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LINEAR AND CYCLIC NODE ARRANGEMENT OF CARTESIAN PRODUCT OF CERTAIN GRAPHS

Jessie Abraham

Department of Mathematics,
Loyola College, Chennai 600034, INDIA
E-mail: jessie.abrt@gmail.com

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Abstract: The linear node arrangement of a graph G on n nodes is the embedding of the nodes of the graph onto the line topology L in such a way that the sum of the distance between adjacent nodes in G is optimized. The cyclic node arrangement is the embedding of the nodes of G onto a cycle C in such a way that the optimization is preserved. In this paper we obtain general results to compute the cyclic and linear node arrangement of a class of Cartesian product graphs with C_k and P_k respectively, where C_k, $k \geq 2$, is a cycle on k nodes and P_k is a path on k nodes and their conditional edge faulty graphs.

Keywords and Phrases: Embedding, optimal ordering, edge faulty graph.

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1. Introduction

Let $G = (V_G, E_G)$ be an undirected arbitrary graph with node set $V_G = \{1, 2, \ldots, n\}$. The linear arrangement of G is a bijective mapping λ from V_G to V_L. The cost of a linear arrangement λ is given by

$$LA_\lambda(G) = \sum_{(u,v)\in E_G} |\lambda(u) - \lambda(v)|$$

The linear node arrangement problem is to nd a λ such that $LA_\lambda(G)$ is minimized. The minimum thus obtained is called linear node arrangement of G and is denoted by $LA(G)$ [1]. The cyclic arrangement of G is a bijective mapping λ from V_G to V_C. The cost of a cyclic arrangement λ is given by

$$CA_\lambda(G) = \sum_{(u,v)\in E_G} |\lambda(u) - \lambda(v)|.$$

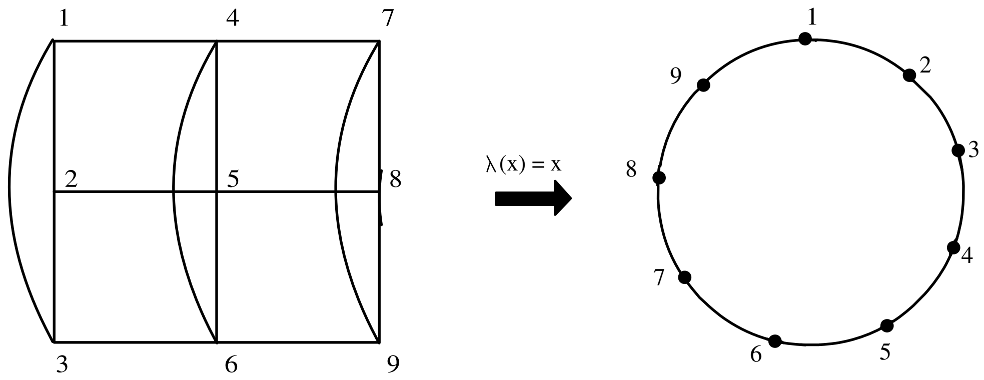

Figure 1: Cyclic node arrangement of cylinder G with $\lambda(x) = x$ and $CA(G) = 30$.

The cyclic node arrangement problem is to find $\min_\lambda CA_\lambda(G)$ and is denoted by $CA(G)$ [3]. Fig 1 illustrates the cyclic node arrangement of a cylinder graph. This problem is also known in the literature as the cyclic bandwidth problem [7]. It has applications in electronic design automation, minimizing communication congestions and smooth signal processing in computer networks and parallel computing [3,5]. The problem has been investigated in the literature for graphs including hypercubes [4], arrangement graphs [3], trees and hypergraphs [5].

Optimal Ordering: For any $k(1 \leq k \leq n)$, an ordering $\mathcal{O}_k$ of the set $V(G)$ is said to be an optimal ordering if the subgraph of G induced by the first k nodes in this order is maximal among any other induced subgraph of the same node cardinality. Then $CA_{\mathcal{O}_k}(G) = CA(G)$ and $LA_{\mathcal{O}_k}(G) = LA(G)$. If G is an r-regular graph, then $CA(G) = LA(G) = \sum\limits_{k=1}^{n} rk - 2|E(G[\mathcal{O}_k])|$ [2,6].

In a large interconnection network, nodes or edges often develop faults. The fault can either be conditional or random. Fault tolerance is an extensively researched feature of interconnection networks which determines the ability of a system to maintain its functionality, even in the presence of faults. It is very important to take fault tolerance into account when developing embedding algorithms and hardware modules. The graph embedding where all faulty nodes and edges have been removed is called fault-tolerant embedding [8].

2. Main Results

Definition 2.1. *Let H be a graph of size n with optimal order $\mathcal{O} = \{v_1, v_2, \ldots, v_n\}$. Suppose for the Cartesian product $H \times C_k$ (or) $H \times P_k$, where C_k and P_k, $k \geq 2$, is a cycle and path on k nodes, with the node set of the i^{th} copy as $V(H_i) =$*

$\{v_{i1}, v_{i2}, \ldots, v_{in}\}$, $\mathcal{O}_{ij} = \langle v_{i1}, , \ldots, v_{in}, v_{j1}, \ldots, v_{jn}\rangle$ *is an optimal order in* $H_i \cup H_j$, $1 \le i < j \le k$, *then* H *is called a sequentially optimal order graph denoted as SO-graph.*

Notation. For any two non-empty subsets A and B of V_G, $E_{A\wedge B} = \{(x, y) \in E_G : x \in A, y \in B\}$.

The node arrangement problem of SO-graphs nds application in the implementation of parallel algorithms in parallel processors wherein the processors in the network are homogenous multi-core processors like Intel, IBM, Sun and AMD.

Theorem 2.2. *Let* H *be a SO-graph and* $G = H \times C_k$, $k \ge 2$. *Then* $CA(G) = k[CA(H \times P_2) - CA(H)]$.

Proof. We first prove that $\mathcal{O} = \langle v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{k1}, \ldots, v_{kn}\rangle$ is an optimal ordering of $V(G)$. Let $\mathcal{B}$ be any ordering of $V(G)$. Let $\mathcal{B}_{i(i+1)}$ be an order taken from $\mathcal{F}$ corresponding to $H_i \cup H_{i+1}, 1 \le i \le k-1$. Since $\mathcal{O}_{i(i+1)} = \Big\langle v_{i1}, v_{i2}, \ldots, v_{in}, v_{(i+1)1}, v_{(i+1)2}, \ldots, v_{(i+1)n}\Big\rangle$ is the optimal ordering of $V(H_i \cup H_{i+1})$, we have

$$CA_{\mathcal{O}_{i(i+1)}}(H_i \cup H_{i+1}) \le CA_{\mathcal{B}_{i(i+1)}}(H_i \cup H_{i+1}).$$

Since $\mathcal{O}_{1k} = \langle v_{11}, v_{12}, \ldots, v_{1n}, v_{k1}, v_{k2}, \ldots, v_{kn}\rangle$ is the optimal ordering of $V(H_1 \cup H_k)$,

$$CA_{\mathcal{O}_{1k}}(H_1 \cup H_k) \le CA_{\mathcal{B}_{1k}}(H_1 \cup H_k).$$

Then

$$\sum_{i=1}^{k-1} CA_{\mathcal{O}_{i(i+1)}}(H_i \cup H_{i+1}) + CA_{\mathcal{O}_{1k}}(H_1 \cup H_k) \le \sum_{i=1}^{k-1} CA_{\mathcal{B}_{i(i+1)}}(H_i \cup H_{i+1}) + CA_{\mathcal{B}_{1k}}(H_1 \cup H_k)$$

$$\Rightarrow CA_{\mathcal{O}}(G) + \sum_{i=1}^{k-1} CA_{\mathcal{O}_i}(H_i) \le CA_{\mathcal{B}}(G) + \sum_{i=1}^{k-1} CA_{\mathcal{B}_i}(H_i) \Rightarrow CA_{\mathcal{O}}(G) \le CA_{\mathcal{B}}(G).$$

We have

$$\begin{aligned} CA_{\mathcal{O}}(G) &= \sum_{i=1}^{k} CA_{\mathcal{O}_i}(H_i \cup H_{i+1}) - \sum_{i=1}^{k} CA_{\mathcal{O}_i}(H_i) \\ &= k[CA_{\mathcal{O}}(H \times P_2) - CA_{\mathcal{O}}(H)]. \end{aligned}$$

Therefore,

$$CA(G) = k[CA(H \times P_2) - CA(H)].$$

When C_k is replaced by P_k, $k \ge 2$ and the same proof technique as in 2.2 is employed, we get the following result.

Theorem 2.3. *Let H be a SO-graph and $G = H \times P_k$, $k \geq 2$. Then $LA(G) = (k-1)LA(H \times P_2) - (k-2)LA(H)]$.*

Let $\{H_i\}_{i=1}^{k}$ be a set of finite pairwise disjoint subgraphs of the Cartesian product graph with node set $V(H_i) = \{v_{i1}, v_{i2}, \ldots, v_{im_i}\}$, $1 \leq i \leq k$.

Definition 2.4. *A bridge-path graph $BP(H_1, H_2, \ldots, H_k)$ is a conditional edge faulty graph obtained by deleting the first $n-1$ edges in each $E_(H_i \wedge H_{i+1})$, $1 \leq i \leq k-1$ of $H \times P_k$. A bridge-cycle graph $BC(H_1, H_2, \ldots, H_k)$ is a conditional edge faulty graph obtained by deleting the first $n-1$ edges in each $E_(H_i \wedge H_{i+1})$, $1 \leq i \leq k-1$ and $E(H_k \wedge H_1)$ of $H \times C_k$. Fig. 2 illustrates the bridge-path and bridge-cycle graph.*

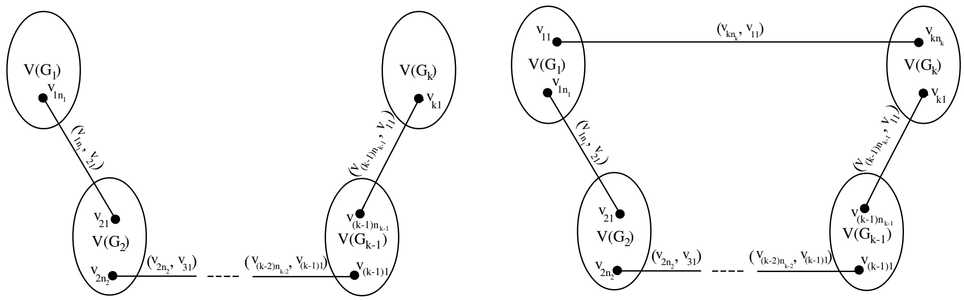

Figure 2: (a) A bridge-path graph (b) A bridge-cycle graph

Theorem 2.5. *Let $G = BC(G_1, G_2, \ldots, G_k)$ be a bridge-cycle graph. If each $G_i, 1 \leq i \leq k$, has an optimal order, then*

$$CA(G) = \sum_{i=1}^{k} CA(G_i) + k.$$

Proof. Let $\mathcal{R}$ be any ordering of $V(G)$. Let $\mathcal{R}_i$ be an order taken from $\mathcal{R}$ corresponding to $G_i, 1 \leq i \leq k$. Let $\mathcal{O}_i = \langle v_{i1}, v_{i2}, \ldots, v_{in_i} \rangle$ be the optimal order for $V(G_i)$. We prove that $\mathcal{O} = \langle\, v_{11}, v_{12}, \ldots, v_{1n_1}, v_{21}, v_{22}, \ldots, v_{2n_2}, \ldots, v_{k1}, v_{k2}, \ldots, v_{kn_k} \,\rangle$ is an optimal order of $V(G)$. Since $\mathcal{O}_i$ is the optimal order of $V(G_i)$, we have

$$CA_{\mathcal{O}_i}(G_i) \leq CA_{\mathcal{R}_i}(G_i), 1 \leq i \leq k.$$

Hence

$$\sum_{i=1}^{k} CA_{\mathcal{O}_i}(G_i) \leq \sum_{i=1}^{k} CA_{\mathcal{R}_i}(G_i)$$

and as a result,

$$CA_{\mathcal{O}}(G) \leq CA_{\mathcal{R}}(G).$$

Therefore, $\mathcal{O}$ is the optimal ordering of G.
We have

$$CA_{\mathcal{O}}(G) = \sum_{i=1}^{k} CA_{\mathcal{O}_i}(G_i) + k.$$

Therefore,

$$CA(G) = \sum_{i=1}^{k} CA(G_i) + k.$$

Proceeding in the same way, we get the linear node arrangement of the bridge-path graph as follows.

Theorem 2.6. *Let $G = BP(G_1, G_2, \ldots, G_k)$ be a bridge-path graph. If each $G_i, 1 \leq i \leq k$, has an optimal order, then*

$$LA(G) = \sum_{i=1}^{k} LA(G_i) + k - 1.$$

3. Conclusion

In this paper we have obtained general results to compute the cyclic and linear node arrangement of the Cartesian product of a sequentially optimal order graph with cycle C_k and path P_k respectively and their conditional edge faulty graphs.

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