

$$(\frac{\alpha^2}{\beta})^2 h_1^* - f^* = 0.$$

Thus, $f^* = (\frac{\alpha^2}{\beta})^2 h_1^*(y)$. Hence, $(\alpha^4/\beta^2)/f^*$.

Proof the theorem 5.3: Since a Berwald metric is Ricci quadratic. We need only prove the 'only if ' part.

By equation (3.1), we deduce from above lemma that, if the Ricci scalar is quadratic, then

$$(\alpha^2/\beta)^2/(r_{00})^2.$$

Since α^2/β is an irreducible polynomial of y, there exists a constant c' such that

$$r_{00}\beta = c'\alpha^2.$$

Then, from lemma 2.1 and proposition 4.3. Equivalently,

$$C^j_{ni} + C^i_{nj} = 0. \tag{5.1}$$

In particular, considering the value at $y = u$ and taking into account the fact that $< u, u >= 1$, we get

$$C^n_{ij} = 0. \tag{5.2}$$

Equation (5.1) and (5.2), we have for any i, j

$$\langle [u, u_i]_m, u_j\rangle + \langle [u, u_j]_m, u_i\rangle = 0,$$
$$\langle [u_i, u_j]_m, u\rangle = 0.$$

By the conditions in [9], we conclude that F is a Berwald metric. This complete the proof of the theorem.

6. Conformal Deformation of the Kropina metric $K(\alpha, \beta)$

Consider the Kropina metric $K(\alpha, \beta) = \frac{\alpha^2}{\beta}$. Now, first we will study the Perturbation theory of this metric:

The Perturbation of Kropina metric is given by,

$$F(\alpha, \beta) = K(\alpha, \beta) + \varepsilon(x)\beta,$$

where $\varepsilon : M \to R^*$ is differentiable function, we remark that the particular case $\varepsilon(x) = \pm 1$ was studied by R. Miron, H. Shimada and V.s. Sabau in [19].

First we checked that F to be positive on $TM\backslash\{0\}$, through the following proposition;

Proposition 6.4. *If F is positive on $TM\backslash\{0\}$ if and only if it satisfies $||b|| < \frac{1}{\sqrt{|\varepsilon|}}$ and $\beta > 0$.*

Proof. Let F is positive on $TM\backslash\{0\}$, which implies that,

$$K(\alpha, \beta) + \varepsilon(x)\beta > 0, \quad \forall y \neq 0.$$

It follows that,

$$\frac{\alpha^2 + \varepsilon\beta^2}{\beta} > 0, \quad \forall y \neq 0.$$

Suppose positivity holds. Substitute $y^i = -b^i = -a^{ij}b_j$ in above inequality, we obtain the relation $||b|| < \frac{1}{\sqrt{|\varepsilon|}}$ and as $\beta > 0$.
Conversely, suppose that the conditions holds and using the Cauchy-Buniakowski Schwarza inequality, we have

$$|a_{ij}b^ib^j| \leq \sqrt{a_{pq}b^pb^q}\sqrt{a_{rs}y^ry^s}.$$

Thus, we obtain the positivity of F.
Now, using the formula for the fundamental tensor in [19], we have the fundamental tensor g_{ij} of the Kropina metric (M, F) is given by

$$\begin{aligned} g_{ij} &= 2\frac{\alpha^2}{\beta^2}a_{ij} + 4(\frac{\alpha}{\beta})^2 - 4(\frac{\alpha}{\beta})^3(b_il_j + b_jl_i) \\ &+ \left(\varepsilon^2 + 3(\frac{\alpha}{\beta})^4\right) b_ib_j, \end{aligned}$$

where $l_i = \frac{1}{\alpha}a_{ij}y^j$.
The contravariant tensor g^{ij} is expressed in the following form,

$$\begin{aligned} g^{ij} &= \frac{\beta^2}{2\alpha^2}a^{ij} - \Theta\left[\alpha^4b^ib^j - 2\alpha^3\beta(b^il^j + b^jl^i)\right. \\ &\left.-2\alpha^2(b^2(\alpha^2 - \varepsilon\beta^2) - 2\beta^2)l^il^j\right], \end{aligned}$$

where $b^2 = a^{ij}b_ib_j$ and $\Theta = b^2(A + B)$, where $A = 2(\alpha^3 + \varepsilon\alpha\beta^2)(3\alpha^4 + \varepsilon^2\beta^4)$ and $B = 2(\alpha^4 - \varepsilon\alpha^2\beta^2)$.
The covariant and contravariant metric tensor of Conformal Kropina metric $\bar{F}$ are given as;

$$\bar{g}_{ij} = e^{\sigma(x)}g_{ij} \text{ and } \bar{g}^{ij} = e^{-\sigma(x)}g^{ij}.$$

Therefore, the covariant $\bar{g}_{ij}$ and contravariant $\bar{g}^{ij}$ metric tensor of $\bar{F} = e^{\sigma(x)}F$ are given by,

$$\begin{aligned}\bar{g}_{ij} &= e^{\sigma(x)}\left\{2\frac{\alpha^2}{\beta^2}a_{ij} + 4(\frac{\alpha}{\beta})^2 - 4(\frac{\alpha}{\beta})^3(b_il_j + b_jl_i)\right. \\ &\quad \left. + \left(\varepsilon^2 + 3(\frac{\alpha}{\beta})^4\right)b_ib_j\right\},\end{aligned}$$

and

$$\begin{aligned}\bar{g}^{ij} &= e^{-\sigma(x)}\left\{\frac{\beta^2}{2\alpha^2}a^{ij} - \Theta[\alpha^4 b^ib^j - 2\alpha^3\beta(b^il^j + b^jl^i).\right. \\ &\quad \left. . -2\alpha^2(b^2(\alpha^2 - \varepsilon\beta^2) - 2\beta^2)l^il^j]\right\}.\end{aligned}$$

Now, taking into accounts the invariants [19] related to $\bar{F}$ are,

$$\bar{\rho} = e^{\sigma}(2(\frac{\alpha}{\beta})^2 + 2\varepsilon), \quad \bar{\rho}_0 = -e^{\sigma}(3(\frac{\alpha}{\beta})^4 + \varepsilon^2),$$
$$\bar{\rho}_{-1} = -e^{\sigma}(4\frac{\alpha^2}{\beta^3}), \quad \bar{\rho}_{-2} = e^{\sigma}(\frac{4}{\beta^2})$$

By direct computation we get,

Proposition 6.5. *If k_{ij} is the fundamental tensor of the Kropina Space (M, K), then the fundamental tensor field $\bar{g}_{ij}$ of the conformal deformation space $(M, \bar{F})$ is expressed by*

$$\bar{g}_{ij} = \bar{k}_{ij} + 2\varepsilon\bar{a}_{ij} + \varepsilon^2\bar{b}_i\bar{b}_j.$$

7. Single Colored Kropina metric

In this section, we characterize the single colored Kropina metric.

The Finslerian L. Berwald studied [4] a class of Finsler metrics $F = F(x, y)$ on a manifold M, whose geodesics are determined by second order ODEs similar to the Riemannian case. More precisely the geodesics in local co-ordinates satisfy

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i(x, y) = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are called quadratic in $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. Finsler metric with this property are called Berwald metrics. It can be shown that Berwald manifolds are modeled on a single norm space, i.e., all the tangent spaces T_xM with induced norm F_x are linearly isometric to each other. Intuitively speaking, if one assigns a single color to each tangent norm space (T_xM, F_x) depending on the geometric shape of the unit tangent sphere S_xM then the Berwald manifolds has uniform color.

Definition 7.3. *(29) On the manifold M, a Finsler metric F is called single colored, if for every point x, there is a neihborhood U and a local frame field $\{e_i\}$ defined on U such that $F(y^ie_i) = F(y^1, y^2, ..., y^m)$ is a function of y^i's.*

Remark: As we know, Finsler metric is single colored if and only if the tangent spaces are linearly isometric to each other (as Minkowski spaces). The author, Z. Shen has said, in many occasions, that a Finsler manifold is usually colorful, because if we assign a color to each kind of Minkowski space, then a Finsler manifold can admit many colors and also author M. Matsumoto [18] suggested to consider 1-form metrics, whose definition is similar to our single colored ones, but with additional requirement that the underlying manifold being parallelizable, that is exactly in [15], Libing Huang has proved the following lemma;

Lemma 7.3. *If F is a single colored Finsler metric on M, then any conformal deformation $e^{\rho(x)}F$ is also single colored.* Combining theorem 5.3 , proposition 6.5 and lemma 7.3, we state

Theorem 7.4. *Let $F = \frac{\alpha^2}{\beta}$ be single colored Kropina metric on M, then the conformal deformation $\bar{F}$ is also single colored.*

Proof. Since α is Riemannian metric and β is 1-form, which has a non zero constant [4] and proposition 5.3, F be single colored. Then by proposition 6.5, implies that the conformal deformation of F. Hence, from lemma 7.4, we conclude that $\bar{F}$ is single colored.

Example: Suppose that $F = \frac{\alpha^2}{\beta}$ be homogeneous Kropina metric, which is single colored. Now, consider the orthogonal invariance of this metric, which is of the form

$$\bar{F}(x, y) = |x|^\sigma \left(\frac{|y|^2|x|}{< x, y >} \right),$$

where $|,|$ and $<,>$ denote the usual norm and inner product on a Euclidean space respectively. By elementary properties of linear algebra in particularly, Grams-Schmidt orthogonalization process, which would conclude that F is the conformal deformation of the single colored Kropina metric with $\alpha = |y|$ and $\beta =< x, y > /|y|$.

5. Conclusion

The Kropina metric is an (α, β)- metric, which was considered by V. K. Kropina. This metric is of physical interest in the sense that it describes the general dynamical system represented by a Lagrangian function. On Einstein Kropina metric has been investigated by X. Zhang and Yi-Bing Shen.

The theory of homogeneous/symmetric Riemannian manifolds has become the basis of many branches of mathematics, including group, geometry analysis and representation theory. That, theory was called Lie theory to study the Finsler geometry, which has developed by the theory of homogeneous/symmetric Finsler spaces.

There are several interesting curvatures in Finsler geometry, among them the Ricci curvature and flag curvature are most important. The flag curvature is the natural generalization of sectional curvature. The notion of S-curvature of Finsler spaces, comes under the Riemannian-Finsler geometry.

As we know, Finsler manifold is usually colorful because if we assign a color to each kind of Minkowski space then the Finsler manifold can admit many colors. It is an very interesting to study those Finsler manifolds with a single color.

In this paper, we consider the homogeneous Kropina metric, first we found the formula of Ricci curvature for homogeneous Kropina metric. Using this formula, we proved a necessary condition related ϕ for F to be Einstein. Then, we shown that if ϕ is normal. Moreover, on the compactness, we obtained the sufficient and necessary condition for a homogeneous Kropina metric to be Einstein with vanishing S-curvature. Further, we studied the conformal deformation of this metric. Under this, we proved that the conformal deformation of Kropina metric is single colored.

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